Wan Muhamad Saridan Wan Hassan

Physics—Problems, Solutions, and Computer Calculations

Vol. 1 Mechanics, Properties of Matter, and Heat

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To my parents, Wan Hassan Wan Yusoff and Wan Saudah Wan Ahmad

Preface

The book is an introductory physics study book. The book offers students of science and engineering the basic concepts and principles of introductory physics, presenting problems and their solutions by analytical and computer calculations. It is for the introductory physics learning at early undergraduate university education.

The introductory physics topics are grouped into two volumes,

Volume I Mechanics, Properties of Matter, and Heat—which is the present volume,
Volume II Waves, Sound, Electricity, Magnetism, and Optics.

Each chapter begins with main points of the topic. These are summary of concepts, principles, definitions, and formulae of the topic. Then, problems are posed and solved. Steps are detailed, so that reasoning and understanding are built. Many figures are drawn to help in visualizing the physics problems and solutions. Calculations and solutions are also performed by computer using wxMaxima to instill computational skill. An Appendix *Introduction to wxMaxima* is provided to get student start the software. Calculations by wxMaxima achieved the solutions itself or for rechecking the values obtained analytically.

Chapter 1 is on measurement, uncertainty, conversion of units, and dimensional analysis, while Chap. 2 is on vectors. These are basic knowledge, tool, and skill that are required for studying and doing introductory physics.

Problems on description of motion in one dimension in terms of displacement, velocity, and acceleration are considered in Chap. 3, while description of projectile motion in two dimensions is considered in Chap. 4.

Chapter 5 is on problems that involve Newton's second and third laws of motion. Physical situations of blocks on flats and inclines with or with negligible friction, light pulley, and tensions in light strings are considered. Analyses include getting the net force, setting and solving equations using the Newton's law.

Problems involving uniform motion in circle are solved in Chap. 6. Uniform circular motion in horizontal and vertical circles relative to the earth, conical pendulum, and motion of satellite is considered. Centripetal acceleration and centripetal force related to the uniform circular motion are emphasized.

Chapter 7 solves problems related to mechanical work, energy, and power. Work as defined by vector dot product of force and displacement is applied. Kinetic energy of a moving mass, gravitational potential energy of a mass due to its height, and elastic potential energy of a stretched or compressed spring are considered. Conservation of mechanical energy and work–energy theorem is applied.

Chapter 8 considers problems on linear momentum and collisions in one and two dimensions. Conservations of linear momentum and kinetic energy apply to an elastic collision, while conservation of linear momentum applies to an inelastic collision.

Chapter 9 solves problems involving rotational motion, moment of inertia, radius of gyration, torque, angular acceleration, rotational kinetic energy, and conservation of angular momentum. For rigid or extended body, its rotational motion and translational motion of its center of mass are considered.

Chapter 10 solves problems on rigid body in mechanical equilibrium. Two mechanical equilibrium conditions—net force on a body and net torque about any point on the body are zero—are applied to arrive at solutions.

Problems on simple harmonic motion and oscillation are discussed in Chap. 11. Simple harmonic motion is defined as a motion whose acceleration is proportional to negative of its displacement. Oscillations of simple pendulum, mass–spring system, and physical pendulum are considered as simple harmonic motions.

Problems on general laws of gravity are discussed in Chap. 12. These include problems related to gravitational force between two masses, their gravitational potential energy, and gravitational field. Motions of earth around the sun and that of satellite around the earth are studied.

Problems related to elastic properties of material are solved in this Chap. 13. These elastic properties are characterized by Young's modulus, shear modulus, bulk modulus, or Poisson's ratio. Energy stored in deformed elastic material is also considered.

Chapter 14 solves problems on hydrostatics of fluid. These include problems related to fluid density and pressure, Pascal's law, floating and Archimedes principle, and surface tension. The equilibrium of forces in physical situations is applied to solve the problems.

Chapter 15 solves problems on fluid in motion. These include problems related to non-turbulent flow of incompressible fluid, continuity and Bernoulli's equations, laminar flow of viscous fluid, Poiseuille's equation, Stokes' law, and terminal speed.

Chapter 16 discusses problems on temperature and thermal expansion. Different temperature scales and conversion of temperature to different temperature scales are discussed. Problems related to linear, area, and volume thermal expansions and their coefficients of thermal expansion are solved.

Problems on heat and calorimetry are solved in Chap. 17. Heat as rise in temperature ($Q = mc\ \Delta\theta$) and as phase change ($Q = mL$) and conservation of energy are applied to solve the problems.

Chapter 17 solves problems on heat transfer by conduction, convection, and radiation. For conduction of heat through a material, the rate of heat conduction is proportional to coefficient of thermal conduction of the material. For forced convection, the rate of heat loss from a body is proportional to temperature difference of

the body over the surrounding. Rate of radiation emission from a body follows the Stefan's law.

Chapter 18 solves problems on heat transfer by conduction, convection, and radiation. For conduction of heat through a material, the rate of heat conduction is proportional to coefficient of thermal conduction of the material. For forced convection, the rate of heat loss from a body is proportional to temperature difference of the body over the surrounding. Rate of radiation emission from a body follows the Stefan's law.

Chapter 19 considers problems related to gas laws and kinetic theory of gases. Ideal gas law is used to solve many of the problems. Problems on Maxwell's speed distribution of a gas are also considered.

The last chapter solves problems related to first law of thermodynamics which is a statement of conservation of energy. Problems on heat engines and entropy are also discussed.

I wish to acknowledge the advice from several of my past colleagues and undergraduate students on the idea of the book. I am also grateful to my wife for her continuing encouragement and editorial staff of Springer Nature for their support.

Johor Bahru, Malaysia
2023

Wan Muhamad Saridan Wan Hassan

About This Book

The book offers students of science and engineering the basic concepts and principles of introductory physics, presenting problems and their solutions by analytical and computer calculations. It is for the introductory physics learning at first undergraduate year of higher education. This volume covers topics of physical mechanics, properties of matter, and heat. Each chapter begins with main points of the topic. These are summary of concepts, principles, definitions, and formulae of the topic. Then, problems are posed and solved. Steps are detailed, so that reasoning and understanding are built. There are more than 300 worked problems and 100 exercises in this volume. There are 306 figures drawn to help visualize the physics problem and solution. Calculation and solution are also performed by computer using wxMaxima to provide insight and instill computational skill. The knowledge and skill presented by the book are important foundations for further studies in science or engineering. Physics teachers would also find the book useful for their instruction.

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Chapter 1
Measurement, Uncertainty, Conversion of Units, and Dimensional Analysis

1.1 Basic Concepts and Formulae

(1) Most of the physical quantities, in mechanics for example, can be expressed in terms of three fundamental quantities: mass, m, length, l, and time, t. These quantities can be physically measured using instruments or devices.
(2) If u that is to be determined as a function of x and y i.e. $u = u(x, y)$, while $x \pm \Delta x$ and $y \pm \Delta y$ are the measured quantities and their uncertainties, the uncertainty of u, Δu can be estimated as shown in Table 1.1.
(3) Dimensions of m, l, and t are M, L, and T, respectively. SI units for mass, length, and time are kilogram (kg), meter (m), and second (s), respectively.
(4) Dimensional analysis can be used to check the consistency of an equation and derive an expression. The dimension of the left side of an equation must be the same as the one given on the right side.
(5) Linear least square fit of a line: If (x_i, y_i), $i = 1 \ldots n$ are points to be fitted to line $y = mx + c$, then the slope, m and intercept, c are,

$$m = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \quad \text{and} \quad c = \frac{\sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i}{n},$$

and the estimates of uncertainty of the slope, Δm and the intercept, Δc are,

$$\Delta m = \sqrt{n\frac{\sigma^2}{\Delta}} \quad \text{and} \quad \Delta c = \sqrt{\frac{\sigma^2}{\Delta}\sum_{i=1}^{n} x_i^2},$$

where $\Delta = n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2$ and $\sigma^2 = \frac{1}{n-2}\sum_{i=1}^{n}[y_i - (mx_i + c)]^2$.

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Table 1.1 Functions and their uncertainties

Function, $u(x, y)$	Uncertainty approximation, Δu	Uncertainty, Δu (based on error propagation theory)
$u = x + y$	$\Delta u \approx \Delta x + \Delta y$	$\Delta u = \sqrt{(\Delta x)^2 + (\Delta y)^2}$
$u = x - y$	$\Delta u \approx \Delta x + \Delta y$	$\Delta u = \sqrt{(\Delta x)^2 + (\Delta y)^2}$
$u = xy$	$\frac{\Delta u}{u} \approx \frac{\Delta x}{x} + \frac{\Delta y}{y}$	$\frac{\Delta u}{u} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$
$u = \frac{x}{y}$	$\frac{\Delta u}{u} \approx \frac{\Delta x}{x} + \frac{\Delta y}{y}$	$\frac{\Delta u}{u} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$
$u = x^n$	$\frac{\Delta u}{u} = n\frac{\Delta x}{x}$	$\frac{\Delta u}{u} = n\frac{\Delta x}{x}$

(6) Error propagation theory states that if $u = u(x, y, z, \ldots)$, then the error in u i.e. σ_u is,

$$\sigma_u = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial u}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial u}{\partial z}\right)^2 \sigma_z^2 + \cdots}$$

Table 1.1 is obtained by applying this formula. Entries in column three are derived from entries in column one. Entries in column two are the approximations of entries in column three.

1.2 Problems and Solutions

Problem 1.1 Given that,

$$1\,\text{mile}^2 = 640\,\text{acre}$$
$$1\,\text{km}^2 = 100\,\text{ha}$$
$$1\,\text{mile} = 1.609\,\text{km}.$$

Express 1 ha in acre.

Solution

Conversion of units is as follows,

$$100\,\text{ha} = 1\,\text{km}^2 \times \frac{1\,\text{mile}^2}{(1.609)^2\,\text{km}^2} \times \frac{640\,\text{acre}}{1\,\text{mile}^2} = 247\,\text{acre}.$$

This means that 1 ha = 2.47 acre.

- wxMaxima codes:

```
(%i1)  fpprintprec:5;
(fpprintprec)      5
(%i2)  640/1.609^2;
(%o2)  247.21
```

Comments on the codes:

(%i1) Set floating point print precision to 5 digits.
(%i2) Calculate 640 divided by 1.609^2.
(%o2) Result of the calculation.

Example 1.2 The length and width of a metal plate are 12.30 ± 0.04 cm and 4.24 ± 0.03 cm, respectively. Calculate the area of the plate and its uncertainty.

Solution

Using A, l, and w for area, length, and width, respectively, the area of the plate is,

$$A = lw = 12.30\,\text{cm} \times 4.24\,\text{cm} = 52.15\,\text{cm}^2.$$

Uncertainty in area, ΔA, can be estimated using the calculus as follows,

$$\begin{aligned} A &= lw, \\ \Delta A &= l\,\Delta w + w\,\Delta l, \\ \frac{\Delta A}{A} &= \frac{l\,\Delta w}{lw} + \frac{w\Delta l}{lw} = \frac{\Delta w}{w} + \frac{\Delta l}{l}, \\ \Delta A &= \left(\frac{\Delta w}{w} + \frac{\Delta l}{l}\right) A = \left(\frac{0.03\,\text{cm}}{4.24\,\text{cm}} + \frac{0.04\,\text{cm}}{12.30\,\text{cm}}\right) \times 52.15\,\text{cm}^2 \\ &= 0.54\,\text{cm}^2. \end{aligned}$$

The area of the metal plate is written as $A \pm \Delta A = 52.15 \pm 0.54\ \text{cm}^2$.

- wxMaxima codes:

```
(%i3)  fpprintprec:5; l:12.3; w:4.24;
(fpprintprec)      5
(l)    12.3
(w)    4.24
(%i4)  A: l*w;
(A)    52.152
```

```
(%i6) dw:0.03; dl:0.04;
(dw)  0.03
(dl)  0.04
(%i7) dA: (dw/w + dl/l)*A;
(dA)  0.5386
```

Comments on the codes:

(%i3) Set floating point print precision to 5 and assign values of length l as 12.3 cm and width w as 4.24 cm.
(%i4) Calculate area, $A = l \times w$.
(%i6) Assign values of Δw as 0.03 cm and Δl as 0.04 cm.
(%i7) Calculate uncertainty, $\Delta A = (\Delta w/w + \Delta l/l)A$.

Problem 1.3 The mass and volume of an object are measured to be 2.40 ± 0.01 g and 4.35 ± 0.03 cm^3, respectively. Calculate the density of the object and its uncertainty.

Solution

The density, ρ of the object is,

$$\rho = \frac{m}{V} = \frac{2.40\,\text{g}}{4.35\,\text{cm}^3} = 0.552\,\text{g cm}^{-3},$$

where m is the mass and V is the volume. Using the calculus, one finds,

$$\begin{aligned}
\rho &= \frac{m}{V} = mV^{-1}, \\
\Delta\rho &= \Delta m V^{-1} - mV^{-2}\Delta V = \frac{\Delta m}{V} - \frac{m\Delta V}{V^2}, \\
\frac{\Delta\rho}{\rho} &= \frac{\Delta m}{V(m/V)} - \frac{m\Delta V}{V^2(m/V)} = \frac{\Delta m}{m} - \frac{\Delta V}{V}.
\end{aligned}$$

For the estimation of uncertainty, the minus sign in the last equation is replaced with plus sign, because uncertainties add up, they do not getting smaller. The uncertainty in density, $\Delta\rho$ is,

$$\begin{aligned}
\Delta\rho &= \left(\frac{\Delta m}{m} + \frac{\Delta V}{V}\right)\rho = \left(\frac{0.01\,\text{g}}{2.40\,\text{g}} + \frac{0.03\,\text{cm}^3}{4.35\,\text{cm}^3}\right) \times 0.552\,\text{g cm}^{-3} \\
&= 0.006\,\text{g cm}^{-3}.
\end{aligned}$$

The density of the object is $\rho \pm \Delta\rho = 0.552 \pm 0.006$ g cm^{-3}.

- wxMaxima codes:

```
(%i5)  fpprintprec:5; m:2.4; dm:0.01; V:4.35; dV:0.03;
(fpprintprec)      5
(m)    2.4
(dm)   0.01
(V)    4.35
(dV)   0.03
(%i6)  rho: m/V;
(rho)  0.55172
(%i7)  delta_rho: (dm/m + dV/V)*rho;
(delta_rho)  0.0061038
```

Comments on the codes:

(%i5) Set floating point print precision to 5 and assign values of m, Δm, V, and ΔV.
(%i6), (%i7) Calculate density, $\rho = m/V$ and its uncertainty, $\Delta\rho$.

Problem 1.4 The diameter of a ball bearing measured by a micrometer is 9.72 ± 0.01 mm. Calculate the volume of the ball bearing and its uncertainty.

Solution

Using V, r, and d for volume, radius, and diameter of the ball, respectively, its volume is,

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 = \frac{1}{6}\pi d^3 = \frac{1}{6}\pi(9.72\,\text{mm})^3 = 480.84\,\text{mm}^3.$$

Using the calculus, we find,

$$\Delta V = \frac{1}{6}\pi(3)d^2\Delta d,$$
$$\frac{\Delta V}{V} = \frac{\frac{1}{6}\pi(3)d^2\Delta d}{\frac{1}{6}\pi d^3} = 3\frac{\Delta d}{d}.$$

The uncertainty of volume, ΔV of the ball is,

$$\Delta V = 3\frac{\Delta d}{d}V = 3 \times \frac{0.01\,\text{mm}}{9.72\,\text{mm}} \times 480.84\,\text{mm}^3 = 1.48\,\text{mm}^3.$$

The volume of the ball bearing is $V \pm \Delta V = 481 \pm 1\ \text{mm}^3$.

- wxMaxima codes:

```
(%i3)  fpprintprec:5; d:9.72; delta_d:0.01;
(fpprintprec)      5
(d)   9.72
(delta_d)  0.01
(%i4)  V: 1/6*float(%pi)*d^3;
(V)    480.84
(%i5)  delta_V: 3*delta_d/d*V;
(delta_V) 1.4841
```

Comments on the codes:

(%i3) Set floating point print precision to 5 and assign values of d and Δd.
(%i4), (%i5) Calculate volume, V and its uncertainty, ΔV.

Problem 1.5 Five physical quantities and their uncertainties are: $v \pm \Delta v$, $w \pm \Delta w$, $x \pm \Delta x$, $y \pm \Delta y$, and $z \pm \Delta z$. A quantity u in terms of these five quantities is given by,

$$u = 657\pi \frac{v^4 w^3 x}{y^2 z}.$$

What is the uncertainty of u?

Solution

Here, a quantity is related by multiplications and divisions of other quantities. The fractional uncertainty of the quantity is the sum of fractional uncertainties of the other related quantities. Thus,

$$\frac{\Delta u}{u} = 4\frac{\Delta v}{v} + 3\frac{\Delta w}{w} + \frac{\Delta x}{x} + 2\frac{\Delta y}{y} + \frac{\Delta z}{z}.$$

The uncertainty of u is,

$$\Delta u = \left(4\frac{\Delta v}{v} + 3\frac{\Delta w}{w} + \frac{\Delta x}{x} + 2\frac{\Delta y}{y} + \frac{\Delta z}{z}\right)u.$$

Problem 1.6 The rotational inertia of a solid cylinder about its symmetry axis is given by,

$$I = \frac{1}{2}mr^2, \tag{1.1}$$

where m and r are the mass and radius of the cylinder, respectively. From the measurement, the mass of the cylinder is 3.20 ± 0.01 kg and the radius is 0.350 ± 0.003 m. Calculate the rotational inertia of the cylinder and its uncertainty.

Solution

The rotational inertia, I of the cylinder is,

$$I = \frac{1}{2}mr^2 = \frac{1}{2}(3.20\,\text{kg})(0.350\,\text{m})^2 = 0.196\,\text{kg m}^2.$$

The uncertainty, ΔI is calculated from Eq. (1.1) as follows,

$$\begin{aligned}
\Delta I &= \frac{1}{2}(\Delta m\, r^2 + m\, 2r\, \Delta r),\\
\frac{\Delta I}{I} &= \frac{\frac{1}{2}(\Delta m\, r^2 + m\, 2r\, \Delta r)}{\frac{1}{2}mr^2} = \left(\frac{\Delta m}{m} + 2\frac{\Delta r}{r}\right),\\
\Delta I &= \left(\frac{\Delta m}{m} + 2\frac{\Delta r}{r}\right) I\\
&= \left(\frac{0.01\,\text{kg}}{3.20\,\text{kg}} + 2\frac{(0.003\,\text{m})}{(0.350\,\text{m})}\right)(0.196\,\text{kg m}^2)\\
&= 0.00397\,\text{kg m}^2.
\end{aligned}$$

The rotational inertia of the solid cylinder is $I \pm \Delta I = 0.196 \pm 0.004$ kg m^2.

- wxMaxima codes:

```
(%i5)  fpprintprec:5; m:3.2; dm:0.01; r:0.35; dr:0.003;
(fpprintprec)      5
(m)    3.2
(dm)   0.01
(r)    0.35
(dr)   0.003
(%i6)  I: 1/2*m*r^2;
(I)    0.196
(%i7)  dI:(dm/m + 2*dr/r)*I;
(dI)   0.0039725
```

Comments on the codes:

(%i5) Set floating point print precision to 5 and assign values of m, Δm, r, and Δr.
(%i6), (%i7) Calculate rotational inertia, I and its uncertainty, ΔI.

Problem 1.7 The displacement, s of a moving particle in terms of time, t and its acceleration, a is given as,

$$s = ka^x t^y. \tag{1.2}$$

Use dimensional analysis to find how displacement relates to time and acceleration.

Solution

Equation (1.2) writes,

$$s = ka^x t^y,$$

where s, a, t, and k are the displacement, acceleration, time, and a dimensionless constant, respectively, while x and y are constants to be determined. Dimensions of s, a, and t are $[s] = \mathrm{L}$, $[a] = \mathrm{LT}^{-2}$, and $[t] = \mathrm{T}$. In terms of dimensions, the equation is written as,

$$[s] = [a]^x [t]^y,$$
$$\mathrm{L} = (\mathrm{LT}^{-2})^x \mathrm{T}^y = \mathrm{L}^x \mathrm{T}^{-2x+y}.$$

Equating the powers (or indices) of each term on both sides of the equation gives,

$$\text{for L}: \quad 1 = x,$$
$$\text{for T}: \quad 0 = -2x + y.$$

Solving these two equations simultaneously one gets $x = 1$ and $y = 2$. Thus, the displacement relates to time and acceleration as,

$$s = kat^2.$$

- wxMaxima codes:

```
(%i1)  solve([1 = x, 0 = -2*x + y], [x,y]);
(%o1)  [[x = 1,y = 2]]
```

Comments on the codes:

(%i1) Solve $1 = x$ and $0 = -2x + y$ for x and y.
(%o1) The solutions.

Problem 1.8

(a) According to Newton's second law of motion, the acceleration of an object is directly proportional to the force applied on the object and inversely proportional to its mass. What is the dimension of the force?
(b) The SI unit of force is newton (N). Express newton in terms of fundamental SI units of m, kg, and s.

Solution

(a) From Newton's second law one writes,

$$\text{acceleration} \propto \frac{\text{force}}{\text{mass}},$$

or,

$$\text{force} \propto \text{mass} \times \text{acceleration}.$$

This means that,

$$\text{force} = k \times \text{mass} \times \text{acceleration},$$

where k is a dimensionless constant. Thus, the dimension of force is,

$$[\text{force}] = [\text{mass}][\text{acceleration}] = \text{MLT}^{-2}.$$

(b) From part (a), the dimension of force is MLT^{-2}. This translates into SI units as,

$$1\,\text{N} = 1\,\text{kg m s}^{-2}.$$

Problem 1.9 The period, τ of a satellite orbiting the earth depends on universal gravitational constant, G, radius of orbit, r, and mass of earth, m, that is, $\tau = kG^a r^b m^c$. Determine the relationship of these quantities by dimensional analysis.

Solution

The relationship is in the form,

$$\tau = kG^a r^b m^c, \tag{1.3}$$

where k is a dimensionless constant, while a, b, and c are the constants to be determined. Dimensions of τ, r, and m are $[\tau] = \text{T}$, $[r] = \text{L}$, and $[m] = \text{M}$, respectively. The SI unit for universal gravitational constant, G is $\text{N m}^2\ \text{kg}^{-2} = \text{m}^3\ \text{kg}^{-1}\ \text{s}^{-2}$, so that its dimension is $[G] = \text{L}^3\text{M}^{-1}\ \text{T}^{-2}$. In terms of dimensions, Eq. (1.3) takes the form,

$$[\tau] = [G]^a [r]^b [m]^c,$$

$$\mathrm{T} = (\mathrm{L}^3\mathrm{M}^{-1}\mathrm{T}^{-2})^a\mathrm{L}^b\mathrm{M}^c = \mathrm{L}^{3a+b}\mathrm{M}^{-a+c}\mathrm{T}^{-2a}.$$

Equating powers of each term on the left and right of the equation gives,

$$\begin{aligned} \text{for T}: &\quad 1 = -2a, \\ \text{for L}: &\quad 0 = 3a + b, \\ \text{for M}: &\quad 0 = -a + c. \end{aligned}$$

Solving the three equations, one gets,

$$a = -\frac{1}{2}, \quad b = \frac{3}{2}, \quad c = -\frac{1}{2}.$$

Substituting the values into Eq. (1.3), gives the relationship between period of the satellite with gravitational constant, radius of orbit, and mass of the earth as,

$$\tau = kG^{-1/2}r^{3/2}m^{-1/2} = k\left(\frac{r^3}{Gm}\right)^{1/2}.$$

- wxMaxima codes:

```
(%i1)  solve([1 = -2*a, 0 = 3*a + b, 0 = -a + c], [a,b,c]);
(%o1)  [[a = -1/2,b = 3/2,c = -1/2]]
```

Comments on the codes:

(%i1) Solve $1 = -2a$, $0 = 3a + b$, and $0 = -a + c$ for a, b, and c.
(%o1) The solutions.

Problem 1.10 The period of oscillation of a simple pendulum is given by the formula,

$$T = 2\pi\sqrt{\frac{l}{g}}, \tag{1.4}$$

where l is the length of the string and g is the acceleration of gravity. Show that the formula is consistent dimensionally.

Solution

To show the formula is consistent dimensionally, show that the dimension in the left hand side of the equation is the same as the one of the right. We find that in Eq. (1.4),

Table 1.2 List of (x, y) values

x	y
0	− 0.8
2	− 0.6
4	− 0.2
6	− 0.1
8	− 0.2
10	0.1
12	0.2

left: the dimension is $[T] = \mathrm{T}$,

right: the dimension is $[l]^{1/2}[g]^{-1/2} = \mathrm{L}^{1/2}\left(\mathrm{LT}^{-2}\right)^{-1/2} = \mathrm{T}$.

The formula is consistent because the dimensions of both sides are T.

Problem 1.11 Table 1.2 lists values of (x, y) pairs. Determine the best line $y = mx + c$ by the linear least square method.

Solution

First calculate Σx, Σy, Σx^2, and Σxy as shown in Table 1.3. This can be done by a spreadsheet software.

With Σx, Σy, Σx^2, and Σxy values, the slope, m and intercept, c of the best line by the linear least square method can be calculated.

The slope, m is,

$$m = \frac{n\Sigma xy - \Sigma x \Sigma y}{n\Sigma x^2 - (\Sigma x)^2} = \frac{7(-0.8) - 42(-1.6)}{7(364) - 42^2} = 0.078571,$$

where n is the number of value pairs.

The intercept, c is,

Table 1.3 Calculation table for Problem 1.11

x	y	x^2	xy	$mx + c$	$[y - (mx + c)]^2$
0	− 0.8	0	0	− 0.7	0.01
2	− 0.6	4	− 1.2	− 0.542857143	0.003265306
4	− 0.2	16	− 0.8	− 0.385714286	0.034489796
6	− 0.1	36	− 0.6	− 0.228571429	0.016530612
8	− 0.2	64	− 1.6	− 0.071428571	0.016530612
10	0.1	100	1	0.085714286	0.000204082
12	0.2	144	2.4	0.242857143	0.001836735
$\Sigma x = 42$	$\Sigma y = -1.6$	$\Sigma x^2 = 364$	$\Sigma xy = -0.8$		$\Sigma\,[y - (mx + c)]^2 = 0.082857143$

$$c = \frac{\Sigma y - m\Sigma x}{n} = \frac{-1.6 - 0.078571(42)}{7} = -0.7.$$

The best line is $y = 0.079x - 0.7$. With m and c known, column 5 "$mx + c$" and 6 "$[y - (mx + c)]^2$" of Table 1.3 are completed, and the sum $\Sigma[y - (mx + c)]^2$ is calculated. Next, calculate Δ, σ^2, and uncertainties of the slope, Δm and the intercept, Δc.

$$\Delta = n\Sigma x^2 - (\Sigma x)^2 = 7(364) - 42^2 = 784,$$
$$\sigma^2 = \frac{1}{n-2}\Sigma[y - (mx + c)]^2 = \frac{1}{7-2}(0.082857143) = 0.016571429,$$
$$\Delta m = \sqrt{n\frac{\sigma^2}{\Delta}} = \sqrt{7 \times \frac{0.016571429}{784}} = 0.012,$$
$$\Delta c = \sqrt{\frac{\sigma^2}{\Delta}\Sigma x^2} = \sqrt{\frac{0.016571429}{784} \times 364} = 0.088.$$

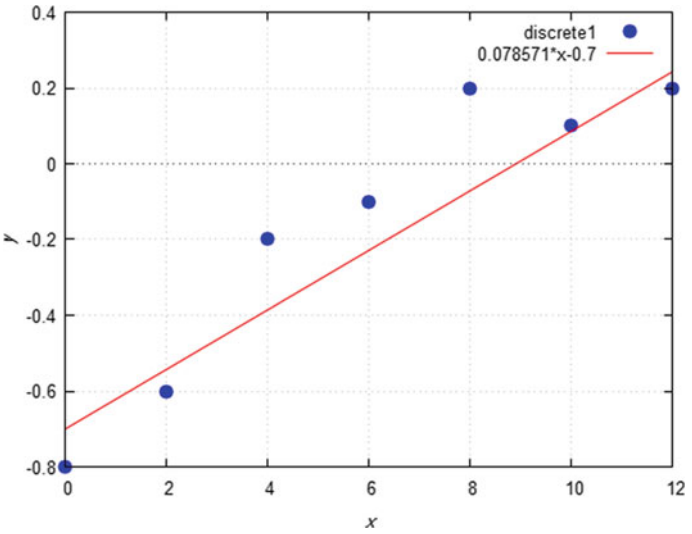
The slope is $m \pm \Delta m = 0.079 \pm 0.012$ and the intercept is $c \pm \Delta c = -0.700 \pm 0.088$.

To do a linear least square line fit by wxMaxima, the (x, y) data are entered as *matrix*, then the *lsquares* package is loaded by *load*("*lsquares*"); command, and lastly the predefined command *lsquares_estimates* is called. The wxMaxima calculation shows that $m = 0.078571$ and $c = -0.7$. The (x, y) discrete data points and the fitted line are then plotted.

- wxMaxima codes:

```
(%i1)   fpprintprec:5;
(fpprintprec)       5
(%i2)    data: matrix([0,-0.8], [2,-0.6], [4,-0.2], [6,-
0.1],
            [8,-0.2],[10,0.1],[12,0.2]);
(data)      matrix(
            [0,  -0.8],
            [2,  -0.6],
            [4,  -0.2],
            [6,  -0.1],
            [8,  -0.2],
            [10,  0.1],
            [12,  0.2])
(%i3)  load("lsquares");
(%o3)       "C:/maxima-5.43.0/share/maxima/5.43.0/share/
lsquares/lsquares.mac"
(%i5)  lsquares_estimates(data,[x,y],y = m*x + c,[m,c])$
float(%);
```

```
(%o5)  [[m = 0.078571,c = -0.7]]
(%i7)  m: rhs(%o5[1][1]); c: rhs(%o5[1][2]);
(m)    0.078571
(c)    -0.7
(%i8)  xy: [[0,-0.8], [2,-0.6], [4,-0.2], [6,-0.1],
       [8,0.2],[10,0.1],[12,0.2]];
(xy)                    [[0,-0.8],[2,-0.6],[4,-0.2],[6,-
0.1],[8,0.2],[10,0.1],[12,0.2]]
(%i9)  y: m*x + c;
(y)    0.078571*x-0.7
(%i10)    wxplot2d( [[discrete,xy], y],[x,0,12],[style,
[points],[lines]],
[xlabel,"{/Helvetica-Italic x}"],
[ylabel,"{/Helvetica-Italic y}"],grid2d);
```



Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2) Assign "data" as matrix of x, y values.
(%i3) Load "lsquares" package.
(%i5) Calculate fit values of gradient, m and y axis intercept, c.
(%o5) The results.
(%i7) Assign values of m and c.
(%i8) Assign discrete (x, y) as list xy.
(%i9) Assign fitted line y.

(%i10) Plot discrete xy and line *y*.

To calculate uncertainties of the slope and the intercept by wxMaxima is tedious and is not shown here.

Problem 1.12 How is the error propagation formula related to the error estimates calculated in Problems 1.2 to 1.6?

Solution

Error propagation formula states that if $u = u(x, y, z, \ldots)$, then,

$$\sigma_u^2 = \left(\frac{\partial u}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial u}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial u}{\partial z}\right)^2 \sigma_z^2 + \cdots$$

$$\sigma_u = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial u}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial u}{\partial z}\right)^2 \sigma_z^2 + \cdots}$$

where σ_u^2 is the variance of u, and σ_u is the standard deviation of u and this is considered as error in u. One writes,

$$(\Delta u)^2 = \left(\frac{\partial u}{\partial x}\right)^2 (\Delta x)^2 + \left(\frac{\partial u}{\partial y}\right)^2 (\Delta y)^2 + \left(\frac{\partial u}{\partial z}\right)^2 (\Delta z)^2 + \cdots$$

$$\Delta u = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 (\Delta x)^2 + \left(\frac{\partial u}{\partial y}\right)^2 (\Delta y)^2 + \left(\frac{\partial u}{\partial z}\right)^2 (\Delta z)^2 + \cdots}$$

where Δu, Δx, Δy, Δz are the uncertainties in u, x, y, and z, respectively. Using the formula, uncertainty can be derived. Table 1.1 gives formula to calculate the uncertainty. In Problems 1.2–1.6, formulae in the second column of Table 1.1 are used to estimate the errors as approximations of the formulae in the third column obtained by error propagation theory. Better error estimates can be obtained by the formulae given in the third column.

Consider $u = x - y$ as an example. Applying the formula,

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = -1,$$

$$\Delta u = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 (\Delta x)^2 + \left(\frac{\partial u}{\partial y}\right)^2 (\Delta y)^2} = \sqrt{1^2 (\Delta x)^2 + (-1)^2 (\Delta y)^2}$$

$$= \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

Using the binomial expansion (Appendix D), one gets,

$$\Delta u = \left((\Delta x)^2 + (\Delta y)^2\right)^{1/2} = \Delta x \left(1 + \frac{(\Delta y)^2}{(\Delta x)^2}\right)^{1/2} = \Delta x \left(1 + \frac{1}{2}\frac{(\Delta y)^2}{(\Delta x)^2} + \cdots\right)$$

$$\approx \Delta x + \Delta y.$$

These are the entries of row 2 in Table 1.1.
Consider $u = xy$ as another example. Applying the formula,

$$\frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x,$$

$$\Delta u = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 (\Delta x)^2 + \left(\frac{\partial u}{\partial y}\right)^2 (\Delta y)^2} = \sqrt{y^2(\Delta x)^2 + x^2(\Delta y)^2},$$

$$\frac{\Delta u}{u} = \sqrt{\frac{y^2(\Delta x)^2}{x^2 y^2} + \frac{x^2(\Delta y)^2}{x^2 y^2}} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}.$$

Using the binomial expansion (Appendix D), one gets,

$$\begin{aligned}\frac{\Delta u}{u} &= \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2} = \frac{\Delta x}{x}\left(1 + \left(\frac{\Delta y}{y}\right)^2 \left(\frac{\Delta x}{x}\right)^{-2}\right)^{1/2} \\ &= \frac{\Delta x}{x}\left(1 + \frac{1}{2}\left(\frac{\Delta y}{y}\right)^2 \left(\frac{\Delta x}{x}\right)^{-2} + \cdots\right) \\ &\approx \frac{\Delta x}{x} + \frac{\Delta y}{y}.\end{aligned}$$

These are entries of row 3 in Table 1.1.

Problem 1.13 A tennis ball is released from rest from the top of a building of height, H and the time, t it reaches height, h is measured (Fig. 1.1). The experiment is repeated at different height, h and time, t is measured. Table 1.4 shows the data collected.

Plot a curve of h against t^2 and determine the acceleration due to gravity, g and the height of the building, H. Also, estimate the uncertainties of both quantities, Δg and ΔH.

Solution

The tennis ball is in a free fall from rest. One has,

$$s = \frac{1}{2} g t^2. \tag{1.5}$$

From Fig. 1.1, Eq. (1.5) gives,

$$H - h = \frac{1}{2} g t^2,$$

$$h = -\frac{1}{2} g t^2 + H,$$

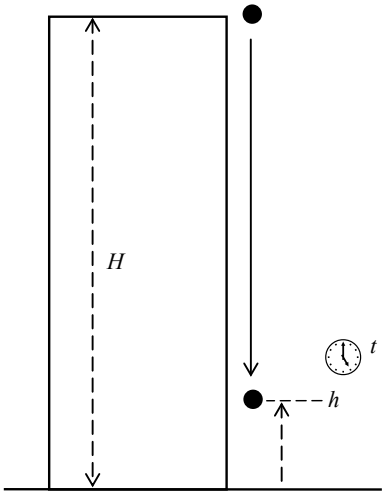


Fig. 1.1 A tennis ball released from a tall building of Problem 1.13

Table 1.4 Values of h and t

h (m)	t (s)
0.50	3.49
10.00	3.35
15.00	3.19
20.00	3.02
25.00	2.86
30.00	2.67
35.00	2.47

$$y = mx + c.$$

If we plot $y \equiv h$ against $x \equiv t^2$, the slope, m is $-\frac{1}{2}g$, and the y intercept, c is H. Following Problem 1.11, we calculate Σx, Σy, Σx^2, Σxy, and $\Sigma[y - (mx + c)]^2$ in Table 1.5.

With Σx, Σy, Σx^2, and Σxy values, the slope, m and intercept, c of the best line by the linear least square method are calculated.

The slope, m is,

$$m = \frac{n\Sigma xy - \Sigma x \Sigma y}{n\Sigma x^2 - (\Sigma x)^2} = \frac{7(1085.25305 - 64.1085(135.5)}{7(615.9821) - 641085^2} = -5.3964\,\text{m s}^{-2},$$

where $n = 7$ is the number of value pairs.

The intercept, c is,

$$c = \frac{\Sigma y - m\Sigma x}{n} = \frac{135.5 + 5.3964(64.1085)}{7} = 68.779\,\text{m}.$$

Table 1.5 Calculation table for Problem 1.13

$y \equiv h$ (m)	t (s)	$x \equiv t^2$ (s^2)	x^2	xy	$mx + c$	$[y - (mx + c)]^2$
0.50	3.49	12.1801	148.35484	6.09005	3.05072	6.506157
10.00	3.35	11.2225	125.94451	112.225	8.21828	3.1745418
15.00	3.19	10.1761	103.55301	152.6415	13.86503	1.2881537
20.00	3.02	9.1204	83.1817	182.408	19.56197	0.1918673
25.00	2.86	8.1796	66.90586	204.49	24.63887	0.1304126
30.00	2.67	7.1289	50.82122	213.867	30.30883	0.095378
35.00	2.47	6.1009	37.22098	213.5315	35.8563	0.733243
$\Sigma y = 135.5$		$\Sigma x = 64.1085$	$\Sigma x^2 = 615.9821$	$\Sigma xy = 1085.25305$		$\Sigma[y - (mx + c)]^2 = 12.11975$

The best line is $y = -5.4x + 68.78$. With m and c known, column 6 "$mx + c$" and 7 "$[y - (mx + c)]^2$" of Table 1.5 are completed, and the sum $\Sigma[y - (mx + c)]^2$ is calculated. Values of Δ, σ^2, and uncertainties of the slope, Δm and the intercept, Δc are,

$$\Delta = n\Sigma x^2 - (\Sigma x)^2 = 7(615.9821) - 64.1085^2 = 201.9749,$$
$$\sigma^2 = \frac{1}{n-2}\Sigma[y - (mx + c)]^2 = \frac{1}{7-2}(12.11975) = 2.42395,$$
$$\Delta m = \sqrt{n\frac{\sigma^2}{\Delta}} = \sqrt{7 \times \frac{2.42395}{201.9749}} = 0.28984\,\text{m s}^{-2},$$
$$\Delta c = \sqrt{\frac{\sigma^2}{\Delta}\Sigma x^2} = \sqrt{\frac{2.42395}{201.9749} \times 615.9821} = 2.719\,\text{m}.$$

The slope is $m \pm \Delta m = -5.4 \pm 0.3$ m s^{-2} and the intercept is $c \pm \Delta c = 68.8 \pm 2.8$ m.

The acceleration due to gravity, g is calculated as follows,

$$m = -\frac{1}{2}g,$$
$$g = -2m = -2(-5.4) = 10.8\,\text{m s}^{-2}.$$

The uncertainty of acceleration due to gravity, Δg is estimated as follows,

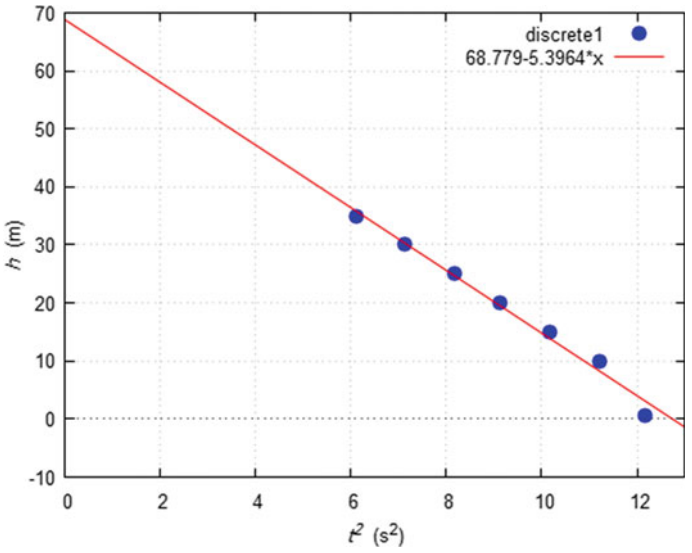
$$\frac{\Delta m}{m} = \frac{\Delta g}{g},$$
$$\Delta g = \frac{\Delta m}{m}g = \frac{0.3}{5.4} \times 10.8 = 0.6\,\text{m s}^{-2}.$$

Thus, the acceleration due to gravity is $g \pm \Delta g = 10.8 \pm 0.6$ m s^{-2}. The height of the building is $H \pm \Delta H = c \pm \Delta c = 68.8 \pm 2.8$ m.

To do a linear least square line fit by wxMaxima, the (x, y) data are entered as *matrix*, the *lsquares* package is loaded, and the predefined command *lsquares_estimates* is called. The wxMaxima calculation gives $m = -5.3964$ m s^{-2} and $c = 68.779$ m. The (x, y) discrete data points and the fitted line are the plotted.

- wxMaxima codes:

```
(%i1)   fpprintprec:5;
(fpprintprec)       5
(%i2)       data:  matrix([12.1801,0.5],  [11.2225,10],
[10.1761,15],  [9.1204,20],  [8.1796,25],  [7.1289,30],
[6.1009, 35]);
(data) matrix(
           [12.18,  0.5],
           [11.223,  10],
           [10.176,  15],
           [9.1204,  20],
           [8.1796,  25],
           [7.1289,  30],
           [6.1009,  35])
(%i3)  load("lsquares");
(%o3)       "C:/maxima-5.43.0/share/maxima/5.43.0/share/
lsquares/lsquares.mac".
(%i5)  lsquares_estimates(data,[x,y],y = m*x + c,[m,c])$
float(%);
(%o5)   [[m = -5.3964,c = 68.779]]
(%i7)   m: rhs(%o5[1][1]); c: rhs(%o5[1][2]);
(m)     -5.3964
(c)     68.779
(%i8)    xy: [[12.1801,0.5], [11.2225,10], [10.1761,15],
[9.1204,20], [8.1796,25], [7.1289,30],[6.1009,35]];
(xy)   [[12.18,0.5],[11.223,10],[10.176,15],[9.1204,20],
[8.1796,25],[7.1289,30],[6.1009,35]]
(%i9)  y: m*x + c;
(y)   68.779-5.3964*x
(%i10)    wxplot2d( [[discrete,xy], y],[x,0,13],[style,
[points],[lines]],
[xlabel,"{/Helvetica-Italic t^2} (s^2)"],
[ylabel,"{/Helvetica-Italic h} (m)"], grid2d);
```



Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2) Assign "data" as matrix of x, y values, where x is t^2 and y is h.
(%i3) Load "lsquares" package.
(%i5) Calculate fit values of gradient, m and y axis intercept, c.
(%o5) The results.
(%i7) Assign values of m and c.
(%i8) Assign discrete (x, y) as list xy.
(%i9) Assign fitted line y.
(%i10) Plot discrete xy and line y.

1.3 Summary

- Measurement of physical quantities (such as length, mass, and time) are defined in terms of the standards and unit of various measures (such as meter, kilogram, and second).
- The International System of Units (SI) is adopted for the physical quantities.
- Dimensional analysis is used to deduce relation among physical quantities.
- Uncertainties in the measurement are considered in physics.

1.4 Exercises

Exercise 1.1 Determine the dimensions of the kinetic energy, K, pressure, p, and the universal gravitational constant, G.

(Answer: $[K] = \mathrm{ML^2T^{-2}}$, $[p] = \mathrm{ML^{-1}\ T^{-2}}$, $[G] = \mathrm{M^{-1}L^3T^{-2}}$)

Exercise 1.2 The SI and CGS units of energy are joule and erg. Determine their relationship.

(Answer: $1\ \mathrm{J} = 10^7$ erg)

Exercise 1.3 The centripetal force, F, on an object moving uniformly in a circular path is related to the object's mass, m, its speed, v, and the radius, r, of the circular path as,

$$F = km^a v^b r^c,$$

where k, a, b, and c are the dimensionless constants. By dimensional analysis determine the values of a, b, and c.

(Answer: $a = 1, b = 2, c = -1$)

Exercise 1.4 The speeds of a hi-speed train and a cheetah are 574 km $\mathrm{h^{-1}}$ and 93.0 km $\mathrm{h^{-1}}$, respectively, while a fastest human is capable to run a 100 m race in 9.58 s. Express all the speeds in m $\mathrm{s^{-1}}$.

(Answer: 159, 25.8, 10.4 m $\mathrm{s^{-1}}$)

Exercise 1.5 Fit the (x, y) data in Table 1.6 to the line $y = mx + c$ by the linear least square fit. Get the slope, m, the y axis intercept, c, and their uncertainties, Δm and Δc.

Table 1.6 Values of (x, y) pairs

x	y
1.0	0.5
2.0	− 0.2
3.0	0.1
4.0	0.2
5.0	0.3
6.0	0.5
7.0	0.6

(Answer: $m \pm \Delta m = 0.17 \pm 0.02$, $c \pm \Delta c = -0.6 \pm 0.1$)

Chapter 2
Vector

2.1 Basic Concepts and Formulae

(1) A vector is a quantity with both magnitude and direction. A scalar is a quantity with magnitude only.
(2) Vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ can be added by two equivalent geometry methods like triangle method and parallelogram method.
(3) In a two dimensional (2D) space, the x component of vector $\boldsymbol{A}$, written as A_x is the projection of the vector on x axis; while the y component of the vector, written as A_y is the projection of the vector on y axis,

$$A_x = A\cos\theta, \tag{2.1}$$

$$A_y = A\sin\theta, \tag{2.2}$$

where A is the magnitude of the vector and θ is the angle between the vector and x axis.
(4) Addition of vectors in two dimensions is performed analytically by resolving each vector into x and y components. Then, adding each component separately and using Pythagoras theorem to get the magnitude of the resultant vector. The angle between the resultant vector and x axis can also be calculated.
(5) Unit vector has magnitude of 1. If $\boldsymbol{A}$ is any vector, a unit vector parallel to $\boldsymbol{A}$ is,

$$\hat{\mathbf{u}} = \frac{\boldsymbol{A}}{A}. \tag{2.3}$$

(6) Unit vectors for Cartesian coordinate system in three dimensions (3D) are $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.
(7) Dot product or scalar product of the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as,

$$\boldsymbol{A}\cdot\boldsymbol{B} = AB\ \cos\ \theta, \tag{2.4}$$

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where A and B are magnitudes of $\boldsymbol{A}$ and $\boldsymbol{B}$, and θ is the angle between the two vectors.

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are given by,

$$\boldsymbol{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}, \tag{2.5}$$

$$\boldsymbol{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}, \tag{2.6}$$

then,

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z. \tag{2.7}$$

The magnitudes A and B are,

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}, \tag{2.8}$$

$$B = \sqrt{B_x^2 + B_y^2 + B_z^2}. \tag{2.9}$$

Dot products of unit vectors are as follows,

$$\boxed{\begin{aligned}\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1,\\ \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0.\end{aligned}}$$

(8) Cross product or vector product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as,

$$\boldsymbol{A} \times \boldsymbol{B} = AB\sin\theta\ \hat{\mathbf{n}}, \tag{2.10}$$

where θ is the angle between vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, and $\hat{\mathbf{n}}$ is the unit vector perpendicular to vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. This means that vector $\boldsymbol{A} \times \boldsymbol{B}$ is perpendicular to the plane containing vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.

The right hand rule can be used to determine the direction of vector $\boldsymbol{A} \times \boldsymbol{B}$ or the direction of unit vector $\hat{\mathbf{n}}$. Figure 2.1 shows the right hand rule for the cross product of two vectors. The thumb is vector $\boldsymbol{A}$, the index finger is vector $\boldsymbol{B}$, and the middle finger is the direction of vector $\boldsymbol{A} \times \boldsymbol{B}$.

If vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are,

$$\boldsymbol{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k},$$

$$\boldsymbol{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k},$$

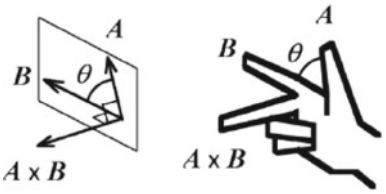

Fig. 2.1 The right hand rule for vector cross product. The thumb is vector $\boldsymbol{A}$, the index finger is vector $\boldsymbol{B}$, and the middle finger is the direction of vector cross product $\boldsymbol{A} \times \boldsymbol{B}$. The angle between $\boldsymbol{A}$ and $\boldsymbol{B}$ is θ

then,

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$
$$= (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}. \quad (2.11)$$

Cross products of unit vectors are as follows,

$$\boxed{\begin{aligned} &\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0, \\ &\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}, \\ &\mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i}, \\ &\mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j}. \end{aligned}}$$

(9) Force is a vector quantity. If a particle is in mechanical equilibrium, the net force on the particle is zero. That is,

$$\Sigma \boldsymbol{F} = 0.$$

In two dimensions, this means x and y components of the net force are zero,

$$\Sigma F_x = 0, \quad \Sigma F_y = 0.$$

2.2 Problems and Solutions

Problem 2.1 The magnitude of vector $\boldsymbol{A}$ is 5.0 units and the angle between $\boldsymbol{A}$ and x axis is 40°.

(a) Calculate the x and y components of $\boldsymbol{A}$.
(b) Express $\boldsymbol{A}$ in terms of unit vectors.

Solution

(a) Using Eqs. (2.1) and (2.2), the x and y components are given by,

$$A_x = A\ \cos\ \theta = 5\ \cos\ 40^\circ = 3.8,$$

$$A_y = A\ \sin\theta = 5 \sin 40^\circ = 3.2.$$

- wxMaxima codes:

```
(%i3)    fpprintprec:5; A:5; theta:float(40/180*%pi);
(fpprintprec)        5
(A)      5
(theta)     0.69813
(%i4)    Ax: A*cos(theta);
(Ax)     3.8302
(%i5)     Ay: A*sin(theta);
(Ay)     3.2139
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign $A = 5$, and $\theta = 40/180 \times \pi$ rad.
(%i4) Calculate $A_x = A\ \cos\ \theta$.
(%i5) Calculate $A_y = A\ \sin\ \theta$.

(b) From part (a), vector $\boldsymbol{A}$ is expressed as,

$$\boldsymbol{A} = A_x\mathbf{i} + A_y\mathbf{j} = 3.8\mathbf{i} + 3.2\mathbf{j}.$$

Problem 2.2 The polar coordinates of a point in the xy plane are $r = 20$ m and $\theta = 120^\circ$. Determine,

(a) the Cartesian coordinate of the point,
(b) the position vector of the point.

Solution

(a) Figure 2.2 shows the point P at $r = 20$ m and $\theta = 120^\circ$. The x and y coordinates of P are,

$$x = r\cos\theta = 20\cos 120^\circ = -10\,\text{m},$$

$$y = r\sin\theta = 20\sin 120^\circ = 17.3\,\text{m}.$$

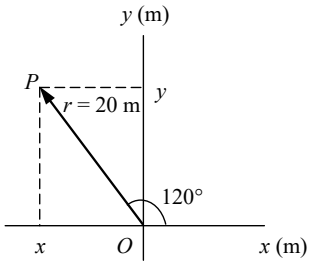

Fig. 2.2 Polar coordinates of a point of Problem 2.2

The Cartesian coordinate of P is (–10 m, 17.3 m).

- wxMaxima codes:

```
(%i3)    fpprintprec:5; r:20; theta:float(120/180*%pi);
(fpprintprec)         5
(r)         20
(theta)      2.0944
(%i4)    x: r*cos(theta);
(x)        -10.0
(%i5)    y: r*sin(theta);
(y)        17.321
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of r andθ.
(%i4), (%i5) Calculate x and y.

(b) Position vector $\overrightarrow{OP}$ is $-10\mathbf{i} + 17.3\mathbf{j}$.

Problem 2.3 The dimensions of a room are length, $l = 3.0$ m, width, $w = 2.0$ m, and height, $h = 4.0$ m. A bee flies from a corner to opposite corner of the room. What is the magnitude of displacement vector of the bee?

Solution

The room, its length, l, width, w, height, h, and the bee's displacement, $\overrightarrow{OC}$ are shown in Fig. 2.3.

The magnitude of displacement vector is the length of OC. By Pythagoras theorem,

$$OB^2 = OA^2 + AB^2,$$

and

$$OC^2 = OB^2 + BC^2 = OA^2 + AB^2 + BC^2.$$

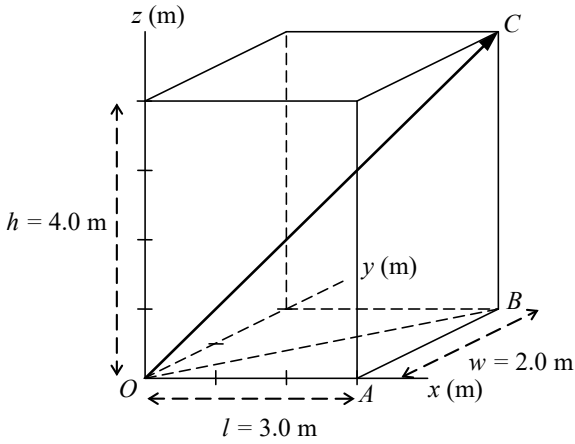

Fig. 2.3 Displacement vector $\overrightarrow{OC}$ of Problem 2.3

So,

$$OC = \sqrt{OA^2 + AB^2 + BC^2} = \sqrt{(3\,\mathrm{m})^2 + (2\,\mathrm{m})^2 + (4\,\mathrm{m})^2} = 5.4\,\mathrm{m}.$$

Alternatively,

$$\overrightarrow{OC} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}\,\mathrm{m},$$
$$\overrightarrow{OC} \cdot \overrightarrow{OC} = 3^2 + 2^2 + 4^2\,\mathrm{m}^2 = 29\,\mathrm{m}^2,$$
$$|\overrightarrow{OC}| = \sqrt{3^2 + 2^2 + 4^2}\,\mathrm{m} = 5.4\,\mathrm{m}.$$

- wxMaxima codes:

```
(%i1)    fpprintprec:5;
(fpprintprec)         5
(%i3)    sqrt(3^2 + 2^2 + 4^2); float(%);
(%o2)    sqrt(29)
(%o3)    5.3852
```

We can as well define the vector $\overrightarrow{OC}$ and then calculate the length of the vector as follows.

- wxMaxima codes:

```
(%i1)       fpprintprec:5;
(fpprintprec)            5
(%i2)       OC: [3, 2, 4];
(OC)        [3,2,4]
(%i3)       OC.OC;
(%o3)       29
(%i4)       float(sqrt(%));
(%o4)       5.3852
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2) Assign vector $\overrightarrow{OC}$.
(%i3) Calculate vector dot product.
(%i4) Calculate square root of the dot product.

Problem 2.4 Determine the angles between the vector, $\boldsymbol{A} = 3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ and x, y, and z axes.

Solution

Referring to Eq. (2.4), the angles between any vector $\boldsymbol{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ and the x, y, and z axes are,

$$\theta_x = \cos^{-1}\frac{A_x}{A}, \quad \theta_y = \cos^{-1}\frac{A_y}{A}, \quad \text{and} \quad \theta_z = \cos^{-1}\frac{A_z}{A},$$

respectively, where A is the magnitude of the vector. This is because,

$$\begin{aligned}
\boldsymbol{A}\cdot\mathbf{i} = A\cos\theta_x = A_x \Rightarrow \cos\theta_x = \frac{A_x}{A} \Rightarrow \theta_x = \cos^{-1}\frac{A_x}{A},\\
\boldsymbol{A}\cdot\mathbf{j} = A\cos\theta_y = A_y \Rightarrow \cos\theta_y = \frac{A_y}{A} \Rightarrow \theta_y = \cos^{-1}\frac{A_y}{A},\\
\boldsymbol{A}\cdot\mathbf{k} = A\cos\theta_z = A_z \Rightarrow \cos\theta_z = \frac{A_z}{A} \Rightarrow \theta_z = \cos^{-1}\frac{A_z}{A}.
\end{aligned}$$

The magnitude is,

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{3^2 + 5^2 + 2^2} = \sqrt{38}.$$

The angles are,

$$\theta_x = \cos^{-1}\frac{3}{\sqrt{38}} = 60.9^\circ,$$

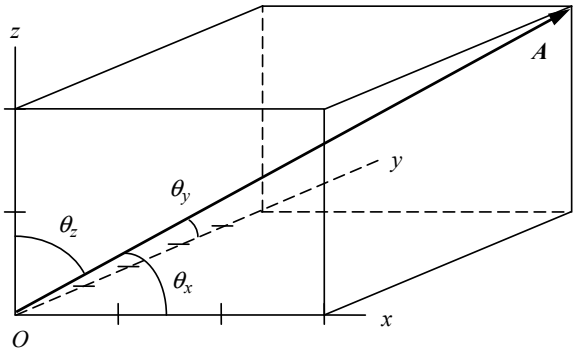

Fig. 2.4 Angles between a vector and the x, y, and z axes, Problem 2.4

$$\theta_y = \cos^{-1}\frac{5}{\sqrt{38}} = 35.8°,$$
$$\theta_z = \cos^{-1}\frac{2}{\sqrt{38}} = 71.1°.$$

Figure 2.4 shows the vector and the angles.

- wxMaxima codes:

```
(%i4)     fpprintprec:5; Ax:3; Ay:5; Az:2;
(fpprintprec)         5
(Ax)      3
(Ay)      5
(Az)      2
(%i5)     A: sqrt(Ax^2 + Ay^2 + Az^2);
(A)       sqrt(38)
(%i6)     theta_x: acos(Ax/A);
(theta_x)   acos(3/sqrt(38))
(%i7)      float(%*180/%pi);
(%o7)     60.878
(%i8)      theta_y:acos(Ay/A);
(theta_y)   acos(5/sqrt(38))
(%i9)     float(%*180/%pi);
(%o9)     35.796
(%i10)     theta_z:acos(Az/A);
(theta_z)    acos(2/sqrt(38))
(%i11)    float(%*180/%pi);
(%o11)    71.068
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of $A_x = 3$, $A_y = 5$, and $A_z = 2$.
(%i5) Calculate A.
(%i6), (%i7) Calculate $\theta_x = \cos^{-1}(A_x/A)$ and convert θ_x from radian to degree.
(%i8), (%i9) Calculate $\theta_y = \cos^{-1}(A_y/A)$ and convert θ_y from radian to degree.
(%i10), (%i11) Calculate $\theta_z = \cos^{-1}(A_z/A)$ and convert θ_z from radian to degree.

Problem 2.5 Show that the vector $\boldsymbol{A} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is perpendicular to the vector $\boldsymbol{B} = -6\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$.

Solution

If vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are perpendicular to each other, their dot product is zero, that is $\boldsymbol{A} \cdot \boldsymbol{B} = AB\cos 90^\circ = 0$. This can be used to prove that the two vectors are perpendicular to each other. For this problem,

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z = (2)(-6) + (1)(4) + (2)(4) = 0.$$

Because the dot product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is zero, then $\boldsymbol{A}$ is perpendicular to $\boldsymbol{B}$.

- wxMaxima codes:

```
(%i1)   2*(-6) + 1*4 + 2*4;
(%o1)   0
```

Another check: Define vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, compute the dot product $\boldsymbol{A} \cdot \boldsymbol{B}$.

- wxMaxima codes:

```
(%i1)    A: [2, 1, 2];
(A)      [2,1,2]
(%i2)    B: [-6, 4, 4];
(B)       [-6,4,4]
(%i3)    A.B;
(%o3)    0
```

Comments on the codes:

(%i1), (%i2) Assign vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.
(%i3) Calculate vector dot product $\boldsymbol{A} \cdot \boldsymbol{B}$.

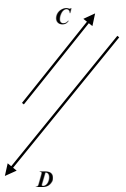

Fig. 2.5 Two anti-parallel vectors, Problem 2.6

Problem 2.6 Show that the vector $\boldsymbol{A} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is anti-parallel to the vector $\boldsymbol{B} = -6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}$.

Solution

Two anti-parallel vectors $\boldsymbol{C}$ and $\boldsymbol{D}$ are shown in Fig. 2.5.

The vectors are anti-parallel if $\boldsymbol{C} = -k\boldsymbol{D}$, where k is a positive number. In this problem,

$$\boldsymbol{A} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k},$$

and

$$\begin{aligned}\boldsymbol{B} &= -6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k},\\ &= -3(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}),\\ &= -3\boldsymbol{A}.\end{aligned}$$

Clearly, $\boldsymbol{A}$ is anti-parallel to $\boldsymbol{B}$.

Another way to show that $\boldsymbol{A}$ is anti-parallel to $\boldsymbol{B}$ is to calculate unit vector in the direction of $\boldsymbol{A}$, $\hat{\mathbf{u}}_A$ and unit vector in the direction of $\boldsymbol{B}$, $\hat{\mathbf{u}}_B$ and compare the two. We find,

$$\hat{\mathbf{u}}_A = \frac{\boldsymbol{A}}{A} = \frac{2\mathbf{i} - 3\mathbf{j} + \mathbf{k}}{\sqrt{2^2 + (-3)^2 + 1^2}} = \frac{1}{\sqrt{14}}(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}),$$

and

$$\begin{aligned}\hat{\mathbf{u}}_B &= \frac{\boldsymbol{B}}{B} = \frac{-6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}}{\sqrt{(-6)^2 + 9^2 + (-3)^2}} = -\frac{1}{\sqrt{14}}(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})\\ &= -\hat{\mathbf{u}}_A.\end{aligned}$$

It is clear that the vector $\boldsymbol{A}$ is anti-parallel to the vector $\boldsymbol{B}$.

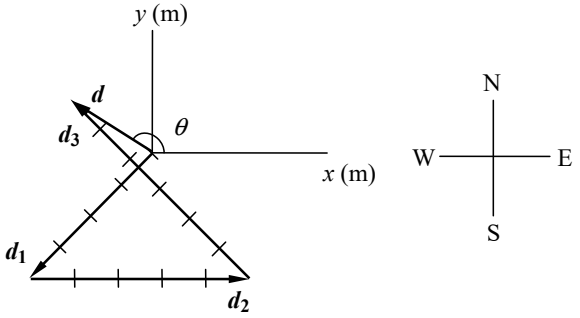

Fig. 2.6 Displacement vectors of Problem 2.7

Problem 2.7 A boy walks 4.0 m south-west, 5.0 m east, and then 6.0 m north-west. Use a two-dimensional coordinate system and the origin as the starting point. Determine,

(a) the three displacement vectors,
(b) the total displacement vector.

Solution

(a) Figure 2.6 shows the path of the boy.

The three displacement vectors are,

$$\boldsymbol{d}_1 = 4\cos 225^\circ\,\mathbf{i} + 4\sin 225^\circ\,\mathbf{j} = -2\sqrt{2}\mathbf{i} - 2\sqrt{2}\mathbf{j} = -2.83\mathbf{i} - 2.83\mathbf{j},$$
$$\boldsymbol{d}_2 = 5\mathbf{i},$$
$$\boldsymbol{d}_3 = 6\cos 135^\circ\,\mathbf{i} + 6\sin 135^\circ\,\mathbf{j} = -3\sqrt{2}\mathbf{i} + 3\sqrt{2}\mathbf{j} = -4.24\mathbf{i} + 4.24\mathbf{j}.$$

(b) The total displacement is the vector addition of the three displacements,

$$\boldsymbol{d} = \boldsymbol{d}_1 + \boldsymbol{d}_2 + \boldsymbol{d}_3 = (-5\sqrt{2} + 5)\mathbf{i} + \sqrt{2}\mathbf{j}\,\text{m} = -2.07\mathbf{i} + 1.42\mathbf{j}\,\text{m}.$$

The magnitude of the total displacement is,

$$d = \sqrt{(5\sqrt{2} + 5)^2 + (\sqrt{2})^2}\,\text{m} = 2.5\,\text{m}.$$

The angle between total displacement and the x axis is,

$$\theta = \tan^{-1}\left(\frac{\sqrt{2}}{-5\sqrt{2} + 5}\right) = 146^\circ.$$

- wxMaxima codes:

```
(%i1)    fpprintprec:5;
(fpprintprec)        5
(%i3)    d1x:float(4*cos(225/180*%pi));
d1y:float(4*sin(225/180*%pi));
(d1x)    -2.8284
(d1y)    -2.8284
(%i5)    d2x:5; d2y:0;
(d2x)    5
(d2y)    0
(%i7)    d3x:float(6*cos(135/180*%pi));
d3y:float(6*sin(135/180*%pi));
(d3x)    -4.2426
(d3y)    4.2426
(%i9)    dx:d1x+d2x+d3x; dy:d1y+d2y+d3y;
(dx)     -2.0711
(dy)     1.4142
(%i10)    d: sqrt(dx^2 + dy^2);
(d)      2.5079
(%i11)    theta: atan(dy/dx);
(theta)      -0.59912
(%i12)    theta_deg: float(theta*180/%pi);
(theta_deg)     -34.327
(%i13)    180+theta_deg;
(%o13)    145.67
(%i14)    float(atan(sqrt(2)/(-5*sqrt(2)+5)));
(%o14)    -0.59912
(%i15)    float(%*180/%pi);
(%o15)    -34.327
(%i16)    180+%;
(%o16)   145.67
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i3) Assign d_{1x} and d_{1y}. Part (a).
(%i5) Assign d_{2x} and d_{2y}. Part (a).
(%i7) Assign d_{3x} and d_{3y}. Part (a).
(%i9), (%i10) Calculate d_x, d_y, and d. Part (b).
(%i11), (%i12), (%i13) Calculate θ. Part (b).
(%i14), (%i15), (%i16) Another calculation of θ. Part (b).

Alternatively, one can first define the displacement vectors and add them. Then, calculate the magnitude and angle.

- wxMaxima codes:

```
(%i1)  fpprintprec:5;
(fpprintprec)      5
(%i2)     d1vec:  [4*cos(225/180*float(%pi)),  4*sin(225/
180*float(%pi))];
(d1vec)    [-2.8284,-2.8284]
(%i3)  d2vec: [5, 0];
(d2vec)    [5,0]
(%i4)     d3vec:  [6*cos(135/180*float(%pi)),  6*sin(135/
180*float(%pi))];
(d3vec)    [-4.2426,4.2426]
(%i5)  dvec: d1vec + d2vec + d3vec;
(dvec)     [-2.0711,1.4142]
(%i6)  d: float(sqrt(dvec.dvec));
(d)     2.5079
(%i7)  theta: atan(dvec[2]/dvec[1]);
(theta)    -0.59912
(%i8)  theta_deg: float( (theta*180/%pi));
(theta_deg)   -34.327
(%i9)  180+theta_deg;
(%o9)   145.67
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2), (%i3), (%i4) Assign vectors $\boldsymbol{d}_1$, $\boldsymbol{d}_2$, and $\boldsymbol{d}_3$. Part (a).
(%i5) Calculate vector addition $\boldsymbol{d}$. Part (b).
(%i6) Calculate magnitude d. Part (b).
(%i7), (%i8), (%i9) Calculate angle θ. Part (b).

Problem 2.8 A cube of sides a is placed on a Cartesian coordinate system as shown in Fig. 2.7. Determine,

(a) the vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$, and their magnitudes
(b) the angles θ_1, θ_2, and θ_3.

Solution

(a) The vectors and their magnitudes are according to Fig. 2.7,

$$\boldsymbol{r}_1 = a\mathbf{i}, \quad r_1 = a,$$
$$\boldsymbol{r}_2 = a\mathbf{i} + a\mathbf{j}, \quad r_2 = \sqrt{a^2 + a^2} = \sqrt{2}a = 1.4a,$$

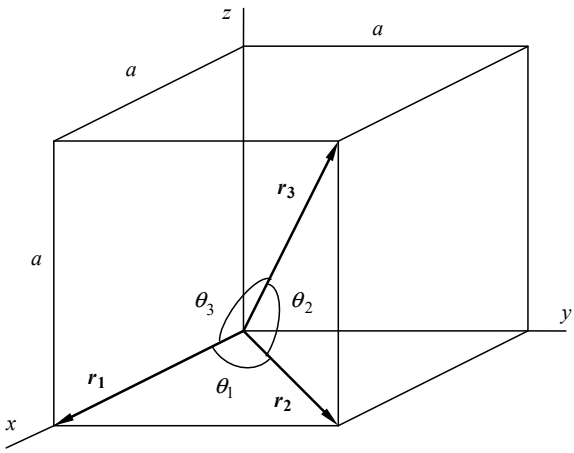

Fig. 2.7 Vectors and angles of Problem 2.8

$$\boldsymbol{r}_3 = a\mathbf{i} + a\mathbf{j} + a\mathbf{k}, \quad r_3 = \sqrt{a^2 + a^2 + a^2} = \sqrt{3}a = 1.7a.$$

(b) The dot product $\boldsymbol{r}_1 \cdot \boldsymbol{r}_2 = r_1 r_2 \cos\theta_1$. This gives,

$$\cos\theta_1 = \frac{\boldsymbol{r}_1 \cdot \boldsymbol{r}_2}{r_1 r_2} = \frac{a^2}{a\sqrt{2}a} = \frac{1}{\sqrt{2}}, \quad \theta_1 = 45^\circ.$$

Similarly, the dot product $\boldsymbol{r}_2 \cdot \boldsymbol{r}_3 = r_2 r_3 \cos\theta_2$. This gives,

$$\cos\theta_2 = \frac{\boldsymbol{r}_2 \cdot \boldsymbol{r}_3}{r_2 r_3} = \frac{a^2 + a^2}{\sqrt{2}a\sqrt{3}a} = \frac{2}{\sqrt{6}}, \quad \theta_2 = 35.3^\circ.$$

The dot product $\boldsymbol{r}_1 \cdot \boldsymbol{r}_3 = r_1 r_3 \cos\theta_3$. This gives,

$$\cos\theta_3 = \frac{\boldsymbol{r}_1 \cdot \boldsymbol{r}_3}{r_1 r_3} = \frac{a^2}{a\sqrt{3}a} = \frac{1}{\sqrt{3}}, \quad \theta_3 = 54.7^\circ.$$

- wxMaxima codes:

```
(%i1)    fpprintprec: 5;
(fpprintprec)         5
(%i2)    theta1: float(acos(1/sqrt(2))*180/%pi);
(theta1)     45.0
(%i3)    theta2: float(acos(2/sqrt(6))*180/%pi);
(theta2)     35.264
(%i4)    theta3: float(acos(1/sqrt(3))*180/%pi);
```

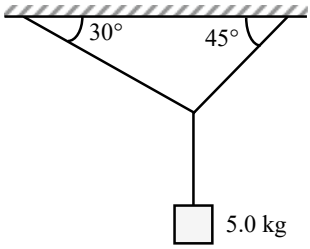

Fig. 2.8 An object suspended by three strings of Problem 2.9

```
(theta3)     54.736
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2), (%i3), (%i4) Calculate θ_1, θ_2, and θ_3.

Problem 2.9 A 5.0 kg object is suspended from a ceiling by three strings as in Fig. 2.8. Determine the tension in each string.

Solution

The forces of the problem are shown in Fig. 2.9. T_1, T_2, and T_3 are the tensions of the strings, while mg is the weight of the object.

The system is in equilibrium, so that,

$$\Sigma \boldsymbol{F} = 0.$$

In two dimensions, this means,

$$\Sigma F_x = 0,$$

$$\Sigma F_y = 0.$$

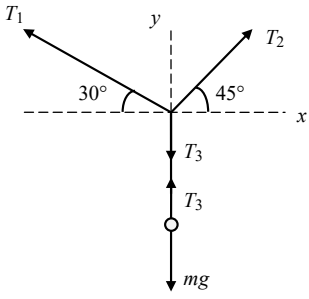

Fig. 2.9 Forces of Problem 2.9

We write,

$$\Sigma F_x = -T_1 \cos 30^\circ + T_2 \cos 45^\circ = 0,$$

$$\Sigma F_y = T_1 \sin 30^\circ + T_2 \sin 45^\circ - T_3 = 0.$$

But $T_3 = mg = (5.0\ \text{kg})(9.8\ \text{N/kg}) = 49\ \text{N}$. The two equations can be written as,

$$-T_1 + T_2 \frac{\cos 45^\circ}{\cos 30^\circ} = 0,$$

$$T_1 + T_2 \frac{\sin 45^\circ}{\sin 30^\circ} = \frac{49\,\text{N}}{\sin 30^\circ}.$$

The two equations are solved to give,

$$T_2 = \frac{49\,\text{N}}{\sin 30^\circ}\left(\frac{\cos 45^\circ}{\cos 30^\circ} + \frac{\sin 45^\circ}{\sin 30^\circ}\right)^{-1} = 44\,\text{N},$$

and

$$T_1 = T_2 \frac{\cos 45^\circ}{\cos 30^\circ} = 36\,\text{N}.$$

- wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)      5
(ratprint)     false
(%i4) solve([-T1 + T2*cos(45/180*%pi)/cos(30/180*%pi)=0,
T1    +    T2*sin(45/180*%pi)/sin(30/180*%pi)=49/sin(30/
180*%pi)], [T1,T2])$ float(%);
(%o4) [[T1=35.87,T2=43.932]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $-T_1 + T_2 \frac{\cos 45^\circ}{\cos 30^\circ} = 0$ and $T_1 + T_2 \frac{\sin 45^\circ}{\sin 30^\circ} = \frac{49}{\sin 30^\circ}$ for T_1 and T_2.

Problem 2.10 Calculate the tensions T_1, T_2, and T_3 of the strings for a system as shown in Fig. 2.10. If each string can only support 200 N, what is the maximum mass that can be suspended?

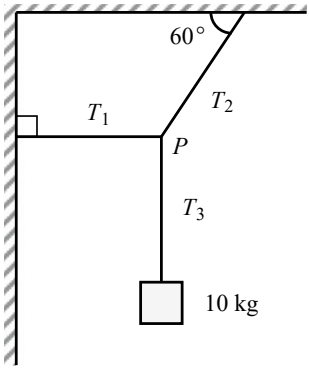

Fig. 2.10 A mass suspended by three strings of Problem 2.10

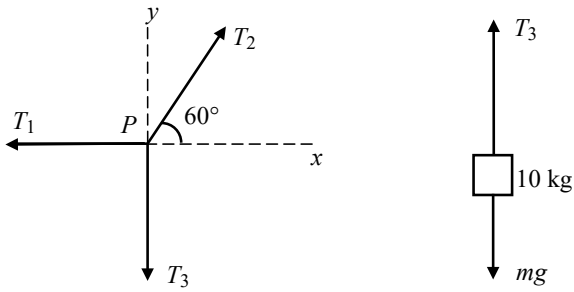

Fig. 2.11 **a** Forces acting at point P. **b** Forces acting on the mass, Problem 2.10

Solution

The forces acting on point P are shown in Fig. 2.11a. The forces are tensions in the strings T_1, T_2, and T_3. This is the free body diagram for point P.

Because the system is in equilibrium (no acceleration), then $\Sigma F_x = 0$. This gives,

$$-T_1 + T_2 \cos 60^\circ = 0.$$

Similarly, $\Sigma F_y = 0$, and this gives,

$$T_2 \sin 60^\circ - T_3 = 0.$$

Forces acting on the 10 kg mass are shown in Fig. 2.11b. The forces are T_3, the tension in the string and mg, the weight of the mass. This is the free body diagram for the mass. Because the mass is in equilibrium, $\Sigma F_y = 0$, and this gives,

$$T_3 - mg = 0.$$

The three equations give,

$$T_3 = mg = (10\,\text{kg})(9.8\,\text{N/kg}) = 98\,\text{N},$$
$$T_2 = \frac{T_3}{\sin\ 60^\circ} = \frac{98\,\text{N}}{\sin 60^\circ} = 113\,\text{N},$$
$$T_1 = T_2 \cos 60^\circ = 56.6\,\text{N}.$$

- wxMaxima codes:

```
(%i2)      fpprintprec:5; ratprint:false;
(fpprintprec)          5
(ratprint)         false
(%i4)     solve([-T1+T2*cos(60*%pi/180)=0, T2*sin(60*%pi/
180)-T3=0, T3=10*9.8], [T1,T2,T3])$ float(%);
(%o4)      [[T1=56.58,T2=113.16,T3=98.0]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $-T_1 + T_2 \cos 60^\text{o} = 0$, $T_2 \sin 60^\text{o} - T_3 = 0$, and $T_3 - mg = 0$ for T_1, T_2, and T_3.

Among the three tensions, T_2 is the largest, so the string with tension T_2 should be considered. Condition that should be satisfied is,

$$T_2 = \frac{T_3}{\sin 60^\circ} = \frac{mg}{\sin 60^\circ} = \frac{m\left(9.8\,\text{m/s}^2\right)}{\sin 60^\circ} < 200\,\text{N},$$
$$m < \frac{200\,\text{N}\sin 60^\circ}{9.8\,\text{m/s}^2} = 17.6\,\text{kg}.$$

This means maximum mass that can be supported is 17.6 kg.

- wxMaxima codes:

```
(%i1)    fpprintprec:5;
(fpprintprec)        5
(%i2)    200*sin(60/180*float(%pi))/9.8;
(%o2)    17.674
```

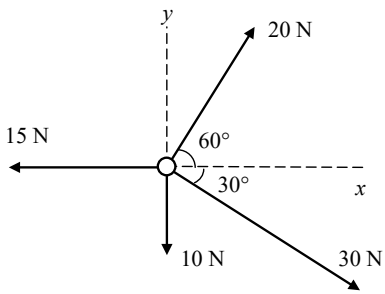

Fig. 2.12 Four coplanar forces acting on an object of Problem 2.11

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2) Calculate $\frac{200 \sin 60^\circ}{9.8}$.

Problem 2.11 Four coplanar forces acting on an object are shown in Fig. 2.12. Determine,

(a) the net force on the object
(b) the fifth force that should be applied on the object so that the net force is zero.

Solution

(a) The x component of the four forces (the net force) is, Fig. 2.12,

$$F_x = -15\,\text{N} + 20\,\text{N}\cos 60^\circ + 30\,\text{N}\cos 30^\circ = 21\,\text{N}.$$

The y component of the net force is,

$$F_y = 20\,\text{N}\sin 60^\circ - 30\,\text{N}\sin 30^\circ - 10\,\text{N} = -7.7\,\text{N}.$$

The net force vector is,

$$\boldsymbol{F} = 21\mathbf{i} - 7.7\mathbf{j}\,\text{N}.$$

The magnitude of the net force is,

$$F = \sqrt{21^2 + (-7.7)^2} = 22\,\text{N}.$$

The angle, θ between $\boldsymbol{F}$ and x axis is,

$$\theta = \tan^{-1}\left(\frac{-7.7}{21}\right) = -0.35\,\text{rad} = -20^\circ = 340^\circ.$$

- wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec)      5
(%i2)  Fx: -15 + 20*cos(60/180*float(%pi)) + 30*cos(30/
180*float(%pi)) + 0;
(Fx)   20.981
(%i3)   Fy:  0  +  20*sin(60/180*float(%pi))  -  30*sin(30/
180*float(%pi)) - 10;
(Fy)   -7.6795
(%i4)  F: sqrt(Fx^2 + Fy^2);
(F)    22.342
(%i5)  theta: atan(Fy/Fx);
(theta)   -0.35088
(%i6)  theta_deg: float(theta*180/%pi);
(theta_deg)  -20.104
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2), (%i3), (%i4) Calculate F_x, F_y, and F.
(%i5), (%i6) Calculate θ and convert the angle to degree.
To get a zero net force,

$$\boldsymbol{F}' = -\boldsymbol{F} = -21\mathbf{i} + 7.7\mathbf{j}\,\mathrm{N},$$

needs to be applied on the object. This is because $\boldsymbol{F} + \boldsymbol{F}' = 0$. Thus, the fifth force is $\boldsymbol{F}' = -21\mathbf{i} + 7.7\mathbf{j}$ **N**.

Problem 2.12 Fig. 2.13 shows two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. Calculate the scalar product $\boldsymbol{A} \cdot \boldsymbol{B}$.

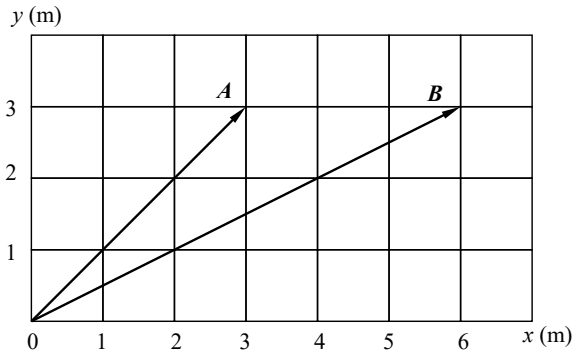

Fig. 2.13 Vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ of Problem 2.12

Solution

The dot product or the scalar product is defined as,

$$\boldsymbol{A} \cdot \boldsymbol{B} = AB\cos\theta,$$

where A and B are the magnitudes of the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, while θ is the angle between vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. From the figure, one gets,

$$A = \sqrt{3^2 + 3^2} = \sqrt{18}\,\text{m} = 4.24\,\text{m},$$
$$B = \sqrt{6^2 + 3^2} = \sqrt{45}\,\text{m} = 6.71\,\text{m}.$$

The angle θ between $\boldsymbol{A}$ and $\boldsymbol{B}$ is,

$$\theta = \tan^{-1}\left(\frac{3}{3}\right) - \tan^{-1}\left(\frac{3}{6}\right) = 18.4^\circ.$$

Therefore,

$$\boldsymbol{A} \cdot \boldsymbol{B} = AB\cos\theta = \sqrt{18}\sqrt{45}\cos 18.4^\circ = 27\,\text{m}^2.$$

- wxMaxima codes:

```
(%i1)  fpprintprec:5;
(fpprintprec)      5
(%i3)  A: float(sqrt(3^2 + 3^2)); B: float(sqrt(6^2 + 3^2));
(A)    4.2426
(B)    6.7082
(%i4)  theta: float(atan(1)-atan(3/6));
(theta)   0.32175
(%i5)  theta_deg: float(theta*180/%pi);
(theta_deg)  18.435
(%i6)  A_dot_B: A*B*cos(theta);
(A_dot_B)  27.0
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i3) Calculate A and B.
(%i4), (%i5) Calculate θ and convert the angle to degree.
(%i6) Calculate $\boldsymbol{A} \cdot \boldsymbol{B}$.

Alternative solution: The given two vectors are written as,

$$\boldsymbol{A} = A_x\mathbf{i} + A_y\mathbf{j} = 3\mathbf{i} + 3\mathbf{j}\ \text{m},$$
$$\boldsymbol{B} = B_x\mathbf{i} + B_y\mathbf{j} = 6\mathbf{i} + 3\mathbf{j}\ \text{m}.$$

Then,

$$\boldsymbol{A}\cdot\boldsymbol{B} = A_xB_x + A_yB_y = (3)(6) + (3)(3) = 27\,\text{m}^2,$$
$$A = \sqrt{3^2 + 3^2} = 4.24\,\text{m},$$
$$B = \sqrt{6^2 + 3^2} = 6.71\,\text{m},$$
$$\theta = \cos^{-1}\left(\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{AB}\right) = \cos^{-1}\left(\frac{27}{4.24\times 6.71}\right) = 0.32\,\text{rad} = 18.4^\circ.$$

To calculate this by wxMaxima, the two vectors are defined, the dot product is executed, the length of A and B are calculated, and the angle θ is computed.

- wxMaxima codes:

```
(%i1)        fpprintprec:5;
(fpprintprec)             5
(%i3)        Avec: [3, 3]; Bvec: [6, 3];
(Avec)             [3,3]
(Bvec)             [6,3]
(%i4)        Avec.Bvec;
(%o4)       27
(%i6)        A: float(sqrt(Avec.Avec));
B: float(sqrt(Bvec.Bvec));
(A)          4.2426
(B)          6.7082
(%i7)        theta: acos(Avec.Bvec/(A*B));
(theta)            0.32175
(%i8)        theta_deg: float(theta*180/%pi);
(theta_deg)       18.435
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i3) Assign vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.
(%i4) Calculate dot product $\boldsymbol{A}\cdot\boldsymbol{B}$.
(%i6) Calculate lengths A and B.
(%i7), (%i8) Calculate angle θ and convert the angle to degree.

Problem 2.13 Given two vectors $\boldsymbol{A} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $\boldsymbol{B} = 2\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$, calculate,

(a) $\boldsymbol{A} + \boldsymbol{B}$
(b) $|\boldsymbol{A} + \boldsymbol{B}|$

(c) $\boldsymbol{A} \cdot \boldsymbol{B}$
(d) The angle between the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.

Solution

(a) $\boldsymbol{A} + \boldsymbol{B} = (3+2)\mathbf{i} + (4-5)\mathbf{j} + (-1+6)\mathbf{k} = 5\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.
(b) $|\boldsymbol{A} + \boldsymbol{B}| = \sqrt{5^2 + (-1)^2 + 5^2} = 7.14$.
(c) $\boldsymbol{A} \cdot \boldsymbol{B} = (3)(2) + (4)(-5) + (-1)(6) = -20$.
(d) Lengths A and B are,

$$A = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{26},$$
$$B = \sqrt{2^2 + (-5)^2 + 6^2} = \sqrt{65}.$$

By definition $\boldsymbol{A} \cdot \boldsymbol{B} = AB\cos\theta$. Using result of (c), the angle can be calculated as,

$$\cos\theta = \frac{\boldsymbol{A} \cdot \boldsymbol{B}}{AB}, \quad \theta = \cos^{-1}\left(\frac{\boldsymbol{A} \cdot \boldsymbol{B}}{AB}\right) = \cos^{-1}\left(\frac{-20}{\sqrt{26}\sqrt{65}}\right) = 119^\circ.$$

- wxMaxima codes:

```
(%i1)     fpprintprec:5;
(fpprintprec)          5
(%i3)     Avec: [3,4,-1]; Bvec: [2,-5,6];
(Avec)      [3,4,-1]
(Bvec)      [2,-5,6]
(%i4)     Avec+Bvec;
(%o4)     [5,-1,5]
(%i5)     lengthAvecplusBvec: float(sqrt((Avec+Bvec).
(Avec+Bvec)));
(lengthAvecplusBvec)       7.1414
(%i6)     Avec.Bvec;
(%o6)     -20
(%i8)     A: float(sqrt(Avec.Avec));
B: float(sqrt(Bvec.Bvec));
(A)        5.099
(B)        8.0623
(%i9)      theta: acos(Avec.Bvec/(A*B));
(theta)       2.0789
(%i10)     theta_deg: float(theta*180/%pi);
(theta_deg)     119.11
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i3) Assign vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.
(%i4) Calculate vector addition $\boldsymbol{A} + \boldsymbol{B}$. Part (a).
(%i5) Calculate length of $\boldsymbol{A} + \boldsymbol{B}$. Part (b).
(%i6) Calculate vector dot product $\boldsymbol{A} \cdot \boldsymbol{B}$. Part (c).
(%i8) Calculate lengths A and B. Part (d).
(%i9), (%i10) Calculate angle θ and convert the angle to degree. Part (d).

Problem 2.14 Add a force of 10 N acting at 30° and a force of 6.0 N acting at 90° geometrically using the triangle and parallelogram methods.

Solution

Figure 2.14a shows the two forces to be added and the result of the addition $\boldsymbol{F}$ using the triangle method. $\overrightarrow{AB}$ is the first force, $\overrightarrow{BC}$ is the second force, and $\overrightarrow{AC}$ is the resultant of the addition. The first vector is drawn. The second vector is drawn by joining the tail of the second vector with the head of the first vector. The resultant is the vector from the tail of first vector to the head of the second vector.

The magnitude, F corresponds to the length AC and can be calculated by the cosine rule (see Appendix D) as follows,

$$\begin{aligned} F &= AC \\ &= \sqrt{AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos 120^\circ} \\ &= \sqrt{10^2 + 6.0^2 - 2(10)(6.0) \cos 120^\circ} \\ &= 14\,\text{N}. \end{aligned}$$

The angle θ can be calculated using the sine rule (Appendix D) as,

$$\frac{\sin\theta}{6} = \frac{\sin 120^\circ}{14},$$

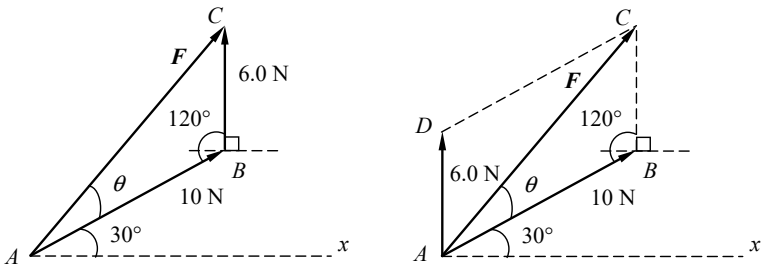

Fig. 2.14 Addition of two vectors by triangle method (**a**) and parallelogram method (**b**), Problem 2.14

$$\theta = 22^{\circ}.$$

The resultant force $\boldsymbol{F}$ is 14 N at 52º.

- wxMaxima codes:

```
(%i2)      fpprintprec:5; ratprint:false;
(fpprintprec)          5
(ratprint)         false
(%i3)      F: sqrt(10^2 + 6^2 - 2*10*6*cos(120/180*%pi));
(F)         14
(%i5)      solve(sin(theta)/6 = sin(120/180*%pi)/14,
theta)$ float(%);
solve: using arc-trig functions to get a solution.
Some solutions will be lost.
(%o5)     [theta=0.38025]
(%i6)      theta: rhs(%[1]);
(theta)       0.38025
(%i7)      theta_deg: float(theta*180/%pi);
(theta_deg)      21.787
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3) Calculate F.
(%i5) Solve $\frac{\sin\theta}{6} = \frac{\sin 120^{\circ}}{14}$ for θ.
(%i6), (%i7) Assign value of θ and convert θ to degree.

Figure 2.14b shows the addition of the two forces by parallelogram method. $\overrightarrow{AB}$ is the first force, $\overrightarrow{AD}$ is the second force, and $\overrightarrow{AC}$ is the resultant of the addition. In parallelogram method, both vectors are drawn with their tails at the origin (AB and AD). Next, parallel line of each vector is drawn at the head of the other vector (DC and BC). This forms a parallelogram $ABCD$. The resultant is the diagonal of the parallelogram (AC). It can be seen that the same triangle ABC appears here as in the triangle method. Thus, the same values of F and θ will be obtained from the parallelogram method.

Alternative solution: The forces are expressed in terms of their components and then added. The first force is,

$$\boldsymbol{F}_1 = 10\cos 30^\circ\,\mathbf{i} + 10\sin 30^\circ\,\mathbf{j} = 8.7\mathbf{i} + 5\mathbf{j}\ \text{N}.$$

The second force is,

$$\boldsymbol{F}_2 = 6.0\mathbf{j}\ \text{N}.$$

Addition of the two forces is,

$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 = 8.7\mathbf{i} + 11\mathbf{j}\ \text{N}.$$

The magnitude of $\boldsymbol{F}$ is,

$$F = \sqrt{8.7^2 + 11^2}\ \text{N} = 14\,\text{N}.$$

The angle between $\boldsymbol{F}$ and x axis is,

$$\tan^{-1}\left(\frac{11}{8.7}\right) = 51.8^\circ.$$

- wxMaxima codes:

```
(%i1)     fpprintprec:5;
(fpprintprec)          5
(%i3)     F1x: 10*cos(30/180*float(%pi)); F1y: 10*sin(30/
180*float(%pi));
(F1x)     8.6603
(F1y)     5.0
(%i5)     F2x: 0; F2y: 6;
(F2x)     0
(F2y)     6
(%i7)     Fx: F1x + F2x; Fy: F1y + F2y;
(Fx)      8.6603
(Fy)      11.0
(%i8)     F: sqrt(Fx^2 + Fy^2);
(F)       14.0
(%i9)     angle: atan(Fy/Fx);
(angle)      0.90385
(%i10)    angle_deg: float(angle*180/%pi);
(angle_deg)     51.787
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i3) Assign values of F_{1x} and F_{1y}.
(%i5) Assign values of F_{2x} and F_{2y}.
(%i7) Calculate F_x and F_y.
(%i8) Calculate F.
(%i9), (%i10) Calculate angle between F and x axis and convert the angle to degree.

- Alternative calculation:

```
(%i1)       fpprintprec:5;
(fpprintprec)            5
(%i2)       F1vec: [10*cos(30/180*float(%pi)),
10*sin(30/180*float(%pi))];
(F1vec)        [8.6603,5.0]
(%i3)       F2vec: [0,6];
(F2vec)        [0,6]
(%i4)       Fvec: F1vec+F2vec;
(Fvec)        [8.6603,11.0]
(%i5)       F: float(sqrt(Fvec.Fvec));
(F)          14.0
(%i6)       angle: atan(Fvec[2]/Fvec[1]);
(angle)         0.90385
(%i7)       angle_deg: float(angle*180/%pi);
(angle_deg)       51.787
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2), (%i3) Assign vectors $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$.
(%i4), (%i5) Calculate vector $\boldsymbol{F}$ and its magnitude F.
(%i6), (%i7) Calculate angle between F and x axis and convert the angle to degree.

Problem 2.15 Determine the magnitudes of the resultant forces as shown in Fig. 2.15.

Solution

The problems are addition of two vectors where the angle between them is given. Let us add any two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ where the angle between them is θ (Fig. 2.16). In Fig. 2.16a, the two vectors to be added are drawn together with the angle. In Fig. 2.16b, the addition of the vectors is performed geometrically by the triangle method. First, draw $\boldsymbol{A}$**.** Next, draw $\boldsymbol{B}$ by joining the tail of $\boldsymbol{B}$ with the head of $\boldsymbol{A}$. The resultant $\boldsymbol{A} + \boldsymbol{B}$ is the vector from tail of $\boldsymbol{A}$ to head of $\boldsymbol{B}$.

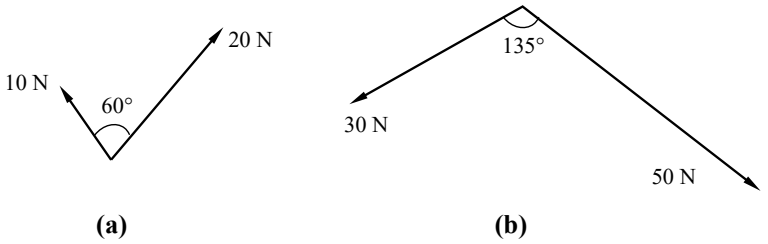

Fig. 2.15 Determining resultant forces of (a) and (b), Problem 2.15

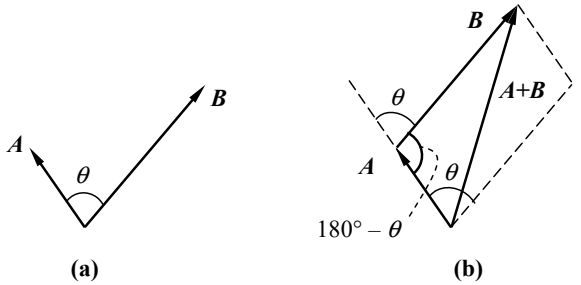

Fig. 2.16 Determining resultant of any two vectors, **a** the vectors, **b** their resultant

From Fig. 2.16b, the magnitude of $\boldsymbol{A} + \boldsymbol{B}$ is obtained by the cosine rule, see Appendix D,

$$\begin{aligned} |\boldsymbol{A} + \boldsymbol{B}| &= \sqrt{A^2 + B^2 - 2AB\cos(180^\circ - \theta)} \\ &= \sqrt{A^2 + B^2 + 2AB\cos\theta}. \end{aligned}$$

The result can also be obtained by the dot product given by,

$$\begin{aligned} |\boldsymbol{A} + \boldsymbol{B}| &= [(\boldsymbol{A} + \boldsymbol{B}) \cdot (\boldsymbol{A} + \boldsymbol{B})]^{1/2} \\ &= (\boldsymbol{A} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{B} \cdot \boldsymbol{A} + \boldsymbol{B} \cdot \boldsymbol{B})^{1/2} \\ &= (A^2 + B^2 + 2\boldsymbol{A} \cdot \boldsymbol{B})^{1/2} \\ &= (A^2 + B^2 + 2AB\cos\theta)^{1/2}. \end{aligned}$$

Using the results, the magnitude of the resultant force in Fig. 2.15a is,

$$\sqrt{10^2 + 20^2 + 2(10)(20)\cos 60^\circ}\,\text{N} = 26\,\text{N}.$$

The magnitude of the resultant force in Fig. 2.15b is,

$$\sqrt{30^2 + 50^2 + 2(30)(50)\cos 135^\circ}\,\text{N} = 36\,\text{N}.$$

- wxMaxima codes:

```
(%i1)      fpprintprec:5;
(fpprintprec)          5
(%i2)       sqrt(10^2 + 20^2 + 2*10*20*cos(60*float(%pi)/
180));
(%o2)      26.458
(%i3)       sqrt(30^2 + 50^2 + 2*30*50*cos(135*float(%pi)/
180));
(%o3)      35.759
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2) Calculate $\sqrt{10^2 + 20^2 + 2(10)(20)\cos 60^\circ}$.
(%i3) Calculate $\sqrt{30^2 + 50^2 + 2(30)(50)\cos 135^\circ}$.

Problem 2.16 Figure 2.17 shows two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. Calculate their vector product $\boldsymbol{A} \times \boldsymbol{B}$.

Solution

The vector product or the cross product of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is given by,

$$\boldsymbol{A} \times \boldsymbol{B} = AB \sin\theta \, \hat{\mathbf{n}}.$$

The magnitude of each vector is,

$$A = \sqrt{3^2 + 3^2}\,\text{m} = \sqrt{18}\,\text{m},$$
$$B = 6\,\text{m}.$$

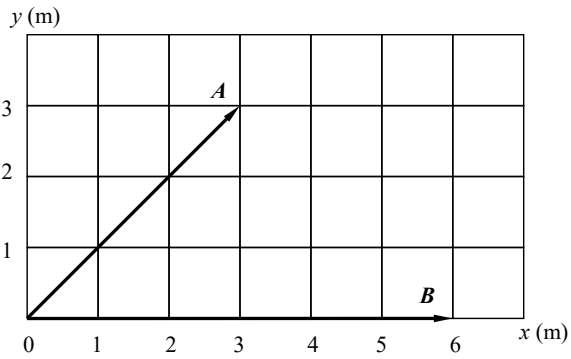

Fig. 2.17 Vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ of Problem 2.16

The angle between these vectors is 45°. The magnitude $|\boldsymbol{A} \times \boldsymbol{B}|$ is,

$$|\boldsymbol{A} \times \boldsymbol{B}| = AB \sin\theta = \sqrt{18}\,\mathrm{m} \times 6\,\mathrm{m} \times \sin 45^\circ = 18\,\mathrm{m}^2.$$

The direction of $\boldsymbol{A} \times \boldsymbol{B}$ is negative z direction, that is $-\mathbf{k}$, or into the plane of the paper. This is determined by the right hand rule (Fig. 2.1). The thumb is the direction of $\boldsymbol{A}$, the index finger is the direction of $\boldsymbol{B}$, and the middle finger is the direction of $\boldsymbol{A} \times \boldsymbol{B}$.

Alternative calculation: Vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are,

$$\boldsymbol{A} = 3\mathbf{i} + 3\mathbf{j}\ \mathrm{m},$$
$$\boldsymbol{B} = 6\mathbf{i}\ \mathrm{m}.$$

Using Eq. (2.11) one gets,

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 0 \\ 6 & 0 & 0 \end{vmatrix} = (3 \times 0 - 3 \times 6)\mathbf{k} = -18\mathbf{k}\,\mathrm{m}^2,$$

by the vector cross product formula given in the equation. The result can also be obtained by direct calculation as follows,

$$\boldsymbol{A} \times \boldsymbol{B} = (3\mathbf{i} + 3\mathbf{j}) \times 6\mathbf{i} = 18\mathbf{i} \times \mathbf{i} + 18\mathbf{j} \times \mathbf{i} = -18\mathbf{k}\,\mathrm{m}^2,$$

because $\mathbf{i} \times \mathbf{i} = \mathbf{0}$ and $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$.

- wxMaxima codes:

```
(%i1)       load("vect");
(%o1)       "C:/maxima-5.43.0/share/maxima/5.43.0/share/
vector/vect.mac"
(%i3)       A: [3,3,0]; B: [6,0,0];
(A)         [3,3,0]
(B)         [6,0,0]
(%i5)       A~B; express(%);
(%o4)       [3,3,0]~[6,0,0]
(%o5)       [0,0,-18]
```

Comments on the codes:

(%i1) Load "vect" package.
(%i3) Assign vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.

(%i5) Calculate vector cross product $\boldsymbol{A} \times \boldsymbol{B}$.
(%o5) The result.

Problem 2.17 Given two vectors,

$$\boldsymbol{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}\ \text{m},$$
$$\boldsymbol{F} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}\ \text{N},$$

where $\boldsymbol{r}$ is the position vector and $\boldsymbol{F}$ is the force. Calculate the torque $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$.

Solution

Using the cross product formula (Eq. 2.11),

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 1 & 3 & -2 \end{vmatrix}$$
$$= [(-2)(-2) - (1)(3)]\,\mathbf{i} - [(3)(-2) - (1)(1)]\,\mathbf{j} + [(3)(3) - (-2)(1)]\mathbf{k}$$
$$= \mathbf{i} + 7\mathbf{j} + 11\mathbf{k}\ \text{N m}.$$

The magnitude of the torque is,

$$\tau = \sqrt{1^2 + 7^2 + 11^2}\ \text{N m} = 13\ \text{N m}.$$

- wxMaxima codes:

```
(%i1)      fpprintprec:5;
(fpprintprec)          5
(%i2)      load("vect");
(%o2)       "C:/maxima-5.43.0/share/maxima/5.43.0/share/
vector/vect.mac"
(%i4)      r: [3,-2,1]; F: [1,3,-2];
(r)         [3,-2,1]
(F)        [1,3,-2]
(%i6)      torque: r~F; express(%);
(torque)       -[1,3,-2]~[3,-2,1]
(%o6)     [1,7,11]
(%i7)      magnitude: float(sqrt(%.%));
(magnitude)      13.077
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i2) Load "vect" package.
(%i4) Assign vectors $\boldsymbol{r}$ and $\boldsymbol{F}$.
(%i6), (%i7) Calculate torque and its magnitude.

2.3 Summary

- Many physical quantities are scalars while some are vectors. Vector has both magnitude and direction, while scalar has only magnitude.
- A vector can be resolved into components.
- Vectors can be summed and subtracted.
- Dot product and cross product of the vector quantities describing physical quantities have specific physical meanings.

2.4 Exercises

Exercise 2.1 Which of these physical quantities are vectors: time, mass, volume, distance, displacement, speed, velocity, acceleration, pressure, temperature?

(Answer: displacement, velocity, acceleration)

Exercise 2.2 A vector $\boldsymbol{A}$ has the x component, $A_x = 12$ m and the y component, $A_y = 5.0$ m. Calculate (a) the magnitude of the vector (b) the angle between the vector and the x axis.

(Answer: (a) 13 m, (b) 23°)

Exercise 2.3 Figure 2.18 shows the forces $F_1 = 8.0$ N, $F_2 = 6.0$ N, and $F_3 = 4.0$ N acting on an object O. Calculate the resultant force acting on the object.

(Answer: 3.5 N at 34° with respect to the positive x axis)

Exercise 2.4 A force $F = 5.0$ N acts on a 6.0 m bar as shown in Fig. 2.19. Calculate the torque on the bar with respect to the point O.

(Answer: 28 N m, out of the page)

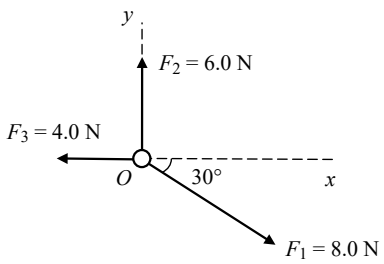

Fig. 2.18 Three forces acting on an object, Exercise 2.3

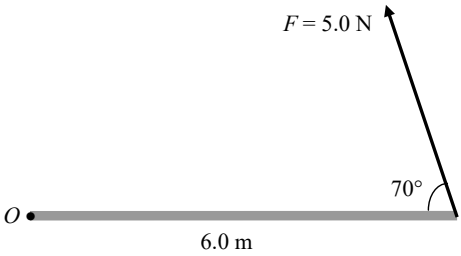

Fig. 2.19 A force acting on a bar, Exercise 2.4

Exercise 2.5 A force $\boldsymbol{F} = 5\mathbf{i} + 2\mathbf{j}$ N acts on an object and the object is displaced by $\boldsymbol{s} = 6\mathbf{i}$ m. What is the work done by the force?

(Answer: 30 J)

Chapter 3
Kinematics of Particle in One Dimension

3.1 Basic Concepts and Formulae

(1) The average velocity, $\overline{v}$, of a particle that has undergone displacement, Δx, in time interval, Δt, is the displacement divided by the time interval,

$$\overline{v} = \frac{\Delta x}{\Delta t}. \tag{3.1}$$

(2) The instantaneous velocity of a particle, v, is the limit of $\Delta x/\Delta t$ as Δt goes to zero. This quantity is the differentiation of the displacement, x, with respect to time, t, or the time rate of change of displacement,

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}. \tag{3.2}$$

The speed is the magnitude or the absolute value of velocity.

(3) The average acceleration of a particle, $\overline{\boldsymbol{a}}$, is the change of velocity, Δv, in time interval, Δt, divided by the time interval,

$$\overline{\boldsymbol{a}} = \frac{\Delta v}{\Delta t}. \tag{3.3}$$

(4) The instantaneous acceleration of a particle, $\boldsymbol{a}$, is the limit of $\Delta v/\Delta t$ as Δt goes to zero. This quantity is the derivative of velocity v with respect to time t, or the time rate of velocity change,

$$\boldsymbol{a} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}. \tag{3.4}$$

(5) The slope of the tangent line to a curve of displacement, x, against time, t, at any time is the instantaneous velocity.

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W. M. S. Wan Hassan, *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-42678-0_3

(6) The slope of the tangent line to a curve of velocity, v, against time, t, at any time is the instantaneous acceleration.
(7) The area under a curve of velocity, v, against time, t, for any time interval is the displacement in the time interval.
(8) Kinematic equations for a particle moving with constant acceleration, a, along x axis are,

$$\begin{array}{ll} v = v_0 + at & \\ x - x_0 = \frac{1}{2}(v_0 + v)t & \text{or} \\ x - x_0 = v_0 t + \frac{1}{2}at^2 & \\ v^2 - v_0^2 = 2a(x - x_0) & \end{array} \qquad \begin{array}{l} v = u + at \\ s = \frac{1}{2}(u + v)t \\ s = ut + \frac{1}{2}at^2 \\ v^2 - u^2 = 2as \end{array}$$

(9) An object in a free fall near the earth surface has a gravitational acceleration, $g = 9.8\ \text{m s}^{-2} = 32\ \text{ft s}^{-2}$, neglecting the air resistance. Choosing the y axis as positive in the upward direction, one writes, $a = -g$, and the kinematic equations for a body undergoing a free fall as,

$$\begin{array}{ll} v = v_0 - gt & \\ y - y_0 = \frac{1}{2}(v_0 + v)t & \text{or} \\ y - y_0 = v_0 t - \frac{1}{2}gt^2 & \\ v^2 = v_0^2 - 2g(y - y_0) & \end{array} \qquad \begin{array}{l} v = u - gt \\ s = \frac{1}{2}(u + v)t \\ s = ut - \frac{1}{2}gt^2 \\ v^2 = u^2 - 2gs \end{array}$$

3.2 Problems and Solutions

Problem 3.1 A man runs along a line at average velocity of 5.0 m s^{-1} for 4.0 min, followed by average velocity of 4.0 m s^{-1} for 6.0 min. Determine,

(a) the total displacement
(b) the average velocity in 10 min.

Solution

(a) The total displacement is the displacement in the first 4.0 min and next 6.0 min,

$$\text{total displacement} = (5\,\text{m/s})(4 \times 60\,\text{s}) + (4\,\text{m/s})(6 \times 60\,\text{s}) = 2640\,\text{m}.$$

(b) The average velocity is the total displacement divided by total time, displacement time

$$\text{average velocity} = \frac{2640\,\text{m}}{10 \times 60\,\text{s}} = 4.4\,\text{m s}^{-1}.$$

- wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec)      5
(%i2) total_displacement: 5*4*60 + 4*6*60;
(total_displacement) 2640
(%i4) average_velocity: total_displacement/(10*60)$
float(%);
(%o4) 4.4
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i1), (%i4) Calculate total displacement and average velocity.

Problem 3.2 The displacement, x, of a particle along the x axis against time, t, is shown in Fig. 3.1. Determine,

(a) the average velocity in the time interval 0–2.0 s, 0–5.0 s, 1.0–4.0 s, and 0–6.0 s.
(b) the instantaneous velocity at $t = 1.0$ s, $t = 2.5$ s, $t = 4.0$ s, and $t = 5.5$ s.
(c) the curve of velocity against time.

Solution

(a) The average velocity is defined as,

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1},$$

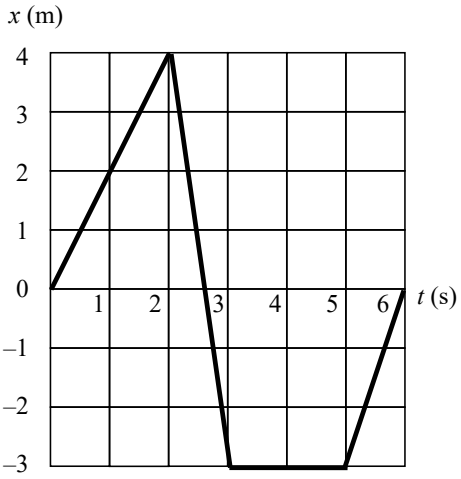


Fig. 3.1 Displacement against time curve of Problem 3.2

where x is the displacement of the particle and t is the time.

In 0–2.0 s, the average velocity is,

$$\overline{v} = \frac{4\,\mathrm{m} - 0\,\mathrm{m}}{2\,\mathrm{s} - 0\,\mathrm{s}} = 2.0\,\mathrm{m\,s^{-1}}.$$

In 0–5.0 s, the average velocity is,

$$\overline{v} = \frac{-3\,\mathrm{m} - 0\,\mathrm{m}}{5\,\mathrm{s} - 0\,\mathrm{s}} = -0.60\,\mathrm{m\,s^{-1}}.$$

In 1.0–4.0 s, the average velocity is,

$$\overline{v} = \frac{-3\,\mathrm{m} - 2\,\mathrm{m}}{4\,\mathrm{s} - 1\,\mathrm{s}} = -1.7\,\mathrm{m\,s^{-1}}.$$

In 0–6.0 s, the average velocity is,

$$\overline{v} = \frac{0\,\mathrm{m} - 0\,\mathrm{m}}{6\,\mathrm{s} - 0\,\mathrm{s}} = 0\,\mathrm{m\,s^{-1}}.$$

(b) The instantaneous velocity is the slope of the displacement against time curve at the time.

At $t = 1.0$ s, the instantaneous velocity is,

$$v = \frac{4\,\mathrm{m} - 0\,\mathrm{m}}{2\,\mathrm{s} - 0\,\mathrm{s}} = 2.0\,\mathrm{m\,s^{-1}}.$$

At $t = 2.5$ s, the instantaneous velocity is,

$$v = \frac{-3\,\mathrm{m} - 4\,\mathrm{m}}{3\,\mathrm{s} - 2\,\mathrm{s}} = -7.0\,\mathrm{m\,s^{-1}}.$$

At $t = 4.0$ s, the instantaneous velocity is,

$$v = \frac{-3\,\mathrm{m} - (-3\,\mathrm{m})}{4\,\mathrm{s} - 3\,\mathrm{s}} = 0\,\mathrm{m\,s^{-1}}.$$

At $t = 5.5$ s, the instantaneous velocity is,

$$v = \frac{0\,\mathrm{m} - (-3\,\mathrm{m})}{6\,\mathrm{s} - 5\,\mathrm{s}} = 3.0\,\mathrm{m\,s^{-1}}.$$

(c) The curve of velocity against time is shown in Fig. 3.2. Results of part (b) are used to mark the values of velocities on the graph.

Problem 3.3 The displacement of a particle is given by,

$$x = 7t^2 + 2t,$$

where x is in meter and t in second. Determine,

(a) the displacement of the particle between 2.0 and 4.0 s
(b) the average velocity of the particle between 2.0 and 4.0 s
(c) the instantaneous velocity of the particle at 2.0 s
(d) the instantaneous acceleration of the particle at 2.0 s.

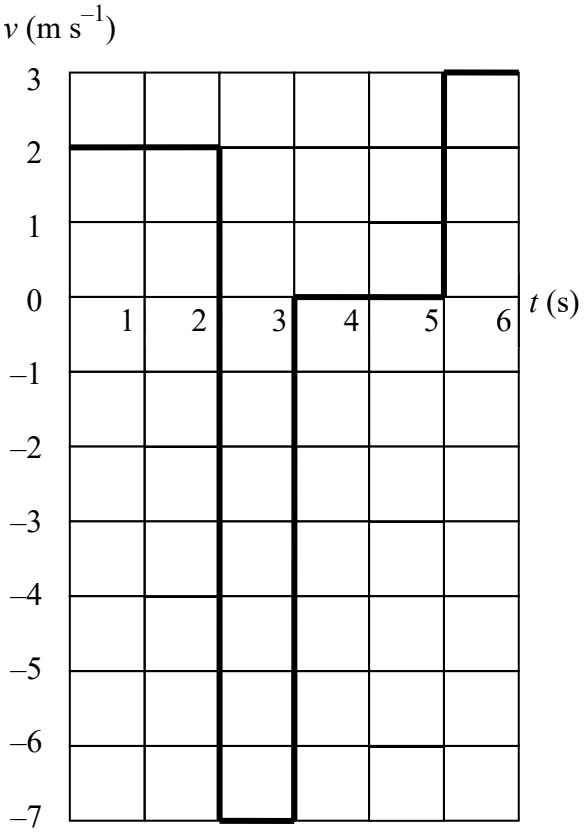


Fig. 3.2 Velocity against time curve of Problem 3.2

Solution

(a) To get the displacement, calculate the positions of the particle at two different times,

$$\text{at } t = 2.0\,\text{s}, \quad x = 7(2)^2 + 2(2) = 32\,\text{m},$$
$$\text{at } t = 4.0\,\text{s}, \quad x = 7(4)^2 + 2(4) = 120\,\text{m}.$$

The displacement of the particle is $\Delta x = 120\,\text{m} - 32\,\text{m} = 88\,\text{m}$.

(b) The average velocity is obtained by dividing the displacement by time interval,

$$\frac{\Delta x}{\Delta t} = \frac{88\,\text{m}}{(4-2)\,\text{s}} = 44\,\text{m s}^{-1}.$$

(c) The instantaneous velocity is the slope of the displacement against the time curve,

$$v = \frac{dx}{dt} = 14t + 2.$$

At $t = 2.0$ s,

$$v = 14(2) + 2 = 30\,\text{m s}^{-1}.$$

(d) The instantaneous acceleration is the slope of the velocity against the time curve,

$$a = \frac{dv}{dt} = 14.$$

The acceleration of the particle at any time is 14 m s^{-2}.

- wxMaxima codes:

```
(%i1)  fpprintprec:5;
(fpprintprec)      5
(%i2)  x(t) := 7*t^2 + 2*t;
(%o2)  x(t):=7*t^2+2*t
(%i3)  x(2);
(%o3)  32
(%i4)  x(4);
(%o4)  120
(%i5)  deltax: x(4)-x(2);
(deltax) 88
(%i6)  deltat:4-2;
(deltat) 2
```

```
(%i7)  average_velocity: deltax/deltat;
(average_velocity) 44
(%i8)  v: diff(x(t),t);
(v)  14*t+2
(%i9)  subst(t=2, v);
(%o9)  30
(%i10) a: diff(v,t);
(a)   14
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2) Define $x(t)$.
(%i3), (%i4), (%i5) Calculate $x(2)$, $x(4)$, Δx. Part (a).
(%i6), (%i7) Calculate Δt and average velocity. Part (b).
(%i8), (%i9) Calculate instantaneous velocity at 2 s. Part (c).
(%i10) Calculate acceleration a. Part (d).

Problem 3.4 A car is moving at a velocity of 35 miles per hour. Suddenly a cow crosses the road. The distance between the car and the cow at the time is 200 ft. Seeing the cow, the driver takes t seconds before he presses the brake and the car decelerates at 9.0 ft s^{-2}. What is the maximum reaction time so that the car does not hit the cow? (1 mile = 5280 ft).

Solution

Units for the distance is feet, time is second, and speed is feet/second. The initial velocity is,

$$v_0 = 35\,\frac{\text{mile}}{\text{h}} = \frac{35 \times 5280\,\text{ft}}{60 \times 60\,\text{s}} = 51.33\,\text{ft/s}.$$

The time of deceleration until the car stops is,

$$t = \frac{v - v_0}{a} = \frac{0 - 51.33\,\text{ft/s}}{-9\,\text{ft/s}^2} = 5.7\,\text{s},$$

and the distance travelled in the time is,

$$s = \frac{v^2 - v_0^2}{2a} = \frac{0 - (51.33\,\text{ft/s})^2}{2(-9\,\text{ft/s}^2)} = 146.4\,\text{ft}.$$

The distance travelled in the reaction time is,

$$200\,\text{ft} - 146.4\,\text{ft} = 53.6\,\text{ft}.$$

The maximum reaction time is,

$$\frac{53.6\,\text{ft}}{51.33\,\text{ft/s}} = 1.04\,\text{s}.$$

- wxMaxima codes:

```
(%i3)   fpprintprec:5; v:0; a:-9;
(fpprintprec)       5
(v)   0
(a)   - 9
(%i4)   v0: float(35*5280/(60*60));
(v0)   51.333
(%i5)   t: (v-v0)/a;
(t)   5.7037
(%i6)   s: (v^2-v0^2)/(2*a);
(s)   146.4
(%i7)   distance_travelled: 200-s;
(distance_travelled)   53.605
(%i8)reaction_time: distance_travelled/v0;
(reaction_time) 1.0443
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of v and a.
(%i4), (%i5), (%i6) Calculate v_0, t, and s.
(%i7) Calculate the distance travelled in the reaction time.
(%i8) Calculate the maximum reaction time.

Problem 3.5 The distance travelled by a racing car during acceleration from velocity of 30–60 m s^{-1} is 120 m. What are the acceleration and the time of the acceleration?

Solution

From equation $v^2 - u^2 = 2as$, the acceleration is,

$$a = \frac{v^2 - u^2}{2s} = \frac{(60\,\text{m/s})^2 - (30\,\text{m/s})^2}{2(120\,\text{m})} = 11.2\,\text{m s}^{-2}.$$

From equation $a = (v - u)/t$, one gets,

$$t = \frac{v - u}{a} = \frac{60\,\text{m/s} - 30\,\text{m/s}}{11.2\,\text{m/s}^2} = 2.7\,\text{s}.$$

- wxMaxima codes:

```
(%i4)   fpprintprec:5; u:30; v:60; s:120;
(fpprintprec)       5
(u)     30
(v)     60
(s)     120
(%i5)   a: float((v^2 - u^2)/(2*s));
(a)     11.25
(%i6)   t: float((v-u)/a);
(t)     2.6667
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of u, v, and s.
(%i5), (%i6) Calculate acceleration, a and time, t.

Problem 3.6 A man falls from level 17 of a building at a height of about 144 ft, and lands on an aluminum box. The box gets crushed by 18 inches and the man suffers minor injury. Determine,

(a) the speed at which he hits the box
(b) the deceleration as he crushes the box
(c) the time of the crush.

Neglect air resistance. Acceleration of gravity is 32 ft s^{-2} and 1 foot = 12 in.

Solution

(a) Using the formula $v^2 - u^2 = 2\,as$, one gets the speed when the man hits the box,

$$v^2 - 0 = 2(32\,\text{ft/s}^2)(144\,\text{ft}) = 9216\,\text{ft}^2/\text{s}^2,$$
$$v = 96\,\text{ft s}^{-1}.$$

- wxMaxima codes:

```
(%i4)  fpprintprec:5; s:144; a:32; u:0;
(fpprintprec)      5
(s)    144
(a)    32
(u)    0
(%i5)  solve(v^2 - u^2 = 2*a*s, v);
(%o5)  [v=-96,v=96]
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of s, a, and u.
(%i5) Solve $v^2 - u^2 = 2as$ for v.

(b) The deceleration of the man is,

$$a = \frac{v^2 - u^2}{2s} = \frac{0^2 - (96\,\text{ft/s})^2}{2\left(\frac{18}{12}\,\text{ft}\right)} = -3072\,\text{ft s}^{-2}.$$

- wxMaxima codes:

```
(%i4)   fpprintprec:5; s:float(18/12); u:96; v:0;
(fpprintprec)       5
(s)    1.5
(u)    96
(v)    0
(%i5)   a: (v^2 - u^2)/(2*s);
(a)    -3072.0
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of s, u, and v.
(%i5) Calculate acceleration, a.

(c) The time of the crush is,

$$t = \frac{v - u}{a} = \frac{0 - 96\,\text{ft/s}}{-3072\,\text{ft/s}^2} = 0.031\,\text{s}.$$

- wxMaxima codes:

```
(%i4)   fpprintprec:5; a:-3072; u:96; v:0;
(fpprintprec)       5
(a)    -3072
(u)    96
(v)    0
(%i5)   t: float((v-u)/a);
(t)    0.03125
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of a, u, and v.
(%i5) Calculate time, t.

Problem 3.7 A stone is thrown vertically upward at velocity 24 m s^{-1}. Determine,

(a) the position and the velocity of the stone at 1.0 and 4.0 s after it is thrown
(b) the velocity at a distance 10 m above the point of throw
(c) the maximum height and the time to reach this height.

Solution

(a) Figure 3.3 shows the path of the stone and reference y axis. The y axis is positive upward and the origin is the point of throw. Acceleration of gravity, $a = -g = -9.8$ m s^{-2}.

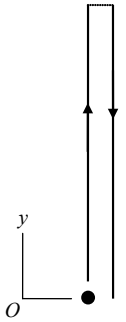


Fig. 3.3 Path of the stone, Problem 3.7

The displacement of the stone after $t_1 = 1.0$ s is,

$$y_1 = ut_1 + \frac{1}{2}at_1^2 = ut_1 - \frac{1}{2}gt_1^2 = (24\,\text{m/s})(1.0\,\text{s}) - \frac{1}{2}(9.8\,\text{m/s}^2)(1.0\,\text{s})^2$$
$$= 19\,\text{m},$$

and after $t_2 = 4.0$ s is,

$$y_2 = ut_2 - \frac{1}{2}gt_2^2 = (24\,\text{m/s})(4.0\,\text{s}) - \frac{1}{2}(9.8\,\text{m/s}^2)(4.0\,\text{s})^2$$
$$= 18\,\text{m}.$$

- wxMaxima codes:

```
(%i4)   fpprintprec:5; g:9.8; u:24; t1:1;
(fpprintprec)       5
(g)    9.8
(u)    24
(t1)    1
(%i5)    y1: u*t1 - 0.5*g*t1^2;
(y1)    19.1
(%i6)    t2: 4;
(t2)    4
(%i7)    y2: u*t2 - 0.5*g*t2^2;
(y2)    17.6
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of g, u, and t_1.
(%i5) Calculate y_1.
(%i6, %i7) Assign t_2 and calculate y_2.

(b) The velocity at $y = 10$ m is calculated as follows,

$$v^2 - u^2 = 2ay$$
$$v^2 - (24\,\text{m/s})^2 = 2(-9.8\,\text{m/s}^2)(10\,\text{m})$$
$$v = \pm 19\,\text{m s}^{-1}.$$

- wxMaxima codes:

```
(%i5)   fpprintprec:5; ratprint:false; y:10; u:24; a:-9.8;
(fpprintprec)       5
(ratprint)  false
(y)      10
(u)      24
(a)      -9.8
(%i7)     solve(v^2 - u^2 = 2*a*y, v)$ float(%);
(%o7)     [v=-19.494,v=19.494]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, rational number print to false, and assign values of y, u, and a.
(%i7) Solve equation $v^2 - u^2 = 2ay$ for v.

(c) The maximum height is attained by the stone when its velocity is zero, $v = 0$, which is obtained as follows,

$$\begin{aligned} v^2 - u^2 &= 2ay \\ 0^2 - (24\,\text{m/s})^2 &= 2(-9.8\,\text{m/s}^2)y \\ y &= 29\,\text{m}. \end{aligned}$$

The time taken to reach the maximum height is,

$$t = \frac{v-u}{a} = \frac{0 - 24\,\text{m/s}}{-9.8\,\text{m/s}^2} = 2.4\,\text{s}.$$

- wxMaxima codes:

```
(%i5)    fpprintprec:5; ratprint:false; v:0; u:24; a:-9.8;
(fpprintprec)        5
(ratprint)  false
(v)      0
(u)      24
(a)      -9.8
(%i7)    solve(v^2 - u^2 = 2*a*y, y)$ float(%);
(%o7)    [y=29.388]
(%i8)    t: (v-u)/a;
(t)      2.449
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false and assign values of v, u, and a.
(%i7) Solve equation $v^2 - u^2 = 2ay$ for y.
(%i8) Calculate t.

Problem 3.8

(a) A car accelerates from the rest to a speed of 100 km h^{-1} is 13 s. Calculate the average acceleration and the distance traversed.
(b) A Boeing 747 jet plane has to accelerate to a speed of 290 km h^{-1} to take off from the runway. If the length of the runway is 1.5 km, calculate the minimum acceleration and the time to accelerate it.

Solution

(a) The acceleration of the car is,

$$a = \frac{v-u}{t} = \frac{\frac{100\times10^3\,\text{m}}{60\times60\,\text{s}} - 0}{13\,\text{s}} = 2.14\,\text{m s}^{-2}.$$

The distance traversed is,

$$s = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2}(2.14\,\text{m/s}^2)(13\,\text{s})^2 = 181\,\text{m},$$

or can also be calculated as follows,

$$v^2 - u^2 = 2as,$$

$$s = \frac{v^2-u^2}{2a} = \frac{\left(\frac{100\times10^3\,\text{m}}{60\times60\,\text{s}}\right)^2 - 0}{2(2.14\,\text{m/s}^2)} = 181\,\text{m}.$$

- wxMaxima codes:

```
(%i1)   fpprintprec:5;
(fpprintprec)       5
(%i2)   v: 100e3/60/60;
(v)     27.778
(%i3)   a: (v-0)/13;
(a)     2.1368
(%i4)   s: 0.5*a*13^2;
(s)     180.56
(%i5)   s: (v^2-0)/(2*a);
(s)     180.56
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2), (%i3), (%i4), (%i5) Calculate v, a, and s.

(b) The minimum acceleration of the plane is calculated as follows,

$$v^2 - u^2 = 2as,$$

$$a = \frac{v^2-u^2}{2s} = \frac{\left(\frac{290\times10^3\,\text{m}}{60\times60\,\text{s}}\right)^2 - 0}{2(1.5\times10^3\,\text{m})} = 2.16\,\text{m s}^{-2}.$$

The acceleration time is,

$$t = \frac{v-u}{a} = \frac{\frac{290\times10^3\,\text{m}}{60\times60\,\text{s}} - 0}{2.16\,\text{m/s}^2} = 37\,\text{s}.$$

- wxMaxima codes:

```
(%i1)   fpprintprec:5;
(fpprintprec)       5
(%i2)   v: 290e3/60/60;
(v)      80.556
(%i3)   a: (v^2-0)/(2*1.5e3);
(a)      2.1631
(%i4)   t: (v-0)/a;
(t)      37.241
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2), (%i3), (%i4) Calculate v, a, and t.

Problem 3.9 A ball is thrown vertically upward at speed u from a height h above the ground.

(a) Show that the time taken by the ball to fall back to the ground is,

$$\frac{u}{g}\left(1+\sqrt{1+\frac{2hg}{u^2}}\right)$$

(b) Calculate the time taken by the ball to fall back to the ground if it is thrown vertically upward at a speed of 10 m s^{-1} from a height of 20 m above the ground.

Solution

Figure 3.4 shows the path of the ball. The y coordinate is positive upward and the origin is at $y = 0$. With respect to the coordinate and the origin, the ground level is $y = -h$. The acceleration of gravity is $a = -g$.

The ball is thrown upward with velocity u from the origin, it moves upward, stops momentarily, and falls to the ground level $y = -h$. The displacement of the ball is $-h$ and we write,

$$y = ut + \frac{1}{2}at^2,$$
$$-h = ut - \frac{1}{2}gt^2,$$
$$t^2 - \frac{2u}{g}t - \frac{2h}{g} = 0.$$

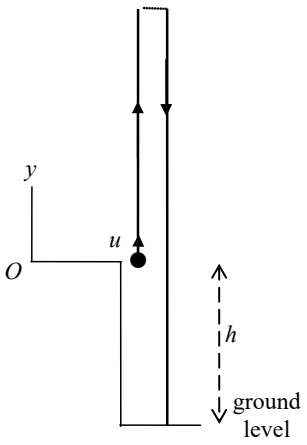


Fig. 3.4 Path of the ball, Problem 3.9

The last equation is a quadratic one and its solution is the flight time,

$$t = \frac{\frac{2u}{g} + \sqrt{\frac{4u^2}{g^2} + 4\left(\frac{2h}{g}\right)}}{2} = \frac{u}{g} + \sqrt{\frac{u^2}{g^2} + \frac{2h}{g}}$$
$$= \frac{u}{g}\left(1 + \sqrt{1 + \frac{2hg}{u^2}}\right).$$

The solution of $ax^2 + bx + c = 0$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is applied.

- wxMaxima codes:

```
(%i1) solve(-h=u*t-1/2*g*t^2, t);
(%o1) [t=-(sqrt(u^2+2*g*h)-u)/g,t=(sqrt(u^2+2*g*h)+u)/g]
```

Comments on the codes:

(%i1) Solve $-h = ut - \frac{1}{2}gt^2$ for t.
(%o1) The solutions.

(b) Inserting the given numerical values, the time for the ball to fall to the ground is,

$$t = \frac{u}{g}\left(1 + \sqrt{1 + \frac{2hg}{u^2}}\right) = \frac{10\,\mathrm{m\,s^{-1}}}{9.8\,\mathrm{m\,s^{-2}}}\left(1 + \sqrt{1 + \frac{2(20\,\mathrm{m})(9.8\,\mathrm{m\,s^{-2}})}{(10\,\mathrm{m\,s^{-1}})^2}}\right)$$

$= 3.3\,\text{s}.$

- wxMaxima codes:

```
(%i5)    fpprintprec:5; ratprint: false; h:20; g:9.8; u:10;
(fpprintprec)        5
(ratprint)  false
(h)       20
(g)       9.8
(u)       10
(%i7)     solve(-h=u*t-1/2*g*t^2, t)$ float(%);
(%o7)     [t=-1.243,t=3.2838]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, and assign values of h, g, and u.
(%i7) Solve $-h = ut - \frac{1}{2}gt^2$ for t.
(%o7) The solutions.

Problem 3.10 A bead can move freely along a wire AB on a vertical circle of radius 1.0 m (Fig. 3.5). The bead is released from rest at the highest point A. Determine,

(a) the acceleration of the bead.
(b) its speed at point B.
(c) the time taken by the bead to reach point B.

Solution

(a) Figure 3.6 shows the bead and the related geometry. AC is the diameter of the circle. From knowledge of geometry, the angle $\angle ABC$ is a right angle. The acceleration of gravity points downward.

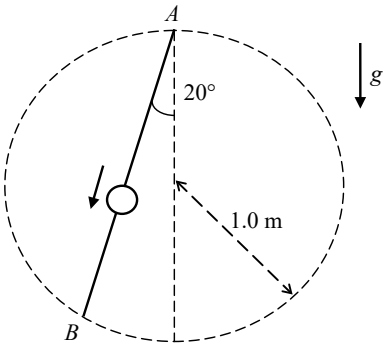


Fig. 3.5 A bead moving along a wire, Problem 3.10

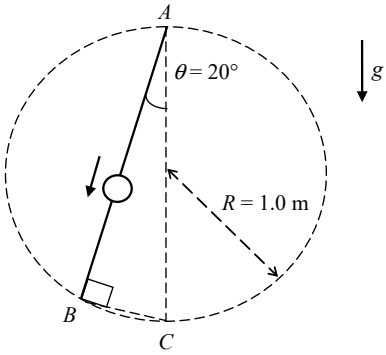


Fig. 3.6 Analysis of the bead motion, Problem 3.10

The acceleration of the bead is,

$$a = g\cos\theta = (9.8\,\text{m s}^{-2})\cos 20^\circ = 9.2\,\text{m s}^{-2}.$$

(b) The length of the wire is

$$AB = s = 2R\cos\theta = 2(1.0\,\text{m})\cos 20^\circ = 1.9\,\text{m}.$$

At point B, the speed of the bead is calculated as follows,

$$v^2 = u^2 + 2as = 0 + 2(g\cos\theta)(2R\cos\theta),$$
$$v = 2\sqrt{gR}\cos\theta = 2\sqrt{(9.8\,\text{m s}^{-2})(1.0\,\text{m})}\cos 20^\circ = 5.9\,\text{m s}^{-1}.$$

(c) The time taken for the bead to reach point B is,

$$t = \frac{v-u}{a} = \frac{2\sqrt{gR}\cos\theta}{g\cos\theta} = 2\sqrt{\frac{R}{g}} = 2\sqrt{\frac{1.0\,\text{m}}{9.8\,\text{m s}^{-2}}} = 0.64\,\text{s}.$$

- wxMaxima codes:

```
(%i5)   fpprintprec:5; g:9.8; theta:float(20/180*%pi);
R:1; u:0;
(fpprintprec)       5
(g)     9.8
(theta)  0.34907
(R)     1
(u)     0
(%i6)    a: g*cos(theta);
(a)     9.209
```

```
(%i7)   s: 2*R*cos(theta);
(s)    1.8794
(%i8)   v: 2*sqrt(g*R)*cos(theta);
(v)    5.8834
(%i9)   t: 2*sqrt(R/g);
(t)    0.63888
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of g, θ, R, and u.
(%i6) Calculate a. Part (a).
(%i7) Calculate s.
(%i8) Calculate v. Part (b).
(%i9) Calculate t. Part (c).

Problem 3.11 When the brake of a vehicle is applied, the speed of the vehicle decreased from 100 to 80 km h^{-1} and the distance traversed is 90 m. What are the deceleration and the time of the deceleration?

Solution

The deceleration of the vehicle is,

$$a = \frac{v^2 - u^2}{2s} = \frac{\left(\frac{80\times10^3\,\text{m}}{3600\,\text{s}}\right)^2 - \left(\frac{100\times10^3\,\text{m}}{3600\,\text{s}}\right)^2}{2(90\,\text{m})} = -1.54\,\text{m s}^{-2}.$$

The time of the deceleration is,

$$t = \frac{v - u}{a} = \frac{\left(\frac{80\times10^3\,\text{m}}{3600\,\text{s}}\right) - \left(\frac{100\times10^3\,\text{m}}{3600\,\text{s}}\right)}{-1.54\,\text{m s}^{-2}} = 3.6\,\text{s}.$$

- wxMaxima codes:

```
(%i4)   fpprintprec:5; u:100e3/3600; v:80e3/3600; s:90;
(fpprintprec)       5
(u)      27.778
(v)      22.222
(s)      90
(%i5)     a: (v^2-u^2)/(2*s);
(a)      -1.5432
(%i6)     t: (v-u)/a;
(t)      3.6
```

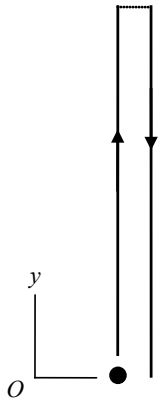


Fig. 3.7 A stone thrown vertically upward, Problem 3.12

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of *u*, *v*, and *s*.
(%5), (%i6) Calculate *a* and *t*.

Problem 3.12 A stone is thrown vertically upward at velocity of 24 m s^{-1}. Plot the curves of displacement, velocity, and acceleration against time for 0 – 4.9 s.

Solution

Figure 3.7 shows the path of the stone and the reference axis. The *y* axis is positive upward and the origin is the point of throw. The acceleration of gravity is $a = -g = -9.8$ m s^{-2}.

The displacement of the stone is,

$$y = ut + \frac{1}{2}at^2 = ut - \frac{1}{2}gt^2 = 24t - 4.9t^2.$$

The velocity of the stone is,

$$v = u + at = u - gt = 24 - 9.8t.$$

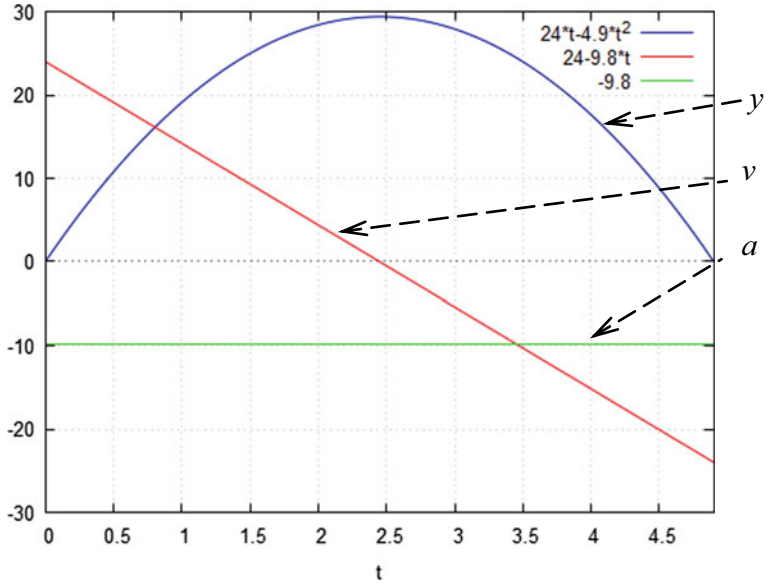
The acceleration of the stone is,

$$a = -g = -9.8\,\text{m s}^{-2}.$$

The curves of displacement, velocity, and acceleration against time for 0–4.9 s are plotted by wxMaxima.

- wxMaxima codes:

```
(%i3)   fpprintprec:5; g:9.8; u:24;
(fpprintprec)       5
(g)     9.8
(u)     24
(%i6)   y: u*t-0.5*g*t^2; v: u-g*t; a: -g;
(y)     24*t-4.9*t^2
(v)     24-9.8*t
(a)     -9.8
(%i7)   wxplot2d([y,v,a],[t,0,4.9], grid2d);
```



Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of g and u.
(%i6) Assign y, v, and a in terms of known values and time t.
(%i7) Plot y, v, and a against t for $0 \leq t \leq 4.9$ s.

Problem 3.13 A marble is thrown vertically downward at velocity of 24 m s^{-1} from a tall building. Plot the curves of displacement, velocity, and acceleration against time for 0 – 2.5 s.

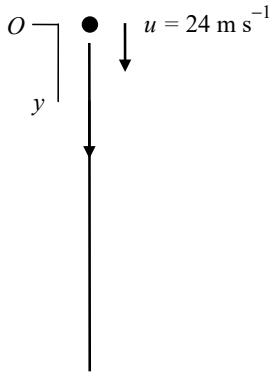


Fig. 3.8 A marble thrown vertically downward, Problem 3.13

Solution

Figure 3.8 shows the marble path, the reference y axis, and the origin. The y axis is downward positive, the origin is the point of throw, and the acceleration of gravity is $a = g = 9.8\ \text{m s}^{-2}$.

The displacement of the marble is,

$$y = ut + \frac{1}{2}at^2 = ut + \frac{1}{2}gt^2 = 24t + 4.9t^2.$$

The velocity of the marble is,

$$v = u + at = u + gt = 24 + 9.8t.$$

The acceleration of the marble is,

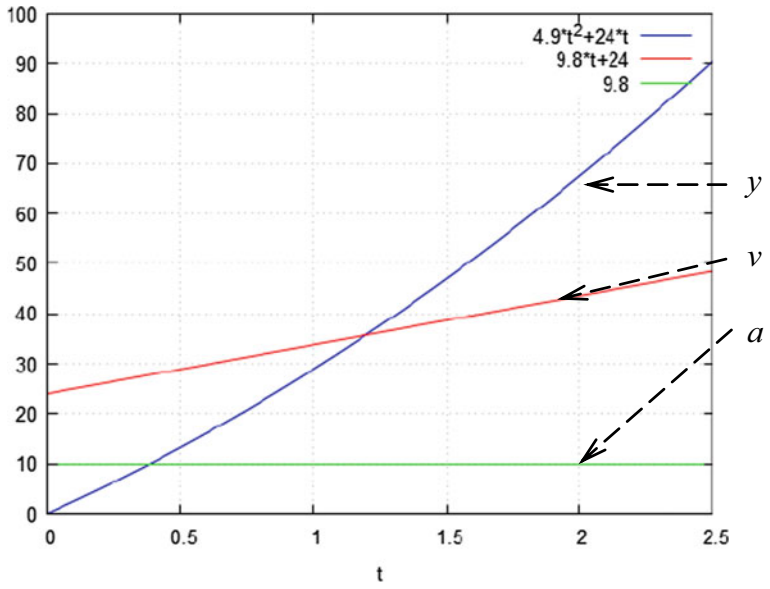
$$a = g = 9.8\,\text{m s}^{-2}.$$

The curves of displacement, velocity, and acceleration against time for $0 \leq t \leq 2.5$ s are plotted by wxMaxima.

- wxMaxima codes:

```
(%i3)   fpprintprec:5; g:9.8; u:24;
(fpprintprec)       5
(g)     9.8
(u)     24
(%i6)   y: u*t+0.5*g*t^2; v: u+g*t; a: g;
(y)   4.9*t^2+24*t
(v)   9.8*t+24
```

```
(a)    9.8
(%i7)  wxplot2d([y,v,a],[t,0,2.5], grid2d);
```



Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of g and u.
(%i6) Assign y, v, and a in terms of known values and time t.
(%i7) Plot y, v, and a against t for $0 \leq t \leq 2.5$ s.

Problem 3.14 A stone is thrown vertically upward at velocity of 24 m s^{-1}, Fig. 3.9.

(a) What is the acceleration of the stone at point A, B, and C?
(b) What is the velocity of the stone at point A, B, and C?

Solution

(a) The acceleration of the stone at A, B, and C is -9.8 m s^{-2}. This is a motion under a constant acceleration of -9.8 m s^{-2}. The acceleration of the stone at B is not zero even though the stone stops momentarily there.
(b) The velocity of the stone at A, B, and C are 24 m s^{-1}, 0, and -24 m s^{-1}, respectively.

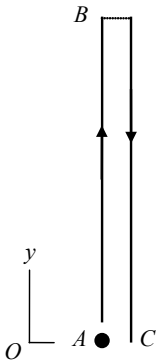


Fig. 3.9 A stone thrown vertically upward, Problem 3.14

Problem 3.15 Case *A*: A stone of 1 kg is dropped from rest from a tall building until it hits the ground.

Case *B*: The same stone is thrown downward with an initial velocity of 24 m s^{-1} from the same building.

Case *C*: A stone of 2 kg is thrown downward with an initial velocity of 24 m s^{-1} from the same building.

Figure 3.10 illustrates the three cases.

(a) What is the acceleration of the stone in each case?
(b) Compare the speeds the stones as they hit the ground.
(c) Compare the time of flight of the stones.
(d) Compare the increase in velocity of the stones.

Solution

(a) The acceleration of the stone in each case is 9.8 m s^{-2}, in downward direction.

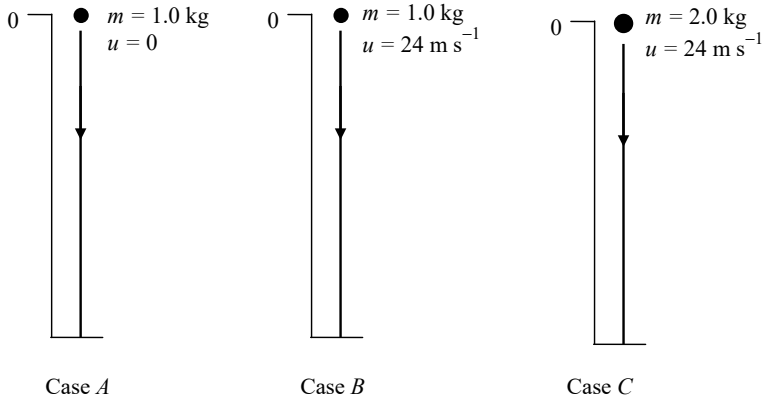


Fig. 3.10 Three cases of Problem 3.15

(b) The stones hit the ground at the same speed in Case *B* and *C*. The different stone masses do not affect their accelerations. The acceleration is still 9.8 m s^{-2} and hence their final speeds are the same. For Case *A*, the speed the stone hitting the ground is smaller than those of Case *B* and *C*.
(c) The time of flight of Case *B* and *C* is shorter than that of Case *A*.
(d) The increase in speed of Case *A* is larger than those of Case *B* and *C*.

Problem 3.16 Verify Problem 3.15 by setting the height of the building to be 10 m and doing the calculation.

Solution

The speeds at which the stones hit the ground are,

Case *A*:

$$\begin{aligned} & v_A^2 - u_A^2 = 2gs \\ & \Rightarrow v_A^2 - 0^2 = 2(9.8\,\text{m s}^{-2})(10\,\text{m}) \\ & \Rightarrow v_A = 14\,\text{m s}^{-1}. \end{aligned}$$

Case *B* and *C*:

$$\begin{aligned} & v_{BC}^2 - u_{BC}^2 = 2gs \\ & \Rightarrow v_{BC}^2 - (24\,\text{m s}^{-1})^2 = 2(9.8\,\text{m s}^{-2})(10\,\text{m}) \\ & \Rightarrow v_{BC} = 27.8\,\text{m s}^{-1}. \end{aligned}$$

The times of the flights are,

Case *A*:

$$t_A = (v_A - u_A)/g = (14\,\text{m s}^{-1} - 0)/9.8\,\text{m s}^{-2} = 1.4\,\text{s}.$$

Case *B* and *C*:

$$t_{BC} = (v_{BC} - u_{BC})/g = (27.8\,\text{m s}^{-1} - 24\,\text{m s}^{-1})/9.8\,\text{m s}^{-2} = 0.39\,\text{s}.$$

The increases in the speeds are,

Case *A*: $v_A - u_A = 14\,\text{m s}^{-1} - 0 = 14\,\text{m s}^{-1}$.

Case *B* and *C*: $v_{BC} - u_{BC} = 27.8\,\text{m s}^{-1} - 24\,\text{m s}^{-1} = 3.8\,\text{m s}^{-1}$.

- wxMaxima codes:

```
(%i4)     fpprintprec: 5; ratprint: false; g:9.8; s:10;
(fpprintprec)          5
(ratprint)   false
(g)       9.8
(s)       10
(%i5)     uA: 0;
(uA)      0
(%i6)     solve(vA^2-uA^2 = 2*g*s, vA); /* speed, case A */
(%o6)     [vA=-14,vA=14]
(%i7)     vA: 14;
(vA)      14
(%i8)     uBC: 24;
(uBC)     24
(%i10)      solve(vBC^2-uBC^2 = 2*g*s, vBC); float(%); /*
speed, case B and C */
(%o9)     [vBC=-2*sqrt(193),vBC=2*sqrt(193)]
(%o10)    [vBC=-27.785,vBC=27.785]
(%i11)    vBC: 27.785;
(vBC)     27.785
(%i12)    tA: (vA-uA)/g; /* flight time, case A */;
(tA)      1.4286
(%i13)    tBC:(vBC-uBC)/g; /* flight time, case B and C */;
(tBC)     0.38622
(%i14)    vA-uA; /* increase in speed, case A */
(%o14)    14
(%i15)    vBC-uBC; /* increase in speed, case B and C */
(%o15)    3.785
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of g and s.
(%i5) Assign value of u_A.
(%i6) Solve $v_A^2 - u_A^2 = 2gs$ for v_A. Part (b).
(%i7), (%i8) Assign values of v_A and u_{BC}.
(%i10) Solve $v_{BC}^2 - u_{BC}^2 = 2gs$ for v_{BC}. Part (b).
(%i11) Assign value of v_{BC}.
(%i12), (%i13) Calculate t_A and t_{BC}. Part (c).
(%i14), (%i15) Calculate $v_A - u_A$ and $v_{BC} - u_{BC}$. Part (d).

Problem 3.17 Stone A is released from rest at a certain height. Two seconds after the release, stone B is released from rest at the same height, Fig. 3.11.

(a) How does the difference in speed of A and B vary with time?
(b) How does the separation distance of A and B vary with time?
(c) What is the time difference they hit the ground?

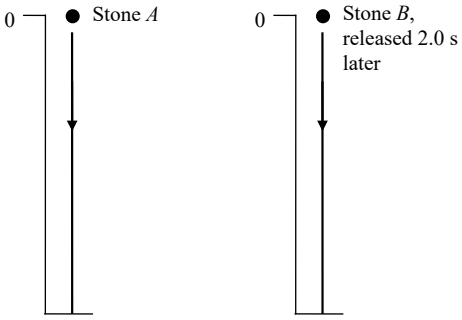


Fig. 3.11 Two stones released from the same height at different times, Problem 3.17

Solution

(a) For a free fall, i.e. a motion with a constant acceleration, speed increases the same amount in the same time interval. Here, it increases 9.8 m s^{-1} every second for both stones. At 2.0 s, the speed of stone A is $gt = (9.8\ \text{m s}^{-2})(2.0\ \text{s}) = 19.6\ \text{m s}^{-1}$, while that of B is zero. Their speed difference is 19.6 m s^{-1}. This difference is sustained after the 2.0 s, because their speeds increase at a constant rate g. After 2.0 s, their difference in speed stays at 19.6 m s^{-1} as time passes.

(b) Because their difference in speed stays at 19.6 m s^{-1}, their separation distance increases 19.6 m every second. This means that the separation distance increases linearly with time. After 2.0 s, the separation distance of stone A and B increases by 19.6 m every second.

(c) Both stones travel the same distance and take the same amount of time to hit the ground. Stone A hits the ground first, followed by stone B two seconds later.

Problem 3.18 Verify Problem 3.17 using the specific physical values and calculations.

Solution

Figure 3.12 shows two stones and the height is set at 40 m. Take the y axis as positive downward and the origin at O.

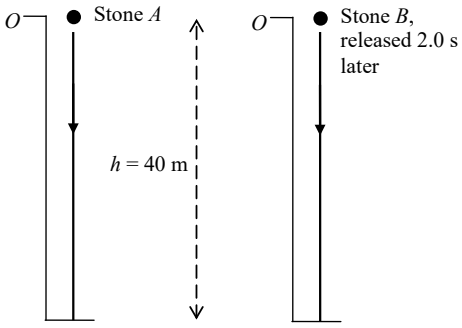


Fig. 3.12 Two stones released from height of 40 m at different times, Problem 3.18

(a) The velocities of the stones A and B are,

$$v_A = gt = 9.8t,$$
$$v_B = g(t-2) = 9.8(t-2).$$

The difference in velocity is, at $t = 2$ s onwards,

$$\begin{aligned} v_{diff} &= v_A - v_B = gt - g(t-2) = 2g \\ &= 2\,\text{s} \times (9.8\,\text{m/s}^2) \\ &= 19.6\,\text{m s}^{-1}. \end{aligned}$$

This indicates that the difference in velocity remains constant as time passes.

(b) Distances travelled by the stones A and B are,

$$y_A = \frac{1}{2}gt^2,$$
$$y_B = \frac{1}{2}g(t-2)^2.$$

The separation is, at $t = 2.0$ s onwards,

$$\begin{aligned} y_{diff} &= y_A - y_B = \frac{1}{2}gt^2 - \frac{1}{2}g(t-2)^2 = 2gt - 2g \\ &= 2(9.8)t - 2(9.8) \\ &= 19.6t - 19.6. \end{aligned}$$

This means that the separation increases with time.

(c) The time for the stone A to hit the ground is calculated as follows,

$$h = \frac{1}{2}gt_A^2,$$
$$t_A = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(40\,\text{m})}{9.8\,\text{m/s}^2}} = 2.9\,\text{s}.$$

The time for the stone B to hit the ground,

$$\begin{aligned} h &= \frac{1}{2}g(t_B - 2)^2 \\ &\Rightarrow \frac{2h}{g} = t_B^2 - 4t_B + 4 \\ &\Rightarrow t_B^2 - 4t_B + 4 - \frac{2h}{g} = 0, \end{aligned}$$

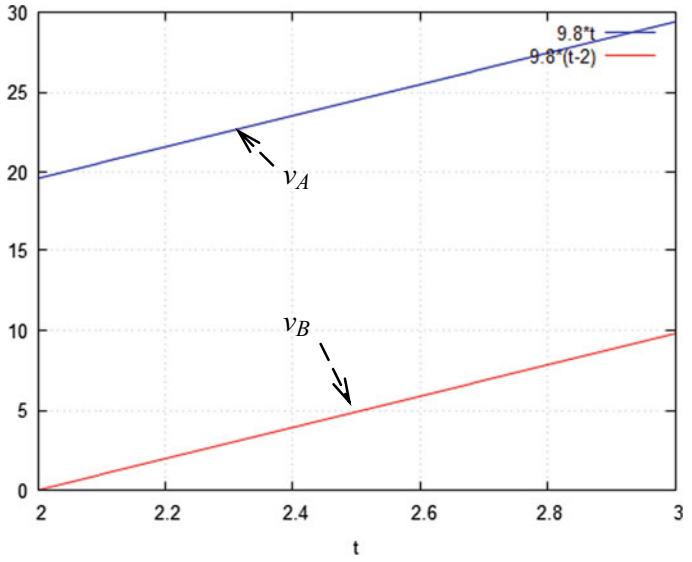
$$t_B = \frac{4 + \sqrt{16 - 4\left(4 - \frac{2h}{g}\right)}}{2} = 2 + \sqrt{\frac{2h}{g}} = 2.0\,\text{s} + \sqrt{\frac{2(40\,\text{m})}{9.8\,\text{m/s}^2}}$$
$$= 4.9\,\text{s}.$$

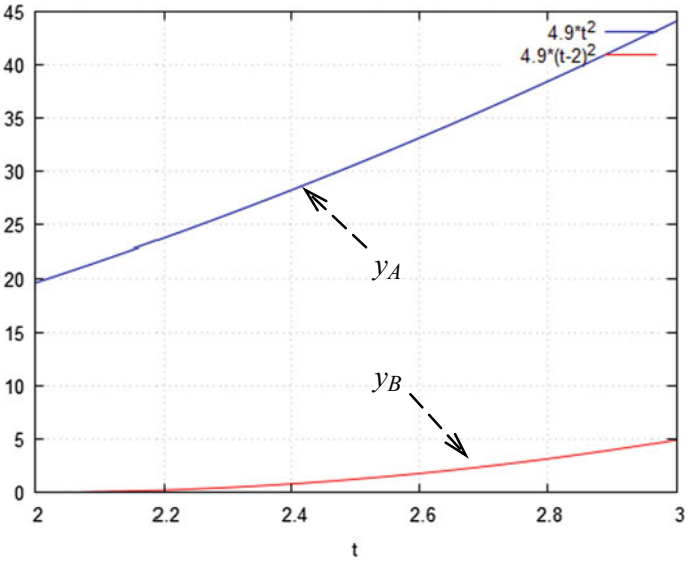
The time difference is,

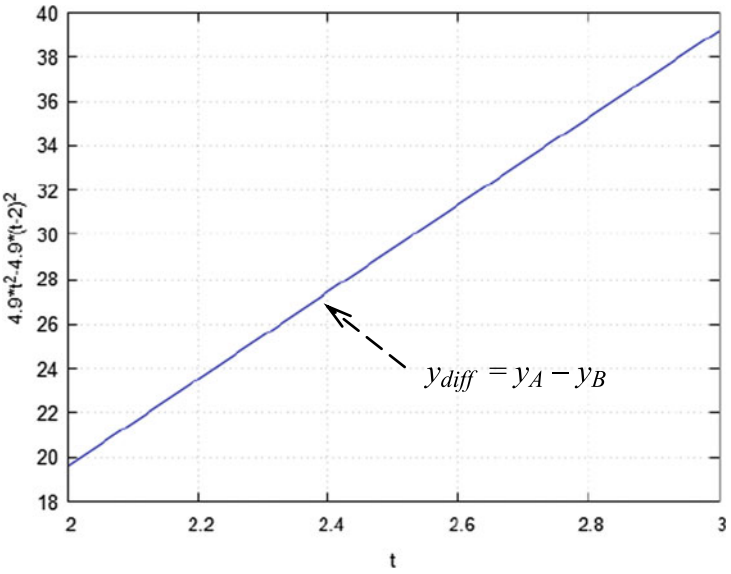
$$t_{diff} = t_B - t_A = 2 + \sqrt{\frac{2h}{g}} - \sqrt{\frac{2h}{g}} = 2.0\,\text{s}.$$

- wxMaxima codes:

```
(%i2)     fpprintprec:5; ratprint:false;
(fpprintprec)         5
(ratprint)  false
(%i3)     vA: 9.8*t;
(vA)      9.8*t
(%i4)     vB: 9.8*(t-2);
(vB)      9.8*(t-2)
(%i5)     wxplot2d([vA,vB], [t,2,3], grid2d);
(%i6)     vdiff: vA-vB;
(vdiff)    9.8*t-9.8*(t-2)
(%i7)     float(ratsimp(vdiff));
(%o7)     19.6
(%i8)     yA: 0.5*9.8*t^2;
(yA)      4.9*t^2
(%i9)     yB: 0.5*9.8*(t-2)^2;
(yB)      4.9*(t-2)^2
(%i10)    wxplot2d([yA,yB], [t,2,3]);
(%i11)    ydiff: yA-yB;
(ydiff)    4.9*t^2-4.9*(t-2)^2
(%i12)    float(ratsimp(ydiff));
(%o12)    0.2*(98.0*t-98.0)
(%i13)    wxplot2d(ydiff, [t,2,3]);
(%i14)    h: 40;
(h)       40
(%i16)    solve(h=0.5*9.8*tA^2, tA)$ float(%);
(%o16)    [tA=-2.8571,tA=2.8571]
(%i17)    tA: rhs(%[2]);
(tA)      2.8571
(%i19)    solve(h=0.5*9.8*(tB-2)^2, tB)$ float(%);
(%o19)    [tB=-0.85714,tB=4.8571]
(%i20)    tB: rhs(%[2]);
(tB)      4.8571
(%i21)    tdiff: tB-tA;
(tdiff)    2.0
```


9.8*t
9.8*(t-2)
v_A
v_B
30
25
20
15
10
5
0
2
2.2
2.4
2.6
2.8
3
t


4.9*t^2
4.9*(t-2)^2
y_A
y_B
45
40
35
30
25
20
15
10
5
0
2
2.2
2.4
2.6
2.8
3
t



Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3), (%i4) Define speed of stone A, v_{A} and speed of stone B, v_{B}.
(%i5) Plot v_{A} and v_{B} against t for $2 \leq t \leq 3$ s. Part (a).
(%i6), (%i7) Calculate difference in speed, v_{diff}. Part (a).
(%i8), (%i9) Define distance, y_{A} and distance, y_{B}.
(%i10) Plot y_{A} and y_{B} against t for $2 \leq t \leq 3$ s. Part (b).
(%i11), (%i12) Calculate separation, y_{diff}. Part (b).
(%i13) Plot separation, y_{diff} against, t for $2 \leq t \leq 3$ s. Part (b).
(%i14) Assign $h = 40$ m. Part (c).
(%i16), (%i19) Solve $h = \frac{1}{2}gt_A^2$ for t_{A} and $h = \frac{1}{2}g(t_B - 2)^2$ for t_{B}. Part (c).
(%i21) Calculate t_{diff}. Part (c).

3.3 Summary

- Physical motion of an object can be described in terms of the displacement, velocity, and acceleration. Velocity is time rate of change of displacement while acceleration is time rate of change of velocity.
- Kinematic equations for a particle moving with constant acceleration, a, in one dimension are:

$$v = v_0 + at,$$

$$x - x_0 = \frac{1}{2}(v_0 + v)t,$$
$$x - x_0 = v_0 t + \frac{1}{2}at^2,$$
$$v^2 = v_0^2 + 2a(x - x_0).$$

- A free fall near the earth surface is a motion with constant acceleration of $g = 9.8\ \mathrm{m\ s^{-2}}$.

3.4 Exercises

Exercise 3.1 A car accelerating from rest moves a distance of 20 m in 4.0 s along a straight road. Calculate the acceleration of the car.

(Answer: 2.5 m $\mathrm{s^{-2}}$)

Exercise 3.2 An object is released from rest at a height in a tall building. What is its average velocity during the first second of time?

(Answer: 4.9 m $\mathrm{s^{-1}}$)

Exercise 3.3 An object is thrown straight up into the air. The positive direction is straight up. Sketch the velocity, v, against time, t, curve of the object.

(Answer: See v against t curve of Problem 3.12)

Exercise 3.4 A stone is thrown straight up from ground level with a speed of 30 m $\mathrm{s^{-1}}$. Calculate its distance above the ground level 2.0 s after the throw.

(Answer: 40 m)

Exercise 3.5 A marble is released from rest at a height in a tall building. How far will it fall during the first and the next seconds of time?

(Answer: 4.9, 19.6 m)

Chapter 4
Motion in Two Dimensions: Projectile Motion

4.1 Basic Concepts and Formulae

(1) For a particle moving in two dimensions (2D), the position, $\boldsymbol{r}$, velocity, $\boldsymbol{v}$, and acceleration, $\boldsymbol{a}$, of the particle are,

$$\boldsymbol{r} = x\,\mathbf{i} + y\,\mathbf{j}, \tag{4.1}$$

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = v_x\mathbf{i} + v_y\mathbf{j}, \tag{4.2}$$

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} = a_x\mathbf{i} + a_y\mathbf{j}. \tag{4.3}$$

(2) Projectile motion: A stone thrown at an angle from the ground is an example of a projectile in 2D. The projectile follows a parabolic path. For a particle launched from the origin (0, 0) at angle of θ_0 and initial speed of v_0, the kinematic equations of the particle are obtained by substituting acceleration in the x direction, $a_x = 0$ and acceleration in the y direction, $a_y = -g = -9.8\ \text{m s}^{-2}$ (Table 4.1, Fig. 4.1). In the x direction, the motion is of constant velocity of $v_0 \cos\theta_0$, while in the y direction it is a motion with constant acceleration, $a_y = -g = -9.8\ \text{m s}^{-2}$ with an initial velocity of $v_0 \sin\theta_0$. In this case, the air friction is neglected.

(3) The maximum height of the projectile, h, and the time to this height, t_h, are,

$$h = \frac{v_0^2 \sin^2\theta_0}{2g}, \tag{4.4}$$

$$t_h = \frac{v_0 \sin\theta_0}{g}. \tag{4.5}$$

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W. M. S. Wan Hassan, *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-42678-0_4

Table 4.1 Equations of projectile motion

	x axis	y axis
Acceleration, a	$a_x = 0$	$a_y = -g = -9.8\ \mathrm{m\ s^{-2}}$
Initial velocity, v_0	$v_0 \cos\theta_0$	$v_0 \sin\theta_0$
Velocity at time t, v	$v_x = v_0 \cos\theta_0$	$v_y = v_0 \sin\theta_0 - gt$
Position at time t, (x, y)	$x = (v_0 \cos\theta_0)t$	$y = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$
Velocity at any position, v	$v_x = v_0 \cos\theta_0$	$v_y^2 = v_0^2 \sin^2\theta_0 - 2gy$

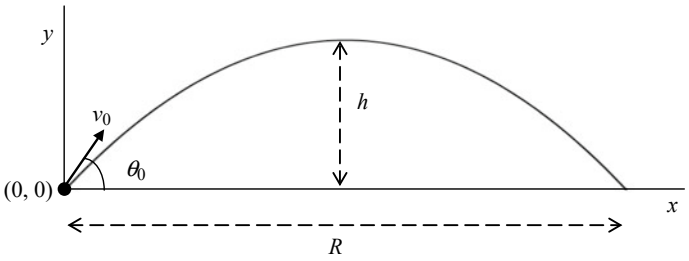


Fig. 4.1 Projectile motion in two dimensions

(4) The range of the projectile, R (horizontal distance from the initial and final points of the flight) and time of flight, t_R are,

$$R = \frac{v_0^2 \sin 2\theta_0}{g}, \tag{4.6}$$

$$t_R = 2t_h = \frac{2v_0 \sin\theta_0}{g}. \tag{4.7}$$

If v_0 is fixed, Eq. (4.6) yields that the range R is a maximum when θ_0 is 45°.

(5) The equation of the projectile trajectory is,

$$y = (\tan\theta_0)x - \frac{g}{2(v_0 \cos\theta_0)^2}x^2, \tag{4.8}$$

which is an equation of a parabola.

4.2 Problems and Solutions

Problem 4.1 The position vector of a particle is,

$$\boldsymbol{r}(t) = (5t^2 + t)\mathbf{i} + (2t + 3)\mathbf{j}\ \text{m}.$$

Calculate,

(a) the average velocity of the particle in time interval 1.0 to 3.0 s,
(b) the instantaneous velocity at 3.0 s,
(c) the acceleration at 3.0 s.

Solution

(a) First, calculate the positions of the particle at 1.0 and 3.0 s.

At $t_1 = 1.0$ s, the position of the particle is,

$$\boldsymbol{r}_1 = [5(1)^2 + 1]\,\mathbf{i} + [2(1) + 3]\,\mathbf{j}\ \text{m} = 6\,\mathbf{i} + 5\,\mathbf{j}\ \text{m}.$$

At $t_2 = 3.0$ s, the position of the particle is,

$$\boldsymbol{r}_2 = [5(3)^2 + 3]\,\mathbf{i} + [2(3) + 3]\,\mathbf{j}\ \text{m} = 48\,\mathbf{i} + 9\,\mathbf{j}\ \text{m}.$$

The average velocity of the particle is,

$$\overline{\boldsymbol{v}} = \frac{\Delta \boldsymbol{r}}{\Delta t} = \frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{t_2 - t_1} = \frac{(48 - 6)\,\mathbf{i} + (9 - 5)\,\mathbf{j}\ \text{m}}{(3 - 1)\ \text{s}} = 21\,\mathbf{i} + 2.0\,\mathbf{j}\ \text{m s}^{-1}.$$

(b) Instantaneous velocity is obtained by differentiating the position vector with respect to time,

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = (10t + 1)\,\mathbf{i} + 2\,\mathbf{j}.$$

The instantaneous velocity at $t = 3.0$ s is,

$$\boldsymbol{v} = [10(3) + 1]\,\mathbf{i} + 2\,\mathbf{j} = 31\,\mathbf{i} + 2.0\,\mathbf{i}\ \text{m s}^{-1}.$$

The magnitude of the velocity or the speed is,

$$v = \sqrt{31^2 + 2^2}\,\mathrm{m\,s^{-1}} = 31.1\,\mathrm{m\,s^{-1}}.$$

(c) The acceleration of the particle is obtained by differentiating the velocity with respect to time,

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = 10\,\mathbf{i}\,\mathrm{m\,s^{-2}}.$$

The acceleration is a constant, does not depend on time and it is $10\,\mathbf{i}\,\mathrm{m\,s^{-2}}$ at any time.

• wxMaxima codes:

```
(%i1)   fpprintprec:5;
(fpprintprec)   5
(%i2)   r(t):= [ 5*t^2+t, 2*t+3];
(%o2)   r(t):=[5*t^2+t,2*t+3]
(%i4)   r1: r(1); r2:r(3);
(r1)   [6,5]
(r2)   [48,9]
(%i5)   v_average: (r2-r1)/(3-1);
(v_average)   [21,2]
(%i6)   v: diff(r(t),t);
(v)   [10*t+1,2]
(%i7)   subst(t=3, v);
(%o7)   [31,2]
(%i9)   v_mag: sqrt(%.%); float(%);
(v_mag)   sqrt(965).
(%o9)   31.064
(%i10)   a: diff(v,t);
(a)   [10,0]
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2) Define vector $\boldsymbol{r}(t)$.
(%i4) Calculate $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.
(%i5) Calculate average velocity. Part (a).
(%i6), (%i7), (%i9) Calculate instantaneous velocity and its magnitude at 3 s. Part (b).
(%i10) Calculate $\boldsymbol{a}$. Part (c).

Problem 4.2 A marble is launched at a speed of 17 m s^{-1} at inclination angle of 58° from the horizontal. Air resistance is negligible. Calculate,

(a) the expression of the marble trajectory,
(b) the time taken to reach the maximum height, t_h,
(c) the maximum height, h,
(d) the horizontal range, R.

Solution

(a) The x–y relation of the marble trajectory is (Eq. 4.8),

$$y = (\tan\theta_0)x - \frac{g}{2(v_0 \cos\ \theta_0)^2}x^2.$$

Substituting the given numerical values, the expression for the trajectory is,

$$\begin{aligned} y &= (\tan\ 58^\circ)x - \frac{9.8}{2(17 \cos\ 58^\circ)^2}x^2 \\ &= 1.6x - 0.06x^2. \end{aligned}$$

• wxMaxima codes:

```
(%i1)   fpprintprec:5;
(fpprintprec)   5
(%i2)   tan(58*float(%pi)/180);
(%o2)   1.6003
(%i3)   9.8/(2*17^2*cos(58*float(%pi)/180)^2);
(%o3)   0.060378
```

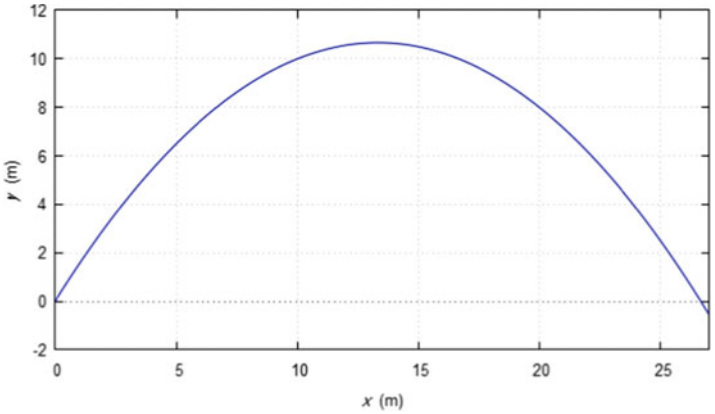
Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2), (%i3) Calculate tan 58° and $\frac{9.8}{2(17 \cos\ 58^\circ)^2}$.

The marble trajectory is visualized by plotting a curve of y against x which is parabolic in shape.

• Plot by wxMaxima:

```
(%i1)   y: 1.6*x-0.06*x^2;
(y)   1.6*x-0.06*x^2
(%i2)   wxplot2d(y, [x,0,27], grid2d, same_xy,
[xlabel,"{/Helvetica-Italic  x}(m)"],[ylabel,"{/Helvetica-
Italic y} (m)"]);
```



Comments on the codes:

(%i1) Define y in terms of x.
(%i2) Plot y against x for $0 \leq x \leq 27$ with grid.

(b) The time taken to reach the maximum height is (Eq. 4.5),

$$t_h = \frac{v_0 \sin\theta_0}{g} = \frac{17\,\text{m/s}\ \ \sin\ 58^\circ}{9.8\,\text{m/s}^2} = 1.5\,\text{s}.$$

• wxMaxima codes:

```
(%i4)      fpprintprec:5; v0:17; theta0:float(58*%pi/180);
g:9.8;
(fpprintprec)    5
(v0)    17
(theta0)    1.0123
(g)    9.8
(%i5)    th: v0*sin(theta0)/g;
(th)    1.4711
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of v_0, θ_0, and g.
(%i5) Calculate the time to maximum height, t_h.

(c) The maximum height is (Eq. 4.4),

$$h = \frac{(v_0 \sin\theta_0)^2}{2g} = \frac{(17\ \text{m/s}\ \ \sin\ 58^\circ)^2}{2(9.8\,\text{m/s}^2)} = 11\,\text{m}.$$

• wxMaxima codes:

```
(%i4)      fpprintprec:5; v0:17; theta0:float(58*%pi/180);
g:9.8;
(fpprintprec)   5
(v0)   17
(theta0)   1.0123
(g)   9.8
(%i5)   h: (v0*sin(theta0))^2/(2*g);
(h)   10.604
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of v_0, θ_0, and g.
(%i5) Calculate the maximum height, h.

(d) The range of the projectile is (Eq. 4.6),

$$R = \frac{v_0^2 \sin 2\theta_0}{g} = \frac{(17\,\text{m/s})^2 \sin\,(2 \times 58°)}{9.8\,\text{m/s}^2} = 27\,\text{m}.$$

• wxMaxima codes:

```
(%i4)      fpprintprec:5; v0:17; theta0:float(58*%pi/180);
g:9.8;
(fpprintprec)   5
(v0)   17
(theta0)   1.0123
(g)   9.8
(%i5)   R: v0^2*sin(2*theta0)/g;
(R)   26.505
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of v_0, θ, and g.
(%i5) Calculate the range, R.

Problem 4.3 The world record in long jump is a length 8.95 m. Assuming a long jump athlete as a point mass and he leaves the ground at 25° from the horizontal, determine,

(a) the speed he leaves the ground,
(b) the maximum height of the long jump.

Solution

(a) The horizontal range of a projectile is (Eq. 4.6),

$$R = \frac{v_0^2 \sin 2\theta_0}{g}.$$

The speed at which the athlete leaves the ground is,

$$v_0 = \sqrt{\frac{Rg}{\sin 2\theta_0}} = \sqrt{\frac{(8.95\,\text{m})\ (9.8\,\text{m/s}^2)}{\sin(2 \times 25^\circ)}} = 10.7\,\text{m}\,\text{s}^{-1}.$$

• wxMaxima codes:

```
(%i4)    fpprintprec:5; R:8.95; g:9.8; theta0:float(25*%pi/
180);
(fpprintprec)   5
(R)   8.95
(g)   9.8
(theta0)   0.43633
(%i5)   v0: sqrt(R*g/sin(2*theta0));
(v0)   10.7
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of R, g, and θ_0.
(%i5) Calculate the initial speed, v_0.

(b) The maximum height is (Eq. 4.4),

$$h = \frac{v_0^2 \sin^2 \theta_0}{2g} = \frac{(10.7\,\text{m/s})^2 \sin^2(25^\circ)}{2\ (9.8\,\text{m/s}^2)} = 1.04\,\text{m}.$$

• wxMaxima codes:

```
(%i4)    fpprintprec:5; v0:10.7; g:9.8; theta0:float(25*%pi/
180);
(fpprintprec)   5
(v0)   10.7
(g)   9.8
(theta0)   0.43633
(%i5)   h: v0^2*sin(theta0)^2/(2*g);
(h)   1.0433
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of v_0, g, and θ_0.
(%i5) Calculate the maximum height, h.

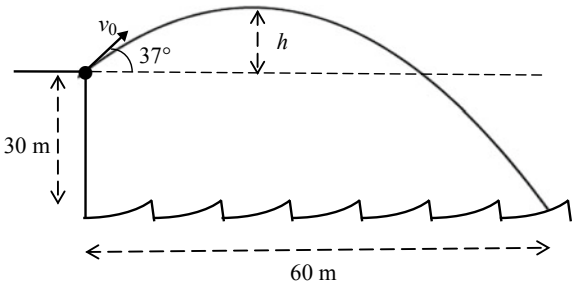


Fig. 4.2 A stone thrown from a cliff, Problem 4.4

Problem 4.4 A stone is thrown from a cliff at an inclination of 37° from the horizontal as shown in Fig. 4.2. The height of the cliff from the water level is 30 m and the stone hits the water at a distance 60 m from the cliff. Determine,

(a) the time of flight, t,
(b) the initial speed of the stone, v_0,
(c) the maximum altitude, h.

Solution

(a) The point of throw is the origin. The stone hits the water at (60 m, –30 m). If the initial speed is v_0, the displacements in the x and y directions are (Table 4.1),

$$x = (v_0 \cos\ \theta_0)t,$$
$$y = (v_0 \sin\ \theta_0)t - \frac{1}{2}gt^2.$$

Substituting the given numerical values, one gets,

$$60 = (v_0 \cos\ 37^\circ)t,$$
$$-30 = (v_0 \sin\ 37^\circ)t - 4.9t^2.$$

The two equations are solved for t. Get v_0 from the first equation and substitute it in the second equation. The result is,

$$-30 = 60 \tan 37^\circ - 4.9t^2,$$
$$t = 3.9\,\text{s}.$$

• wxMaxima codes:

```
(%i2)   fpprintprec:5; ratprint:false;
(fpprintprec)   5
(ratprint)false
(%i4)   solve(-30=60*tan(37*%pi/180)-4.9*t^2, t)$ float(%);
```

```
(%o4)    [t=-3.9179,t=3.9179]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve equation $-30 = 60 \tan 37^\circ - 4.9t^2$ for t.

(b) Using $x = (v_0 \cos\theta_0)t$ (Table 4.1), the initial speed of the stone is,

$$v_0 = \frac{x}{t\cos\theta_0} = \frac{60\,\text{m}}{(3.9\,\text{s})\cos 37^\circ} = 19\ \text{m s}^{-1}.$$

• wxMaxima codes:

```
(%i4)     fpprintprec:5; x:60; t:3.92; theta0:float(37*%pi/
180);
(fpprintprec)    5
(x)    60
(t)    3.92
(theta0)    0.64577
(%i5)    v0: x/(t*cos(theta0));
(v0)    19.165
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of x, t, and θ_0.
(%i5) Calculate the initial speed, v_0.

(c) The maximum altitude is (Eq. 4.4),

$$h = \frac{(v_0 \sin\theta_0)^2}{2g} = \frac{(19.2\,\text{m/s}\ \ \sin\ \ 37^\circ)^2}{2(9.8\,\text{m/s}^2)} = 6.8\,\text{m}.$$

• wxMaxima codes:

```
(%i4)     fpprintprec:5; v0:19.2; theta0:float(37*%pi/180);
g:9.8;
(fpprintprec)    5
(v0)    19.2
(theta0)    0.64577
(g)    9.8
(%i5)    h: (v0*sin(theta0))^2/(2*g);
(h)    6.812
```

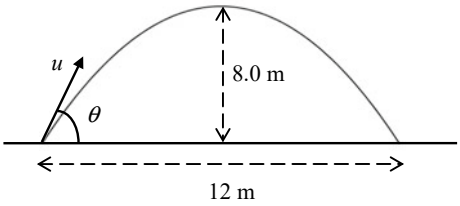


Fig. 4.3 Water coming out of a hose, Problem 4.5

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of v_0, θ_0, and g.
(%i5) Calculate the maximum altitude, h.

Problem 4.5 The range and maximum height of water from a hose are 12 m and 8.0 m, respectively, as shown in Fig. 4.3. What is the speed of the water coming out of the hose? What is the inclination angle?

Solution
Assume the water jet as a point particle and neglect air resistance. The range is (Eq. 4.6),

$$R = \frac{u^2 \sin\ 2\theta}{g} = \frac{2u^2 \sin\ \theta\ \cos\ \theta}{g}.$$

The maximum height is (Eq. 4.4),

$$h = \frac{u^2 \sin^2 \theta}{2g}.$$

These two equations give,

$$\frac{4h}{R} = \frac{4u^2 \sin^2 \theta}{2g} \cdot \frac{g}{2u^2 \sin\ \theta\ \cos\ \theta} = \tan\ \theta.$$

Thus, the inclination angle is,

$$\theta = \tan^{-1}\left(\frac{4h}{R}\right) = \tan^{-1}\frac{4\ (8.0\,\mathrm{m})}{12\,\mathrm{m}} = 1.2\,\mathrm{rad} = 69^\circ,$$

and the initial speed of the water is (Eq. 4.4),

$$u = \frac{\sqrt{2gh}}{\sin\theta} = \frac{\sqrt{2\ (9.8\,\mathrm{m\,s^{-2}})\ (8.0\,\mathrm{m})}}{\sin 69^\circ} = 13\,\mathrm{m\,s^{-1}}.$$

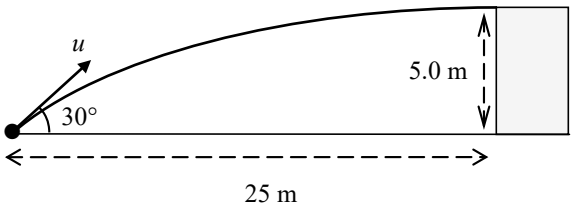


Fig. 4.4 A ball thrown to a building, Problem 4.6

• wxMaxima codes:

```
(%i4)   fpprintprec:5; h:8; R:12; g:9.8;
(fpprintprec)   5
(h)   8
(R)   12
(g)   9.8
(%i5)   theta: float(atan(4*h/R));
(theta)   1.212
(%i6)   theta_degree: float(theta*180/%pi);
(theta_degree)   69.444
(%i7)   u: float( sqrt(2*g*h)/sin(theta));
(u)   13.373
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of h, R, and g.
(%i5), (%i6) Calculate θ and convert the angle to degree.
(%i7) Calculate initial speed of water coming out of the hose, u.

Problem 4.6 A ball is thrown at an angle of 30° and hits a building of 5.0 m high, 25 m from the point of throw as shown in Fig. 4.4. What is the initial speed, u of the ball?

Solution
The initial speed of the ball is u. The velocity of the ball in the x direction is $u \cos 30°$ any time during the flight (Table 4.1 row 3). If the time of flight is t then,

$$t = \frac{x}{u \cos\ \theta} = \frac{25\,\text{m}}{u \cos\ 30°}.$$

The initial velocity of the ball in the y direction is $u \sin 30°$ and the acceleration is $a = -g$ in this direction. Therefore, the displacement in the y direction at time t is,

$$y = (u \sin\ 30°)t - \frac{1}{2}gt^2,$$

$$5.0\,\text{m} = (u\ \sin\ 30^\circ)\frac{25\,\text{m}}{u\ \cos\ 30^\circ} - \frac{1}{2}g\frac{(25\,\text{m})^2}{u^2\cos^2 30^\circ}$$
$$= (25\,\text{m})\ \tan\ 30^\circ - \frac{1}{2}(9.8\,\text{m/s}^2)\frac{(25\,\text{m})^2}{u^2\cos^2 30^\circ}.$$

Solving this equation for u, gives the initial speed of the ball as,

$$u = 21\,\text{m}\,\text{s}^{-1}.$$

• wxMaxima codes:

```
(%i3)       fpprintprec:5; ratprint:false; theta:float(30/
180*%pi);
(fpprintprec)    5
(ratprint)    false
(theta)    0.5236
(%i5)                    solve(5=25*tan(theta)-0.5*9.8*25^2/
(u^2*cos(theta)^2), u)$ float(%);
(%o5)    [u=-20.805,u=20.805]
```

Comments on the codes:

(%i3) Set floating point print precision to 5, internal rational number print to false and assign value of angle, θ.
(%i5) Solve equation $5 = 25\ \tan 30^\circ - \frac{1}{2}(9.8)\frac{25^2}{u^2\cos^2 30^\circ}$ for u.

Alternative solution: Equation of the ball trajectory is (Eq. 4.8),

$$y = (\tan\ \theta)x - \frac{g}{2(u\ \cos\ \theta)^2}x^2.$$

Substituting the known values $y = 5.0$ m, $x = 25$ m, $\theta = 30^\circ$, $g = 9.8$ m s^{-2}, gives,

$$5.0\,\text{m} = (\tan\ 30^\circ)(25\,\text{m}) - \frac{(9.8\,\text{m/s}^2)}{2(u\ \cos\ 30^\circ)^2}(25\,\text{m})^2.$$

This is the same equation as in the previous solution and gives $u = 21$ m s^{-1} as the initial speed of the ball.

Problem 4.7 An air plane is flying at a speed of $u = 300$ m s^{-1} and at height of $s = 1000$ m (Fig. 4.5). A bowling ball falls from the plane. Determine,

(a) the time, t, taken by the ball to reach the ground,
(b) the distance, R, the ball hits the ground,
(c) the speed, v, of the ball when it hits the ground.

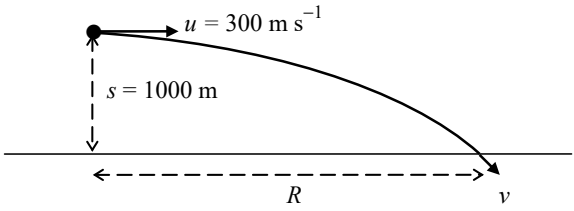


Fig. 4.5 A bowling ball dropped from a plane, Problem 4. 7

Solution

(a) The time, t, taken by the ball to reach the ground is calculated by considering the motion of the ball in the y direction. In the y direction,

$$s = u_y t + \frac{1}{2} g t^2, \quad u_y = 0 \quad \Rightarrow \quad s = \frac{1}{2} g t^2,$$

$$t = \sqrt{\frac{2s}{g}} = \sqrt{\frac{2\ (1000\,\mathrm{m})}{9.8\,\mathrm{m\,s^{-2}}}} = 14.3\ \mathrm{s}.$$

(b) The distance, R, is obtained by considering the motion of the ball in the x direction. In the x direction,

$$R = ut = (300\,\mathrm{m\,s^{-1}})(14.3\,\mathrm{s}) = 4286\,\mathrm{m}.$$

(c) The speed, v, of the ball as it hits the ground is calculated as follows,

$$v_y = gt = (9.8\,\mathrm{m\,s^{-2}})(14.3\,\mathrm{s}) = 140\,\mathrm{m\,s^{-1}},$$
$$v_x = u = 300\,\mathrm{m\,s^{-1}},$$
$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(300\,\mathrm{m\,s^{-1}})^2 + (140\,\mathrm{m\,s^{-1}})^2} = 331\,\mathrm{m\,s^{-1}}.$$

• wxMaxima codes:

```
(%i4)   fpprintprec:5; u:300; s:1000; g:9.8;
(fpprintprec)   5
(u)   300
(s)   1000
(g)   9.8
(%i5)   t: sqrt(2*s/g);
(t)   14.286
(%i6)   R: u*t;
(R)   4285.7
(%i7)   vy: g*t;
(vy)   140.0
(%i8)   vx: u;
(vx)   300
```

```
(%i9)   v: sqrt(vx^2 + vy^2);
(v)   331.06
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of u, s, and g.
(%i5), (%i6) Calculate t and R, parts (a) and (b).
(%i7), (%i8), (%i9) Calculate v_y, v_x, and v, part (c).

Problem 4.8 A motorcycle jumps across a 40 m wide and 100 m deep river from a ramp at an angle of 25° (Fig. 4.6). The point of landing on the opposite side of the river is 15 m lower in height than the ramp. Calculate the speed of the jump.

Solution

The trajectory of the motorcycle is (Eq. 4.8),

$$y = (\tan\ \theta)x - \frac{g}{2(u\ \cos\ \theta)^2}x^2.$$

The coordinate of the ramp is (0, 0) the origin and the point of landing is (40 m, −15 m). Substituting $x = 40$ m, $y = -15$ m, $\theta = 25°$, $g = 9.8$ m s^{-2}, one gets,

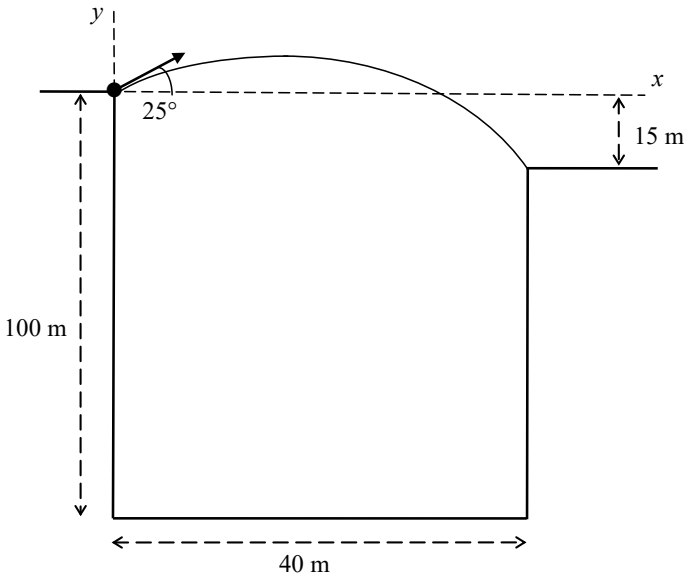


Fig. 4.6 A motorcycle jumps across a river, Problem 4.8

$$-15\,\text{m} = \tan 25^\circ\ (40\,\text{m}) - \frac{9.8\,\text{m/s}^2}{2(u\ \cos\ 25^\circ)^2}(40\,\text{m})^2.$$

The speed of the jump u is obtained from this equation as,

$$u = 17\,\text{m}\,\text{s}^{-1}.$$

• wxMaxima codes:

```
(%i6)        fpprintprec:5;  ratprint:false;  x:40;  y:-15;
theta:float(25/180*%pi); g:9.8;
(fpprintprec)   5
(ratprint)   false
(x)   40
(y)   -15
(theta)   0.43633
(g)   9.8
(%i8)     solve(y=x*tan(theta)-g*x^2/(2*u^2*cos(theta)^2),
u)$ float(%);
(%o8)   [u=-16.841,u=16.841]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of x, y, θ, and g.
(%i8) Solve $y = (\tan\theta)x - \frac{g}{2(u\ \cos\ \theta)^2}x^2$ for u.

Problem 4.9 A shooter has a target 1000 m away and 200 m above the ground. The speed of the bullet is 600 m s^{-1}. Calculate the angle of inclination of the sniper.

Solution
Figure 4.7 shows the bullet, the target, and the geometry of the situation. The initial speed of the bullet is u and the angle of inclination of the sniper is θ.

The trajectory of the bullet is (Eq. 4.8),

$$y = (\tan\theta)x - \frac{g}{2u^2\cos^2\theta}x^2.$$

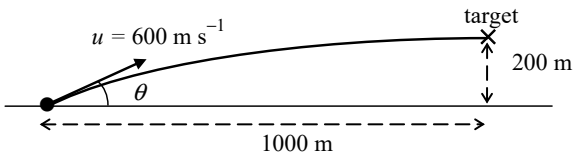


Fig. 4.7 A bullet hits a target, Problem 4.9

The quantities y, x, g, and u are known and the value of θ needs to be calculated. Use the substitution,

$$z = \tan\theta \Rightarrow \cos^2\theta = \frac{1}{1+z^2}.$$

With this substitution and the numerical values given, the trajectory equation becomes,

$$y = zx - \frac{gx^2(1+z^2)}{2u^2},$$
$$200 = 1000z - \frac{9.8(1000^2)(1+z^2)}{2(600^2)}.$$

A solution is,

$$z = \tan\theta = 0.214,$$

and this gives the angle of inclination of the sniper as,

$$\theta = \tan^{-1}(0.214) = 12.1^\circ.$$

• wxMaxima codes:

```
(%i6)   fpprintprec:5; ratprint:false; x:1000; y:200; u:600;
g:9.8;
(fpprintprec)   5
(ratprint)   false
(x)   1000
(y)   200
(u)   600
(g)   9.8
(%i8)   solve(y=z*x-g*x^2*(1+z^2)/(2*u^2), z)$ float(%);
(%o8)   [z=0.21424,z=73.255]
(%i9)   z: rhs(%[1]);
(z)   0.21424
(%i10)   theta: float(atan(z)*180/%pi);
(theta)   12.092
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, and assign values of x, y, u, and g.
(%i8) Solve equation $y = zx - \frac{gx^2(1+z^2)}{2u^2}$ for z.
(%i9), (%i10) Select a solution and calculate the angle of inclination, θ.

Problem 4.10 The speed of launch of a projectile is 80 m s^{-1} and the range is 550 m.

(a) Determine the two possible elevation angles, θ.
(b) Plot the two possible paths of the projectile.
(c) With this launch speed of 80 m s^{-1}, what is the maximum range of the projectile?

Solution

(a) The range of a projectile is (Eq. 4.6),

$$R = \frac{u^2 \sin 2\theta}{g}.$$

Then,

$$2\theta = \sin^{-1}\left(\frac{Rg}{u^2}\right) = \sin^{-1}\frac{(550\,\text{m})(9.8\,\text{m s}^{-2})}{(80\,\text{m s}^{-1})^2} = \sin^{-1} 0.842,$$
$$2\theta = 57.4°, 122.6°,$$
$$\theta = 29°, 61°.$$

To get the 550 m range, the elevation angle is 29° or 61°. The two angles add up to 90°.

• wxMaxima codes:

```
(%i4)   fpprintprec:5; u:80; R:550; g:9.8;
(fpprintprec)   5
(u)   80
(R)   550
(g)   9.8
(%i5)   R*g/u^2;
(%o5)   0.84219
(%i6)   asin(%);
(%o6)   1.0013
(%i7)   twotheta1: float(%*180/%pi);
(twotheta1)   57.372
(%i8)   theta1: twotheta1/2;
(theta1)   28.686
(%i9)   twotheta2: 180-twotheta1;
(twotheta2)   122.63
(%i10)   theta2: twotheta2/2;
(theta2)   61.314
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of u, R, and g.
(%i5) Calculate Rg/u^2.
(%i6), (%i7), (%i8) Calculate the first angle.

(%i9) (%i10) Calculate the second angle.

(b) To plot the two paths, use the trajectory equation (Eq. 4.8),

$$y = (\tan\theta)x - \frac{g}{2(u\ \cos\ \theta)^2}x^2.$$

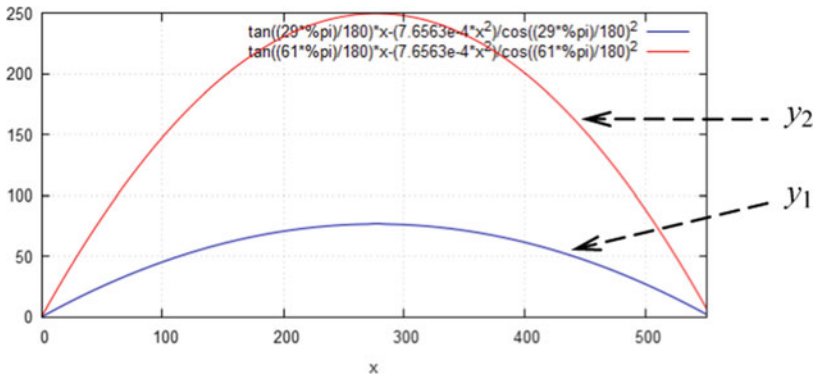
For $\theta = 29°$ and $61°$, define,

$$y_1 = (\tan 29°)x - \frac{9.8}{2(80\cos\ 29°)^2}x^2,$$
$$y_2 = (\tan 61°)x - \frac{9.8}{2(80\cos\ 61°)^2}x^2,$$

and plot y_1 and y_2 against x for $0 \le x \le 550$ m.

• wxMaxima codes:

```
(%i5)    fpprintprec:5; theta1:29; theta2:61; u:80; g:9.8;
(fpprintprec)    5
(theta1)    29
(theta2)    61
(u)    80
(g)    9.8
(%i7)    y1: tan(theta1/180*%pi)*x-0.5*g*x^2/(u*cos(theta1/
180*%pi))^2$ float(%);
(%o7)    0.55431*x-0.0010009*x^2
(%i9)    y2: tan(theta2/180*%pi)*x-0.5*g*x^2/(u*cos(theta2/
180*%pi))^2$ float(%);
(%o9)    1.804*x-0.0032574*x^2
(%i10)    wxplot2d([y1,y2],[x,0,550], same_xy, grid2d);
```



Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of θ_1, θ_2, u, and g.
(%i7), (%i9) Define y_1 and y_2.
(%i10) Plot y_1 and y_2 against x for $0 \leq x \leq 550$ m.

(c) The maximum range corresponds to a launch at an elevation angle of 45° (Eq. 4.6). The maximum range for 80 m s^{-1} is,

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(80\,\text{m/s})^2 \sin\,(2 \times 45°)}{9.8\,\text{m/s}^2} = 650\,\text{m}.$$

• wxMaxima codes:

```
(%i4)   fpprintprec:5; u:80; g:9.8; theta:float(45/180*%pi);
(fpprintprec)   5
(u)   80
(g)   9.8
(theta)   0.7854
(%i5)   R: u^2*sin(2*theta)/g;
(R)   653.06
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of u, g, and θ.
(%i5) Calculate R.

Problem 4.11 A boy throws a ball with an initial speed of 10 m s^{-1}. The point of throw is 6.0 m horizontally away and 1.0 m vertically down the basket (Fig. 4.8). Calculate the speed when the ball hits the basket.

Solution

The path of the ball is given by (Eq. 4.8),

$$y = (\tan\theta)x - \left(\frac{g}{2u^2 \cos^2\theta}\right)x^2,$$

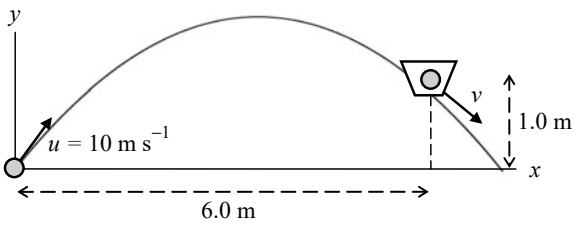


Fig. 4.8 A ball thrown into a basket, Problem 4.11

where the angle of throw is θ and the initial speed is u. Here, x and y are the coordinates of the ball. Given u, g, x, and y, the angle θ can be calculated. To do this, use the substitution $z = \tan\theta$, and we have $\cos^2\theta = (1+z^2)^{-1}$. Inserting the known numerical values into the equation gives,

$$1 = 6z - \frac{9.8(1+z^2)6^2}{2\times 10^2}.$$

Solutions of the equation are,

$$z = 0.549, 2.85.$$

Picking $z = 0.549$, gives the angle of throw $\theta = 0.502$ rad $= 28.8°$. At the point the ball hits the basket, the x component of velocity is,

$$\begin{aligned} v_x &= u\ \cos\ 28.8° = (10\,\mathrm{m\,s^{-1}})\cos\ 28.8° \\ &= 8.76\,\mathrm{m\,s^{-1}}, \end{aligned}$$

while the y component of velocity is,

$$\begin{aligned} v_y &= \sqrt{u_y^2 - 2gs} = \sqrt{u^2\sin^2\theta - 2gs} \\ &= \sqrt{(10\,\mathrm{m/s})^2\sin^2 28.8° - 2(9.8\,\mathrm{m/s^2})(1.0\,\mathrm{m})} \\ &= 1.89\,\mathrm{m\,s^{-1}}. \end{aligned}$$

The ball hits the basket at a speed of,

$$\begin{aligned} v &= \sqrt{v_x^2 + v_y^2} = \sqrt{(8.76\,\mathrm{m/s})^2 + (1.89\,\mathrm{m/s})^2} \\ &= 8.97\,\mathrm{m\,s^{-1}}. \end{aligned}$$

• wxMaxima codes:

```
(%i7)   fpprintprec:5; ratprint:false; x:6; y:1; u:10; g:9.8;
s:1;
(fpprintprec)   5
(ratprint)   false
(x)   6
(y)   1
(u)10
(g)   9.8
(s)   1
(%i9)   solve(y = z*x - g*(1+z^2)/(2*u^2)*x^2, z)$ float(%);
(%o9)   [z=0.54941,z=2.8519]
```

```
(%i10)   z: rhs(%[1]);
(z)   0.54941
(%i11)   theta: float(atan(z));
(theta)   0.50239
(%i12)   degree: float(theta*180/%pi);
(degree)   28.785
(%i15)   vx: u*cos(theta); vy: sqrt(u^2*sin(theta)^2-2*g*s);
v:   sqrt(vx^2 + vy^2);
(vx)   8.7643
(vy)   1.8938
(v)   8.9666
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of x, y, u, g, and s.
(%i9) Solve $y = zx - \frac{g(1+z^2)}{2u^2}x^2$ for z.
(%o9) The solutions.
(%i10), (%i11) Pick $z = 0.54941$ and calculate θ in radian.
(%i12) Convert the angle to degree.
(%i15) Calculate v_x, v_y, and v.

If one picks $z = 2.85$, one arrives at the same result i.e. $v = 8.97\,\mathrm{m\,s^{-1}}$. The angle of throw is $\theta = 1.23$ rad $= 70.7°$. At the point the ball hits the basket, the x component of velocity is,

$$\begin{aligned} v_x &= u\ \cos\ 70.7° = (10\,\mathrm{m\,s^{-1}})\ \cos 70.7° \\ &= 3.31\,\mathrm{m\,s^{-1}}. \end{aligned}$$

The y component of velocity is,

$$\begin{aligned} v_y &= \sqrt{u_y^2 - 2gs} = \sqrt{u^2 \sin^2\theta - 2gs} \\ &= \sqrt{(10\,\mathrm{m/s})^2 \sin^2 70.7° - 2(9.8\,\mathrm{m/s^2})(1.0\,\mathrm{m})} \\ &= 8.33\,\mathrm{m\,s^{-1}}. \end{aligned}$$

The ball hits the basket at a speed of,

$$\begin{aligned} v &= \sqrt{v_x^2 + v_y^2} = \sqrt{(3.31\,\mathrm{m/s})^2 + (8.33\,\mathrm{m/s})^2} \\ &= 8.97\,\mathrm{m\,s^{-1}}. \end{aligned}$$

• wxMaxima codes:

```
(%i7)   fpprintprec:5; ratprint:false; x:6; y:1; u:10; g:9.8;
s:1;
(fpprintprec)   5
(ratprint)   false
(x)   6
(y)   1
(u)   10
(g)   9.8
(s)   1
(%i9)   solve(y = z*x - g*(1+z^2)/(2*u^2)*x^2, z)$ float(%);
(%o9)   [z=0.54941,z=2.8519]
(%i10)   z: rhs(%[2]);
(z)   2.8519
(%i11)   theta: float(atan(z));
(theta)   1.2336
(%i12)   degree: float(theta*180/%pi);
(degree)   70.677
(%i15)   vx: u*cos(theta); vy: sqrt(u^2*sin(theta)^2-2*g*s);
v: sqrt(vx^2 + vy^2);
(vx)   3.3089
(vy)   8.3338
```

(v) 8.9666

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of x, y, u, g, and s.
(%i9) Solve $y = zx - \frac{g(1+z^2)}{2u^2}x^2$ for z.
(%o9) The solutions.
(%i10), (%i11) Pick $z = 2.8519$ and calculate θ in radian.
(%i12) Convert the angle to degree.
(%i15) Calculate v_x, v_y, and v.

Problem 4.12 A gun is aimed straight at a target (Fig. 4.9). A bullet is shot at the same instant the target is dropped from rest. Determine the following,

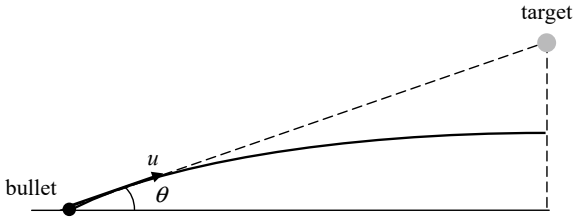


Fig. 4.9 Hitting a falling target, Problem 4.12

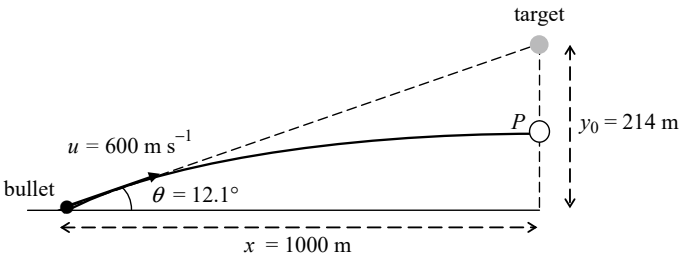


Fig. 4.10 Hitting a falling target, Problem 4.13

(a) will the bullet hit the target?
(b) if the bullet is shot at a higher speed will it hit the target?

Solution

(a) Yes, the bullet hits the falling target. In vertical direction, the bullet falls the same distance as the target does, due to gravity.
(b) Yes, at a higher speed, the bullet still hits the falling target.

Problem 4.13 Verify Problem 4.12 by assuming the values of physical quantities to the problem and do the calculations.

Solution
Figure 4.10 shows the assumed values of the physical quantities. We assumed, the bullet is fired at speed $u = 600$ m s^{-1} at elevation angle $\theta = 21.1°$, $x = 1000$ m away and $y_0 = 214$ m below the target. Let the bullet hits the target at point P, after time t it is fired, or after time t the target falls. Let the bullet initially at the origin (0, 0) and the target at 214 m above the ground. We want to show that the bullet hits the falling target. That is to show that at time t the bullet and the target are at the same position.

(a) Using the x component of the bullet position (at point P), the time, t is calculated as,

$$(u\cos\ \theta)t = x \Rightarrow t = \frac{x}{u\cos\ \theta} = \frac{1000\,\text{m}}{600\,\text{m s}^{-1}\cos\ 12.1°} = 1.7\,\text{s}.$$

At point P, the y coordinate of the bullet is,

$$\begin{aligned} y_{bullet} &= (u\sin\theta)t - \frac{1}{2}gt^2 \\ &= 600\,\text{m s}^{-1}\sin 12.1°(1.7\,\text{s}) - \frac{1}{2}(9.8\,\text{m s}^{-2})(1.7\,\text{s})^2 \\ &= 200\,\text{m}. \end{aligned}$$

It means that at point P and at $t = 1.7$ s, the coordinate of the bullet is (1000 m, 200 m). If there is no acceleration due to gravity the bullet will be at coordinate

(1000 m, 214 m). Due to acceleration of gravity the bullet falls by 214 m − 200 m = 14 m in the vertical direction.

At point P, the y coordinate of the target is,

$$y_{target} = y_0 - \frac{1}{2}gt^2 = 214\,\text{m} - \frac{1}{2}(9.8\,\text{m s}^{-2})(1.7\,\text{s})^2 = 200\,\text{m}.$$

The coordinate of the target at time $t = 1.7$ s and at point P is (1000 m, 200 m). The target has fallen by a distance 214 m − 200 m = 14 m in the vertical direction. Thus, the bullet hits the target.

• wxMaxima codes:

```
(%i7)             fpprintprec:5;   ratprint:false;   u:600;
theta:float(12.1/180*%pi); x:1000; g:9.8; y0:214;
(fpprintprec)   5
(ratprint)   false
(u)   600
(theta)   0.21118
(x)   1000
(g)   9.8
(y0)   214
(%i8)   t: x/(u*cos(theta));
(t)   1.7045
(%i9)   ybullet: u*sin(theta)*t-0.5*g.t^2;
(ybullet)   200.14
(%i10)   fall_bullet: y0-ybullet;
(fall_bullet)   13.855
(%i11)   ytarget: y0-0.5*g*t^2;
(ytarget)   199.76
(%i12)   fall_target: y0-ytarget;
(fall_target)   14.237
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of u, θ, x, g, and y_0.
(%i8) Calculate t.
(%i9), (%i10) Calculate y_{bullet} and the bullet vertical fall distance.
(%i11), (%i12) Calculate y_{target} and the target vertical fall distance.

(b) Let the bullet be fired at a higher speed $u = 700$ m s^{-1}. Repeating the calculation, the time t is calculated from the x component of the bullet position (at point P),

$$(u\cos\,\theta)t = x \Rightarrow t = \frac{x}{u\cos\,\theta} = \frac{1000\,\text{m}}{700\,\text{m s}^{-1}\cos\,12.1^\circ} = 1.46\,\text{s}.$$

At point P, the y coordinate of the bullet is,

$$\begin{aligned} y_{bullet} &= (u \sin\ \theta)t - \frac{1}{2}gt^2 \\ &= 700\,\mathrm{m\,s^{-1}} \sin\ 12.1^\circ(1.46\,\mathrm{s}) - \frac{1}{2}(9.8\,\mathrm{m\,s^{-2}})(1.46\,\mathrm{s})^2 \\ &= 204\,\mathrm{m}. \end{aligned}$$

At point P, the coordinate of the bullet is (1000 m, 204 m). If there is no gravity the bullet will be at coordinate (1000 m, 214 m). Due to acceleration of gravity the bullet falls by 214 m − 204 m = 10 m in the vertical direction.

At point P, the y coordinate of the target is,

$$\begin{aligned} y_{target} &= y_0 - \frac{1}{2}gt^2 \\ &= 214\,\mathrm{m} - \frac{1}{2}(9.8\,\mathrm{m\,s^{-2}})(1.46\,\mathrm{s})^2 \\ &= 204\,\mathrm{m}. \end{aligned}$$

The coordinate of the target at point P is also (1000 m, 204 m). The target has fallen by a distance 214 m − 204 m = 10 m. The bullet still hits the target.

• wxMaxima codes:

```
(%i7)             fpprintprec:5;   ratprint:false;   u:700;
theta:float(12.1/180*%pi); x:1000; g:9.8; y0:214;
(fpprintprec)   5
(ratprint)   false
(u)   700
(theta)   0.21118
(x)   1000
(g)   9.8
(y0)   214
(%i8)   t: x/(u*cos(theta));
(t)   1.461
(%i9)   ybullet: u*sin(theta)*t-0.5*g.t^2;
(ybullet)   203.92
(%i10)   fall_bullet: y0-ybullet;
(fall_bullet)   10.078
(%i11)   ytarget: y0-0.5*g*t^2;
(ytarget)   203.54
(%i12)   fall_target: y0-ytarget;
(fall_target)   10.46
```

The greater the speed of the bullet, the shorter the time it hits the target and the hit is at a greater height from the ground.

To generalize, the y coordinates of the bullet and the target at point P, at time t, are,

$$\begin{aligned} y_{bullet} &= (u\ \sin\ \theta)t - \frac{1}{2}gt^2 \\ &= (u\ \sin\ \theta)(\frac{x}{u\ \cos\ \theta}) - \frac{1}{2}g\ (\frac{x}{u\ \cos\ \theta})^2 \\ &= x \tan\ \theta - \frac{1}{2}g\ (\frac{x}{u\ \cos\ \theta})^2 \\ &= y_0 - \frac{1}{2}g\ (\frac{x}{u\ \cos\ \theta})^2. \end{aligned}$$

$$\begin{aligned} y_{target} &= y_0 - \frac{1}{2}gt^2 \\ &= y_0 - \frac{1}{2}g(\frac{x}{u\ \cos\ \theta})^2. \end{aligned}$$

It means that $y_{bullet} = y_{target}$ for any u, that is, the bullet always hits the falling target for any initial speed of the bullet.

Problem 4.14 Two projectiles are launched simultaneously from O and they hit the ground at points A and B (Fig. 4.11). Which point is hit first?

Solution
Point B is hit first. The time of the projectile in air depends on its maximum height of the parabolic path. The larger the height, the longer the time is. The projectile that hits A stays longer in air than the projectile that hits B. Thus, B is hit first.

Problem 4.15 A plane flying at an altitude of 1000 m at a speed of 300 m s^{-1} releases a bowling ball and the ball hits the ground at point A (Fig. 4.12). What is the angle θ when the ball was released?

Solution
The time taken by the ball to hit the ground is calculated as,

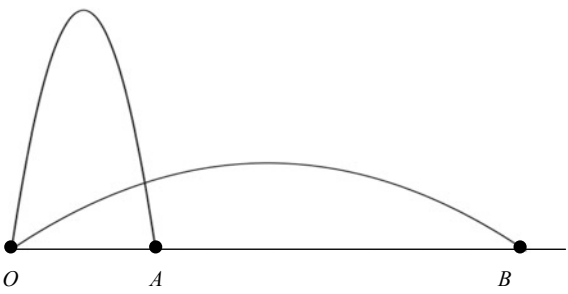


Fig. 4.11 Path of two projectiles, Problem 4.14

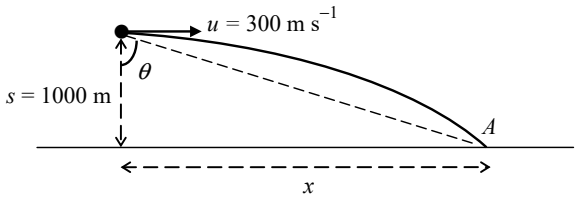


Fig. 4.12 A bowling ball dropped from a plane, Problem 4.15

$$s = \frac{1}{2}gt^2,$$

$$t = \sqrt{\frac{2s}{g}} = \sqrt{\frac{2(1000\,\text{m})}{9.8\,\text{m/s}^2}} = 14.3\,\text{s}.$$

The horizontal distance, x, is,

$$x = ut = u\sqrt{\frac{2s}{g}} = 300\,\text{m/s} \times \sqrt{\frac{2(1000\,\text{m})}{9.8\,\text{m/s}^2}} = 4290\,\text{m}.$$

The angle, θ, is calculated as follows,

$$\tan\theta = \frac{x}{s} = \frac{u\sqrt{\frac{2s}{g}}}{s} = u\sqrt{\frac{2}{sg}},$$

$$\theta = \tan^{-1}\left(u\sqrt{\frac{2}{sg}}\right) = \tan^{-1}\left(300\,\text{m/s} \times \sqrt{\frac{2}{1000\,\text{m}\ (9.8\,\text{m/s}^2)}}\right) = 77^\circ.$$

• wxMaxima codes:

```
(%i4)   fpprintprec:5; u:300; s:1000; g:9.8;
(fpprintprec)   5
(u)   300
(s)   1000
(g)   9.8
(%i5)   t: sqrt(2*s/g);
(t)   14.286
(%i6)   x: u*t;
(x)   4285.7
(%i7)   thetarad: atan(x/s);
(thetarad)   1.3416
(%i8)   theta: float(thetarad/%pi*180);
(theta)   76.866
```

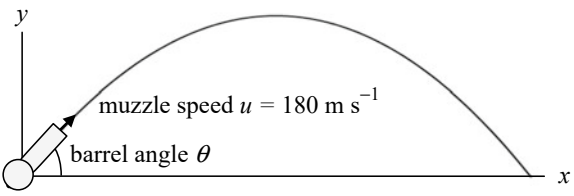


Fig. 4.13 Shots at different barrel angles, Problem 4.16

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of u, s, and g.
(%i5), (%i6) Calculate t and x.
(%i7), (%i8) Calculate angle, θ.

Problem 4.16 The muzzle speed of a canon is 180 m s^{-1}. Four shots are fired at barrel angles of 35°, 40°, 45°, and 50°, Fig. 4.13. Plot the trajectories and calculate the horizontal ranges of the shots.

Solution

The parabolic trajectory of a shot is given by (Eq. 4.8),

$$y = (\tan\theta)x - \frac{g}{2(u\ \cos\theta)^2}x^2,$$

where u is the muzzle speed and θ is the barrel angle. Plotting trajectories of the shots means plotting y against x of the shots. We set the trajectories as,

$$y_1 = (\tan 35°)x - \frac{g}{2(u\ \cos\ 35°)^2}x^2, \tag{4.9}$$

$$y_2 = (\tan 40°)x - \frac{g}{2(u\ \cos\ 40°)^2}x^2, \tag{4.10}$$

$$y_3 = (\tan 45°)x - \frac{g}{2(u\ \cos\ 45°)^2}x^2, \tag{4.11}$$

$$y_4 = (\tan 50°)x - \frac{g}{2(u\ \cos\ 50°)^2}x^2. \tag{4.12}$$

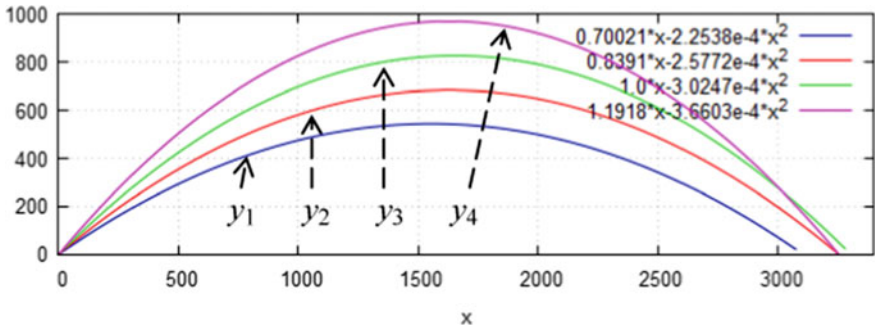
and plot them by wxMaxima.

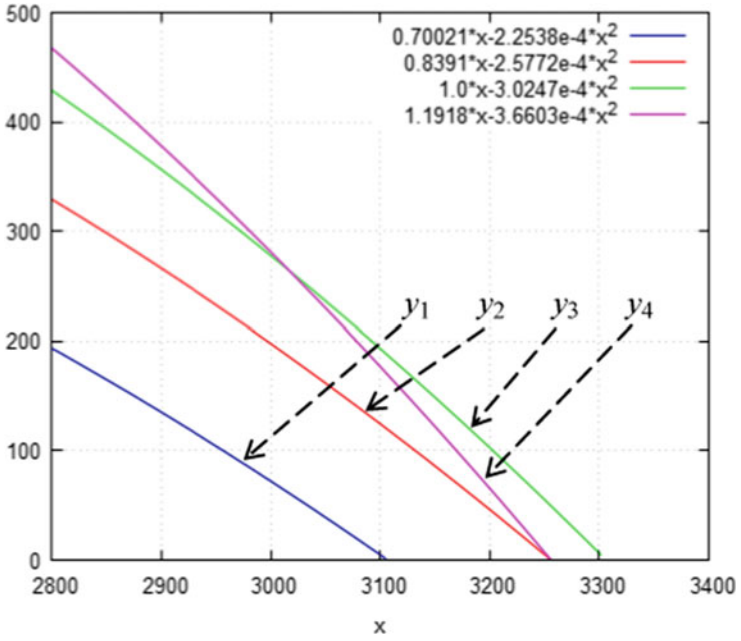
To calculate the horizontal ranges, we set y_1, y_2, y_3, and y_4 to zero and solve for x. Values of x are the ranges. The results are,

$\theta = 35°, x = 3107$ m,
$\theta = 40°, x = 3256$ m,
$\theta = 45°, x = 3306$ m,
$\theta = 50°, x = 3256$ m.

- wxMaxima codes:

```
(%i8)          fpprintprec:5;  ratprint:false;  theta1:35/
180*float(%pi);    theta2:40/180*float(%pi);    theta3:45/
180*float(%pi); theta4:50/180*float(%pi); g:9.8; u:180;
(fpprintprec)   5
(ratprint)   false
(theta1)   0.61087
(theta2)   0.69813
(theta3)   0.7854
(theta4)   0.87266
(g)   9.8
(u)   180
(%i9)   y1: tan(theta1)*x - 0.5*g/(u*cos(theta1))^2*x^2;
(y1)   0.70021*x-2.2538*10^-4*x^2
(%i10)   y2: tan(theta2)*x - 0.5*g/(u*cos(theta2))^2*x^2;
(y2)   0.8391*x-2.5772*10^-4*x^2
(%i11)   y3: tan(theta3)*x - 0.5*g/(u*cos(theta3))^2*x^2;
(y3)   1.0*x-3.0247*10^-4*x^2
(%i12)   y4: tan(theta4)*x - 0.5*g/(u*cos(theta4))^2*x^2;
(y4)   1.1918*x-3.6603*10^-4*x^2
(%i13)     wxplot2d([y1,y2,y3,y4], [x,0,3400], [y,0,1000],
same_xy, grid2d);
(%i14)    wxplot2d([y1,y2,y3,y4], [x,2800,3400], [y,0,500],
same_xy, grid2d);
```






```
(%i16)   solve(y1=0, x)$ float(%);
(%o16)   [x=3106.7,x=0.0]
(%i18)   solve(y2=0, x)$ float(%);
(%o18)   [x=3255.9,x=0.0]
(%i20)   solve(y3=0, x)$ float(%);
(%o20)   [x=3306.1,x=0.0]
(%i22)   solve(y4=0, x)$ float(%);
(%o22)   [x = 3255.9,x = 0.0]
```

Comments on the codes:

(%i8) Set floating point print precision to 5, internal rational number print to false, assign values of θ_1, θ_2, θ_3, θ_4, g, and u.
(%i9), (%i10), (%i11), (%i12) Assign y_1, y_2, y_3, and y_4, as per Eqs. (4.9), (4.10), (4.11), and (4.12), respectively.
(%i13) Plot y_1, y_2, y_3, and y_4 against x.
(%i14) Zoom in part of the plot.
(%i16) Solve $y_1 = (\tan 35°)x - \frac{g}{2(u \cos 35°)^2}x^2 = 0$ for x.
(%i18) Solve $y_2 = (\tan 40°)x - \frac{g}{2(u \cos 40°)^2}x^2 = 0$ for x.
(%i20) Solve $y_3 = (\tan 45°)x - \frac{g}{2(u \cos 45°)^2}x^2 = 0$ for x.
(%i22) Solve $y_4 = (\tan 50°)x - \frac{g}{2(u \cos 50°)^2}x^2 = 0$ for x.

Alternatively, the ranges can be calculated using Eq. (4.6),

$$R = \frac{u^2 \sin 2\theta}{g}.$$

• wxMaxima codes:

```
(%i8)         fpprintprec:5;  ratprint:false;  theta1:  35/
180*float(%pi);    theta2:40/180*float(%pi);    theta3:45/
180*float(%pi); theta4:50/180*float(%pi); g:9.8; u:180;
(fpprintprec)   5
(ratprint)   false
(theta1)   0.61087
(theta2)   0.69813
(theta3)   0.7854
(theta4)   0.87266
(g)   9.8
(u)   180
(%i9)   R(theta) := u^2*sin(2*theta)/g;
(%o9)   R(theta):=(u^2*sin(2*theta))/g
(%i13)   R(theta1); R(theta2); R(theta3); R(theta4);
(%o10)   3106.7
(%o11)   3255.9
(%o12)   3306.1
(%o13)   3255.9
```

Comments on the codes:

(%i8) Set floating point print precision to 5, internal rational number print to false, assign values of θ_1, θ_2, θ_3, θ_4, g, and u.
(%i9) Define $R(\theta) = u^2 \sin(2\theta)/g$.
(%i13) Calculate $R(\theta_1)$, $R(\theta_2)$, $R(\theta_3)$, and $R(\theta_4)$.

The codes calculate the ranges as (%o10), (%o11), (%o12), (%o13),

$\theta = 35°$, $R = 3107$ m,
$\theta = 40°$, $R = 3256$ m,
$\theta = 45°$, $R = 3306$ m,
$\theta = 50°$, $R = 3256$ m.

Problem 4.17 The muzzle speed of a canon is 180 m s^{-1}, Fig. 4.14.

(a) Find maximum horizontal range R_{max} of its projectile
(b) Plot horizontal range R against barrel angle θ.
(c) Find the barrel angles so that the projectile hits the ground at 1000, 2000, and 3000 m away.

Solution

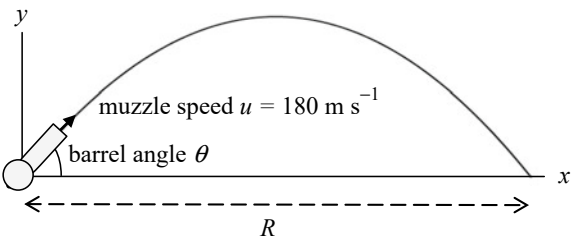


Fig. 4.14 A cannon of Problem 4.17

(a) Equation (4.6) gives the horizontal range R of the projectile as a function of barrel angle θ,

$$R = \frac{u^2 \sin 2\theta}{g}.$$

The maximum of R is obtained as follows,

$$\frac{dR}{d\theta} = \frac{2u^2 \cos 2\theta}{g} = 0,$$
$$\cos 2\theta = 0 \Rightarrow 2\theta = \pi/2 \Rightarrow \theta = \pi/4,$$
$$R_{max} = \frac{u^2 \sin 2\theta}{g} = \frac{u^2 \sin 2(\pi/4)}{g} = \frac{u^2}{g}.$$

Thus,

$$R_{max} = \frac{u^2}{g} = \frac{(180\,\text{m/s})^2}{9.8\,\text{m/s}^2} = 3306\,\text{m}.$$

• wxMaxima codes:

```
(%i3)   fpprintprec:5; g:9.8; u:180;
(fpprintprec)   5
(g)   9.8
(u)   180
(%i4)   Rmax: u^2/g;
(Rmax)   3306.1
```

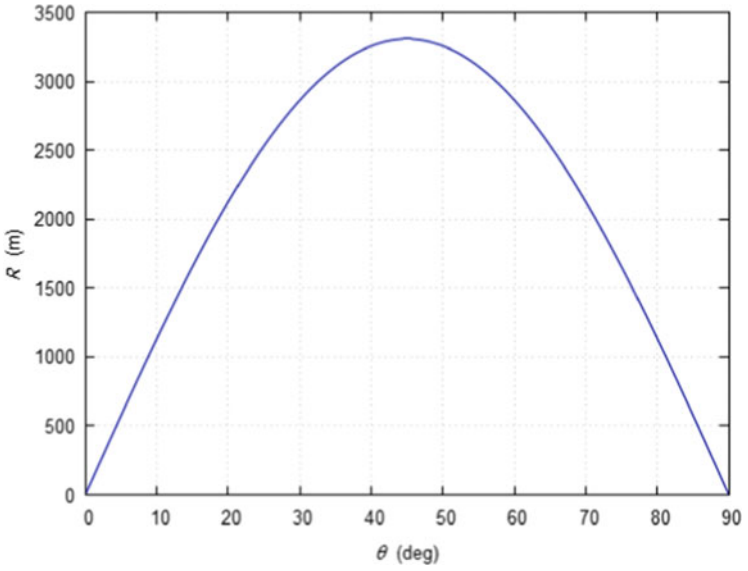
Comments on the codes:

(%i3) Set floating point print precision to 5 and assign values of g and u.
(%i4) Calculate R_{max}.

(b) Plot of R against θ.

• wxMaxima codes:

```
(%i3)   fpprintprec:5; g:9.8; u:180;
(fpprintprec)   5
(g)   9.8
(u)   180
(%i4)   R: u^2*sin(2*deg/180*float(%pi))/g;
(R)   3306.1*sin(0.034907*deg)
(%i5)    wxplot2d(R, [deg,0,90], grid2d, [xlabel,"{/Symbol-
Italic q} (deg)"], [ylabel,"{/Helvetica-Italic R} (m)"]);
```



Comments on the codes:

(%i3) Set floating point print precision to 5 and assign values of g and u.
(%i4), (%i5) Define R and plot R against θ for $0° \leq \theta \leq 90°$.

From the plot, we see that for a specific range (e.g. 1000 m) there are two possible barrel angles (i.e. ~ 10° and ~ 80°), and the two angles add up to 90°. For the maximum range (i.e. 3306 m) the barrel angle is 45°.

(c) Using Eq. (4.6), the barrel angles θ are,

$$R = \frac{u^2 \sin 2\theta}{g},$$

$$\sin 2\theta = \frac{Rg}{u^2},$$
$$\theta = \frac{\sin^{-1}(Rg/u^2)}{2}, \frac{\pi - \sin^{-1}(Rg/u^2)}{2}.$$

This means that to get a specific range R, any one of the two barrel angels is possible. The two barrel angles add up to 90°.

For $R = 1000$ m,

$$\begin{aligned}\theta_1 &= \frac{\sin^{-1}(Rg/u^2)}{2} \\ &= \frac{\sin^{-1}((1000\,\text{m} \times 9.8\,\text{m/s}^2)/(180\,\text{m/s})^2)}{2} \\ &= 0.15\,\text{rad} = 9^\circ, \\ \theta_2 &= 90^\circ - \theta_1 = 81^\circ.\end{aligned}$$

For $R = 2000$ m,

$$\begin{aligned}\theta_1 &= \frac{\sin^{-1}(Rg/u^2)}{2} \\ &= \frac{\sin^{-1}((2000\,\text{m} \times 9.8\,\text{m/s}^2)/(180\,\text{m/s})^2)}{2} \\ &= 0.32\,\text{rad} = 19^\circ, \\ \theta_2 &= 90^\circ - \theta_1 = 71^\circ.\end{aligned}$$

For $R = 3000$ m,

$$\begin{aligned}\theta_1 &= \frac{\sin^{-1}(Rg/u^2)}{2} \\ &= \frac{\sin^{-1}((3000\,\text{m} \times 9.8\,\text{m/s}^2)/(180\,\text{m/s})^2)}{2} \\ &= 0.59\,\text{rad} = 33^\circ, \\ \theta_2 &= 90^\circ - \theta_1 = 57^\circ.\end{aligned}$$

Thus, the projectile hits the ground at,

1000 m away if it is fired at angle 9° or 81°;
2000 m away if it is fired at angle 19° or 71°;
3000 m away if it is fired at angle 33° or 57°.

• wxMaxima codes:

```
(%i3)   fpprintprec:5; g:9.8; u:180;
(fpprintprec)   5
(g)   9.8
(u)   180
(%i7)  R:1000; theta1: 0.5*asin(R*g/u^2); theta1deg: theta1/
float(%pi)*180; theta2deg: 90-theta1deg;
(R)   1000
(theta1)   0.15364
(theta1deg)   8.803
(theta2deg)   81.197
(%i11)      R:2000; theta1: 0.5*asin(R*g/u^2); theta1deg:
theta1/float(%pi)*180; theta2deg: 90-theta1deg;
(R)   2000
(theta1)   0.32484
(theta1deg)   18.612
(theta2deg)   71.388
(%i15)      R:3000; theta1: 0.5*asin(R*g/u^2); theta1deg:
theta1/float(%pi)*180; theta2deg: 90-theta1deg;
(R)   3000
(theta1)   0.56854
(theta1deg)   32.575
(theta2deg)   57.425
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of g and u.
(%i7) Assign value of $R = 1000$, calculate θ_1 in rad, θ_1 in degree, and θ_2 in degree.
(%i11) Assign value of $R = 2000$, calculate θ_1 in rad, θ_1 in degree, and θ_2 in degree.
(%i15) Assign value of $R = 3000$, calculate θ_1 in rad, θ_1 in degree, and θ_2 in degree.

Alternatively, the barrel angle θ can be obtained via Eq. (4.6) by asking wxMaxima to solve the equation for θ.

• wxMaxima codes:

```
(%i4)   fpprintprec:5; ratprint:false; g:9.8; u:180;
(fpprintprec)   5
(ratprint)   false
(g)   9.8
(u)   180
(%i5)   R: u^2*sin(2*theta)/g;
(R)   3306.1*sin(2*theta)
(%i8)   solve(R = 1000, theta)$ float(%); thetadeg:rhs(%[1])/
float(%pi)*180;
solve:   using arc-trig functions to get a solution.
```

```
Some solutions will be lost.
(%o7)   [theta=0.15364]
(thetadeg)   8.803
(%i11)            solve(R   =   2000,   theta)$   float(%);
thetadeg:rhs(%[1])/float(%pi)*180;
solve:   using arc-trig functions to get a solution.
Some solutions will be lost.
(%o10)   [theta=0.32484]
(thetadeg)   18.612
(%i14)            solve(R   =   3000,   theta)$   float(%);
thetadeg:rhs(%[1])/float(%pi)*180;
solve:   using arc-trig functions to get a solution.
Some solutions will be lost.
(%o13)   [theta=0.56854]
(thetadeg)   32.575
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of g and u.
(%i5) Assign R as a function of θ.
(%i8) Solve $R = 1000$ for θ and convert θ to degree.
(%i11) Solve $R = 2000$ for θ and convert θ to degree.
(%i14) Solve $R = 3000$ for θ and convert θ to degree.

The codes say that,

horizontal range $R = 1000$ km, barrel angle $\theta = 9°$;
horizontal range $R = 2000$ km, barrel angle $\theta = 19°$;
horizontal range $R = 3000$ km, barrel angle $\theta = 33°$.

The codes give the smaller angles of angles in part (c).

4.3 Summary

- A projectile motion near the earth surface is a motion in 2D x–y plane. Motion in x direction is a motion with a constant speed of $v_0 \cos\theta_0$ while motion in y direction is a free fall with initial speed of $v_0 \sin\theta_0$ and acceleration of -9.8 m s^{-2}.
- Important formulae:

The maximum height of the projectile:	$h = \frac{v_0^2 \sin^2\theta_0}{2g}$.
The range of the projectile (horizontal distance from the initial and final points of the flight):	$R = \frac{v_0^2 \sin\ 2\theta_0}{g}$.

Time of flight: $t_R = \frac{2v_0 \sin \theta_0}{g}$.

The equation of the projectile trajectory: $y = (\tan \theta_0)x - \frac{g}{2(v_0 \cos \theta_0)^2}x^2$.

4.4 Exercises

Exercise 4.1
A canon is fired at an angle of 35° above the horizontal with a muzzle speed of 800 m s^{-1} (Fig. 4.15). What is the range of its projectile and the projectile time in air?

(Answer: 61,000 m, 94 s)

Exercise 4.2
An object is thrown horizontally from the top of a 20 m high building and it strikes the ground at an angle of 45° (Fig. 4.16). Calculate the initial speed of the throw.

(Answer: 20 m s^{-1})

Exercise 4.3
A ball is thrown horizontally with a speed of 20 m s^{-1} from the edge of a tall building (Fig. 4.17). The ball lands 90 m from the base of the building. What is the height of the building?

(Answer: 99 m)

Exercise 4.4
An object is thrown at an angle of 30° to the horizontal and lands on a 5.0 m tall building 20 m away (Fig. 4.18). What is the initial speed of the throw?

Fig. 4.15 A canon shot of Exercise 4.1

Fig. 4.16 An object thrown horizontally from a building, Exercise 4.2

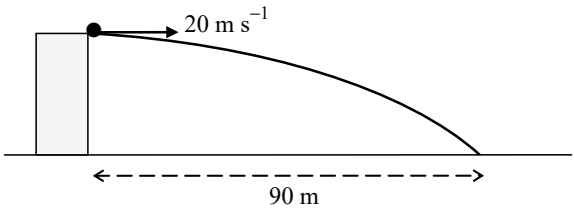


Fig. 4.17 A ball thrown horizontally form a building, Exercise 4.3

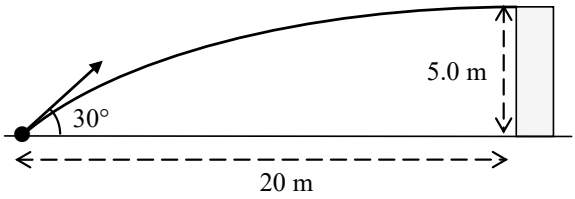


Fig. 4.18 An object thrown to a building, Exercise 4.4

(Answer: 20 m s^{-1})

Exercise 4.5

A projectile is launched with an initial speed of 95 m s^{-1} at an angle of 50° to the horizontal and hits a target 5.0 s later (Fig. 4.19). Calculate the height, h and horizontal distance, x of the target from the launch point.

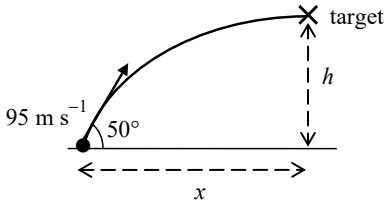


Fig. 4.19 A projectile hits a target, Exercise 4.5

(Answer: 240 m, 310 m)

Chapter 5
Newton's Laws of Motion

5.1 Basic Concepts and Formulae

1. The Newton's laws of motion are,

i. A body at rest remains at rest or a body moving uniformly in a straight line remains as such unless there is a net force acting on the body. This is the Newton's first law or the law of inertia.
ii. The time rate of momentum change of a body is the net force acting on the body. If the mass of the body is a constant, the net or the resultant force is the product of mass and acceleration of the body,

$$\Sigma \boldsymbol{F} = m\boldsymbol{a}, \tag{5.1}$$

where $\Sigma \boldsymbol{F}$ is the net force acting on the body of mass, m and the acceleration, a ensues. This is the Newton's second law of motion.

iii For two interacting bodies, the force on the second by the first is the same and opposite to the force on the first by the second. There is no isolated force. This is the Newton's third law.
2. The first and second laws are valid in inertial reference frame. Inertial reference frame is a frame in which the object in the frame acted by zero net force will move with uniform velocity or be at rest.
3. The SI unit for force is newton (N). One newton is the force acting on a body of 1 kg mass such that the body accelerates at 1 m s^{-2}.

$$1\,\mathrm{N} = 1\,\mathrm{kg\,m\,s^{-2}}$$

4. Mass is a scalar quantity. Mass in the Newton's second law is inertial mass. Weight of a body, W is a product of mass, m with acceleration of gravity, g,

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W. M. S. Wan Hassan, *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-42678-0_5

$$W = mg. \tag{5.2}$$

5. The maximum static frictional force, $f_{s,max}$ between a body and a surface is proportional to the normal force acting on the body. The force exists as the body is about to slip.

$$f_s \leq f_{s,max} = \mu_s N, \tag{5.3}$$

where μ_s is the coefficient of static friction and N is the normal force. When a body slides on a surface, kinetic frictional force, f_k acts in the direction opposite to the direction of the motion. The magnitude of the kinetic friction is,

$$f_k = \mu_k N, \tag{5.4}$$

where μ_k is the coefficient of kinetic friction. Usually, for a body and a surface, one gets,

$$\mu_k < \mu_s.$$

5.2 Problems and Solutions

Problem 5.1 A block of wood slides from rest on an inclined plane of length 2.0 m with negligible friction. The angle between the plane and the horizontal is 10°. Determine,

(a) the acceleration, a, of the block,
(b) the time, t, taken to reach the bottom end of the plane,
(c) the speed, v, of the block as it reaches the bottom end of the plane.

Solution

(a) Figure 5.1 shows the wooden block and the inclined plane. Resolve the weight of the block, mg, into x and y components i.e. $mg \sin\theta$ and $mg \cos\theta$. The $mg \cos\theta$ component is balanced by the reaction, R, of the plane on the block. The resultant force in the y direction is zero. The resultant force in the x direction is $mg \sin\theta$ and this force accelerates the block along the plane.

Applying the Newton's second law, $\Sigma F = ma$, to the wooden block gives,

$$mg \sin\theta = ma,$$

so that the acceleration of the block is,

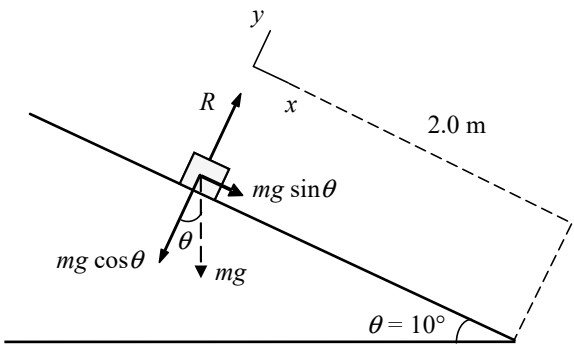

Fig. 5.1 A wooden block on a smooth inclined plane, Problem 5.1

$$a = g\sin\theta = \left(9.8\,\mathrm{m/s^2}\right)\sin 10^\circ = 1.7\,\mathrm{m\,s^{-2}}.$$

(b) The time taken to reach the bottom end of the plane is obtained from $s = ut + \frac{1}{2}at^2$,

$$2.0\,\mathrm{m} = 0 + \frac{1}{2}\left(1.7\,\mathrm{m/s^2}\right)t^2,$$
$$t = 1.5\,\mathrm{s}.$$

(c) The speed of the block as it reaches the bottom end of the incline is obtained from $v^2 - u^2 = 2as$,

$$v^2 - 0^2 = 2\left(1.7\,\mathrm{m/s^2}\right)(2.0\,\mathrm{m}),$$
$$v = 2.6\,\mathrm{m\,s^{-1}}.$$

• wxMaxima codes:

```
(%i6)fpprintprec:5;          ratprint:false;          g:9.8;
theta:float(10*%pi/180); u:0; s:2;
(fpprintprec) 5
(ratprint) false
(g) 9.8
(theta) 0.17453
(u) 0
(s) 2
(%i7)a: g*sin(theta);
(a) 1.7018
(%i9)solve(s=u*t + 1/2*a*t^2, t)$ float(%);
(%o9)[t=-1.5331,t=1.5331]
(%i11)solve(v^2 - u^2 = 2*a*s, v)$ float(%);
(%o11)[v=-2.609,v=2.609]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of g, θ, u, and s.
(%i7) Calculate acceleration, a. Part (a).
(%i9) Solve $s = ut + \frac{1}{2}at^2$ for t. Part (b).
(%i11) Solve $v^2 - u^2 = 2as$ for v. Part (c).

Problem 5.2 A cable is used to pull a 200 kg car. During the pull, the tension in the cable is 500 N. If the car is at rest initially then calculate,

(a) the time t for the car speed to reach 8.0 m s^{-1},
(b) the distance s traversed by the car.

Solution

(a) We assume the friction is negligible. The net force is 500 N. Applying the Newton's second law, gives the acceleration a of the car,

$$F = ma,$$
$$a = \frac{F}{m} = \frac{500\,\text{N}}{200\,\text{kg}} = 2.5\,\text{m}\,\text{s}^{-2}.$$

The final velocity due to an acceleration is $v = u + at$. So the time t is calculated as,

$$t = \frac{v - u}{a} = \frac{8.0\,\text{m/s} - 0}{2.5\,\text{m/s}^2} = 3.2\,\text{s}.$$

(b) The distance traversed s is calculated as follows,

$$v^2 - u^2 = 2as,$$
$$s = \frac{v^2 - u^2}{2a} = \frac{(8.0\,\text{m/s})^2 - 0^2}{2\left(2.5\,\text{m/s}^2\right)} = 12.8\,\text{m}.$$

• wxMaxima codes:

```
(%i3) fpprintprec:5; F:500; m:200;
(fpprintprec) 5
(F) 500
(m) 200
(%i4) a: float(F/m);
(a) 2.5
(%i7)u:0; v:8; t: float((v-u)/a);
(u) 0
(v) 8
(t) 3.2
```

```
(%i8) s: float((v^2 - u^2)/(2*a));
(s) 12.8
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of F and m.
(%i4) Calculate acceleration a.
(%i7) Assign values of u and v, and calculate t, part (a).
(%i8) Calculate s, part (b).

• Alternative calculation by wxMaxima:

```
(%i6) fpprintprec:5; ratprint:false; F:500; m:200; u:0; v:8;
(fpprintprec) 5
(ratprint) false
(F) 500
(m) 200
(u) 0
(v) 8
(%i8) solve(F=m*a, a)$ float(%);
(%o8) [a=2.5]
(%i9) a: rhs(%[1]);
(a) 2.5
(%i11) solve(v=u+a*t, t)$ float(%);
(%o11) [t=3.2]
(%i13) solve(v^2-u^2=2*a*s, s)$ float(%);
(%o13) [s=12.8]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of F, m, u, and v.
(%i8), (%i9) Solve $F = ma$ for a and assign value of a.
(%i11) Solve $v = u + at$ for t.
(%i13) Solve $v^2 - u^2 = 2as$ for s.

Problem 5.3 An object of mass 900 kg moves with velocity of 20 m s^{-1} along the x axis. Calculate the force needed to stop it at a distance of 30 m.

Solution

First, calculate the deceleration of the object,

$$v^2 - u^2 = 2as,$$
$$a = \frac{v^2 - u^2}{2s} = \frac{(20\,\mathrm{m/s})^2 - 0^2}{2(30\,\mathrm{m})} = -6.7\,\mathrm{m\,s^{-2}}.$$

Then, using the Newton's second law, $F = ma$, calculate the needed force to stop the object,

$$F = (900\,\text{kg})\left(-6.7\,\text{m/s}^2\right)$$
$$= -6000\,\text{N}.$$

• wxMaxima codes:

```
(%i5) fpprintprec:5; v:0; u:20; s:30; m:900;
(fpprintprec) 5
(v) 0
(u) 20
(s) 30
(m) 900
(%i6) a: float((v^2-u^2)/(2*s));
(a) -6.6667
(%i7) F: m*a;
(F) -6000.0
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of v, u, s, and m.
(%i6), (%i7) Calculate a and F.

Problem 5.4 What is the force needed to accelerate a locomotive of mass 2.0×10^4 kg moving at an acceleration of 1.5 m s^{-2} on a track? Consider the friction and it is given that the coefficient of rolling friction is 0.03.

Solution

Figure 5.2 shows the locomotive and the relevant forces. This is the free body diagram of the locomotive. The force is F, the rolling friction is F_r, the weight of the locomotive is mg, and the reaction of the tract on the locomotive is N. The net force to the right is $F - F_r$ and this force accelerates the locomotive. The net vertical force is zero because N is balanced by mg.

The coefficient of the rolling friction is,

$$\mu = \frac{F_r}{N} = \frac{F_r}{mg},$$

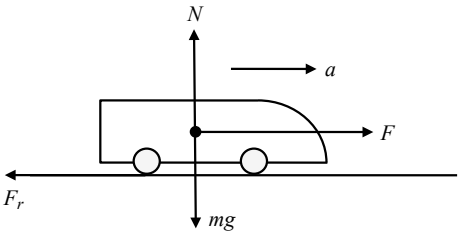

Fig. 5.2 Forces on a locomotive of Problem 5.2

where N is the normal reaction of the track on the locomotive that is equal in magnitude to the weight of the locomotive mg, i.e. $N = mg$. The rolling friction is,

$$F_r = \mu mg.$$

Using the Newton's second law, $\Sigma F = ma$, we get,

$$\begin{aligned} F - F_r &= ma, \\ F - \mu mg &= ma, \\ F &= \mu mg + ma \\ &= (0.03)\left(2.0 \times 10^4\,\text{kg}\right)(9.8\,\text{N/kg}) + \left(2.0 \times 10^4\,\text{kg}\right)\left(1.5\,\text{m/s}^2\right) \\ &= 36000\,\text{N}. \end{aligned}$$

Here, F is the force needed. The force is larger and opposite in direction to the rolling friction F_r. The net force is $F - F_r$ toward the right. The direction of the net force is same as the direction of acceleration of the locomotive.

• wxMaxima codes:

```
(%i5) fpprintprec:5; mu:0.03; m:2e4; g:9.8; a:1.5;
(fpprintprec) 5
(mu) 0.03
(m) 20000
(g) 9.8
(a) 1.5
(%i6) F: mu*m*g + m*a;
(F) 3.588*10^4
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of μ, m, g, and a.
(%i6) Calculate F.

Problem 5.5 Figure 5.3 shows two blocks of masses, $m_1 = 2.0$ kg and $m_2 = 3.0$ kg connected by a light string on a surface with negligible friction. The blocks are pulled to the right by a force of $F = 20$ N. What is the tension in the string that connects the blocks?

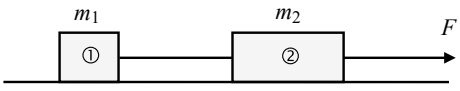

Fig. 5.3 Pulling two blocks on a smoth surface, Problem 5.5

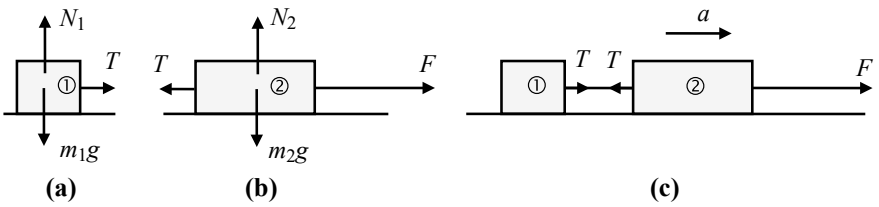

Fig. 5.4 Analysis of pulling two blocks on a smooth surface

Solution

Forces acting on the block 1 are shown in Fig. 5.4a and there are three of them. This is the free body diagram of the block 1. First, the weight of the block, m_1g, acting downward. Second, the normal reaction of the surface on the block, N_1, acting upward. Third, the tension in the string, T, that pulls the block to the right. The normal reaction is equal in magnitude but opposite in direction to the weight. This means that $N_1 = m_1g$ and there is no motion in the y direction. The tension T is the force that accelerates the block with acceleration a, so we write,

$$T = m_1a.$$

Forces acting on the block 2 are shown in Fig. 5.4b. This is the free body diagram of the block 2. The weight of the block is m_2g acting downward, the normal reaction of the surface on the block is N_2 acting upward, the tension in the string is T acting to the left, and the force F acting to the right. There is no motion in the y direction, implying that the normal reaction is equal in magnitude and opposite in direction to the weight of the block, i.e. $N_2 = m_2g$. The net force on the block 2 causing it to move with acceleration, a, to the right is $F - T$. We write,

$$F - T = m_2a.$$

Figure 5.4c shows all the forces acting in the x direction. From these two equations, we get,

$$F - m_1a = m_2a,$$

and the acceleration is,

$$a = \frac{F}{m_1 + m_2} = \frac{20\,\text{N}}{2.0\,\text{kg} + 3.0\,\text{kg}} = 4.0\,\text{m}\,\text{s}^{-2}.$$

The tension in the string is,

$$T = m_1a = \frac{m_1F}{m_1 + m_2} = \frac{(2.0\,\text{kg})(20\,\text{N})}{2.0\,\text{kg} + 3.0\,\text{kg}} = 8.0\,\text{N}.$$

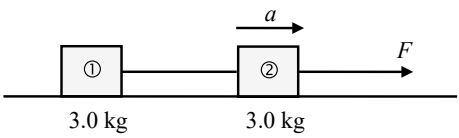

Fig. 5.5 Pulling two blocks on a rough surface, Problem 5.6

• wxMaxima codes:

```
(%i4)fpprintprec:5; m1:2; m2:3; F:20;
(fpprintprec) 5
(m1) 2
(m2) 3
(F) 20
(%i5) solve([T=m1*a, F-T=m2*a],[a,T]);
(%o5) [[a=4,T=8]]
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of m_1, m_2, and F.
(%i5) Solve $T = m_1 a$ and $F - T = m_2 a$ for acceleration, a and tension in the string, T.

Problem 5.6 Two identical blocks each of mass, $m = 3.0$ kg are connected to a string and pulled by a force of $F = 20$ N on a plane surface and they accelerate with an acceleration, $a = 0.50$ m s^{-2} (Fig. 5.5). The frictional force of each block with the surface is the same. Calculate the tension in the string and the frictional force.

Solution

Forces acting on the block 2 in the y direction are its weight, mg, and normal reaction, N (Fig. 5.6a). Both forces are of the same magnitude but opposite in directions. In the x direction, the forces acting on the block are the tension, T, in the string, frictional force, F_{fric}, of the block and the surface, and the pulling force, F. The net force to the right is $F - T - F_{fric}$. Applying the Newton's second law to the block 2, one gets,

$$F - T - F_{fric} = ma,$$

where a is the acceleration of the block.

Forces acting on the block 1 in the y direction are its weight, mg, and normal reaction, N (Fig. 5.6b). Again, both forces are of the same magnitude but opposite in directions. Forces acting on the block 1 in the x direction are tension, T, in the string that pulls the block 1 to the right and friction, F_{fric}, to the left. The net force on the block 1 is $T - F_{fric}$ to the right. The Newton's second law applied to the block 1 is,

$$T - F_{fric} = ma,$$

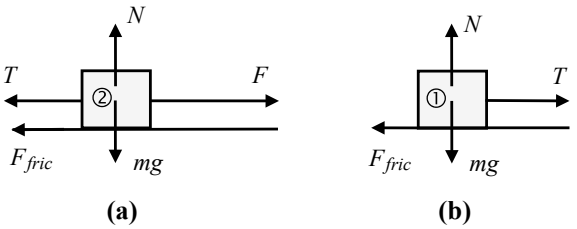

Fig. 5.6 Analysis of pulling two blocks on a rough surface

where a the acceleration of the block.

Substituting known values in the two equations gives,

$$20\,\text{N} - T - F_{fric} = (3.0\,\text{kg})\left(0.50\,\text{m/s}^2\right),$$
$$T - F_{fric} = (3.0\,\text{kg})\left(0.50\,\text{m/s}^2\right).$$

Solving these equations, gives the frictional force and the tension in the string as,

$$F_{fric} = 8.5\,\text{N and } T = 10\,\text{N}.$$

• wxMaxima codes:

```
(%i5)fpprintprec:5; ratprint:false; F:20; a:0.5; m:3;
(fpprintprec) 5
(ratprint) false
(F) 20
(a) 0.5
(m) 3
(%i7)   solve([F-T-Ffric=m*a,   T-Ffric=m*a],   [Ffric,T])$
float(%);
(%o7) [[Ffric=8.5,T=10.0]]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of F, a, and m.
(%i7) Solve $F - T - F_{fric} = ma$ and $T - F_{fric} = ma$ for F_{fric} and T.

Problem 5.7 Two blocks of mass m_1 and m_2 are connected by a string and placed on a flat surface (Fig. 5.7). The coefficient of kinetic friction of the blocks with the surface is μ. A force, F, pulls both blocks to the right. Calculate,

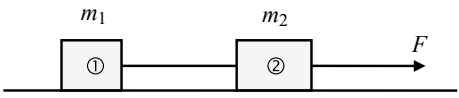

Fig. 5.7 Pulling two blocks on a rough surface, Problem 5.7

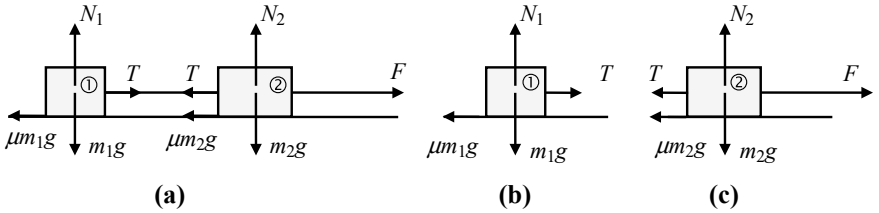

Fig. 5.8 Analysis of pulling two blocks on a rough surface

(a) the acceleration, a, of the blocks
(b) the tension, T, in the string.

Solution

(a) Forces acting on the blocks are shown in Fig. 5.8a. Here, N_1, $m_1 g$, and $\mu m_1 g$ are the normal reaction of the surface on the block 1, weight of the block, and friction of the block with the surface, respectively. Together, N_2, $m_2 g$, and $\mu m_2 g$ are the normal reaction of the surface on the block 2, weight of the block, and friction of the block with the surface, respectively. The tension in the string is T and the applied pull is F. The net force in the x direction to the right is,

$$F - \mu m_1 g - \mu m_2 g = F - \mu g(m_1 + m_2).$$

Applying the Newton's second law to both blocks gives,

$$\begin{aligned} &\Sigma F = ma, \\ &F - \mu g(m_1 + m_2) = (m_1 + m_2)a, \\ &a = \frac{F - \mu g(m_1 + m_2)}{m_1 + m_2} = \frac{F}{m_1 + m_2} - \mu g. \end{aligned}$$

The acceleration, a, is the acceleration of each block.

The net force in the y direction is zero, N_1 cancels with $m_1 g$, while N_2 cancels with $m_2 g$.

(b) Figure 5.8b shows the forces acting on the block 1. This is the free body diagram of the block 1. The net force to the right is,

$$T - \mu m_1 g.$$

The acceleration of block 1 is represented in part (a). Applying the Newton's second law to the block 1 gives,

$$\Sigma F = ma,$$

$$T - \mu m_1 g = m_1\left(\frac{F}{m_1 + m_2} - \mu g\right).$$

Thus, the tension, T, in the string is,

$$T = \mu m_1 g + m_1\left(\frac{F}{m_1 + m_2} - \mu g\right) = \frac{m_1 F}{m_1 + m_2}.$$

As a counter check, Fig. 5.8c shows the free body diagram of the block 2. The net force to the right is,

$$F - T - \mu m_2 g.$$

Applying the Newton's second law to the block 2 gives,

$$\Sigma F = ma,$$
$$F - T - \mu m_2 g = m_2\left(\frac{F}{m_1 + m_2} - \mu g\right),$$

and the tension, T, in the string is,

$$\begin{aligned} T &= F - \mu m_2 g - m_2\left(\frac{F}{m_1 + m_2} - \mu g\right) = F - \frac{m_2 F}{m_1 + m_2} \\ &= \frac{m_1 F + m_2 F - m_2 F}{m_1 + m_2} \\ &= \frac{m_1 F}{m_1 + m_2}. \end{aligned}$$

This is the same result as the one previously calculated.

• wxMaxima codes:

```
(%i1)   solve([F-mu*g*(m1+m2)=(m1+m2)*a,   T-mu*m1*g=m1*a],
[a,T]);
(%o1) [[a=-((g*m2+g*m1)*mu-F)/(m2+m1),T=(F*m1)/(m2+m1)]]
(%i2) expand(%);
(%o2)            [[a=-(g*m2*mu)/(m2+m1)-(g*m1*mu)/(m2+m1)+F/
(m2+m1),T=(F*m1)/(m2+m1)]]
```

Comments on the codes:

(%i1) Solve $F - \mu g(m_1 + m_2) = (m_1 + m_2)a$ and $T - \mu m_1 g = m_1 a$ for a and T.
(%o1), (%o2) The solutions.

Problem 5.8 Determine the force, F, so that blocks A and B move vertically up with an acceleration of 6.0 m s^{-2} (Fig. 5.9). Find also the tension, T, in the string.

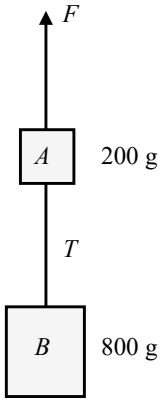

Fig. 5.9 Pulling two blocks vertically upward, Problem 5.8

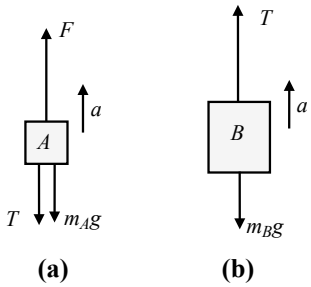

Fig. 5.10 Analysis of the vertical motions of the blocks, Problem 5.8

Solution

Figure 5.10a shows the forces that act on the block A. This is the free body diagram of the block A. The pulling force is F, the tension in the string is T, and the weight of the block is m_Ag.

When the block A moves upward with an acceleration, a, the net force acting on it is,

$$F - T - m_Ag.$$

Applying the Newton's second law to the block A gives,

$$\Sigma F = ma,$$

$$F - T - m_Ag = m_Aa,$$

so that,

$$F = T + m_Ag + m_Aa = T + m_A(g + a).$$

Figure 5.10b shows the forces acting on the block B. This is the free body diagram of the block B. The net force acting of the block B is,

$$T - m_B g,$$

in the upward direction. This force causes the block B to move with the acceleration, a. Applying the Newton's second law to the block B, one has,

$$\Sigma F = ma,$$

$$T - m_B g = m_B a.$$

This means that the tension in the string is,

$$\begin{aligned} T &= m_B g + m_B a = m_B(g + a) \\ &= (0.80\,\text{kg})\left(9.8\,\text{m/s}^2 + 6.0\,\text{m/s}^2\right) \\ &= 12.6\,\text{N}. \end{aligned}$$

Thus, the pulling force, F, is,

$$\begin{aligned} F &= T + m_A(g + a) \\ &= 12.6\,\text{N} + (0.20\,\text{kg})\left(9.8\,\text{m/s}^2 + 6.0\,\text{m/s}^2\right) \\ &= 15.8\,\text{N}. \end{aligned}$$

• wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; mA:0.2; mB:0.8; a:6;
g:9.8;
(fpprintprec) 5
(ratprint) false
(mA) 0.2
(mB) 0.8
(a) 6
(g) 9.8
(%i8) solve([F-T-mA*g=mA*a, T-mB*g=mB*a], [F,T])$ float(%);
(%o8) [[F=15.8,T=12.64]]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m_A, m_B, a, and g.
(%i8) Solve $F - T - m_A g = m_A a$ and $T - m_B g = m_B a$ for the pulling force, F, and the tension in the string, T.

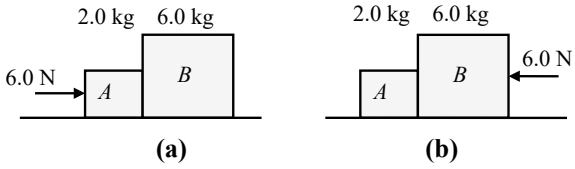

Fig. 5.11 Pushing two blocks to the right and to the left on a smooth floor, Problem 5.9

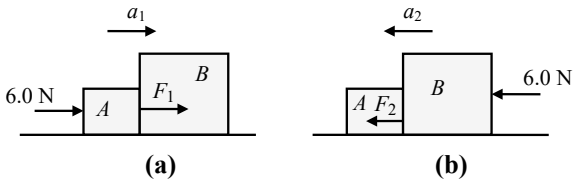

Fig. 5.12 Analyses of the right and left pushes, Problem 5.9

Problem 5.9 Figure 5.11 shows a block A of mass 2.0 kg and a block B of mass 6.0 kg, at rest touching each other on a smooth horizontal floor.

(a) A force to the right of 6.0 N is applied on the block A. Determine the acceleration of the system and the force of block A on the block B (Fig. 5.11a).
(b) A force to the left of 6.0 N is applied on the block B. Determine the acceleration of the system and the force of block B on the block A (Fig. 5.11b).

Solution

(a) Figure 5.12a shows the force of 6.0 N, the force exerted by block A on block B, namely F_1, and the acceleration of the system, a_1.

The force of 6.0 N to the right acts on both blocks and acceleration, a_1 ensues. Using the Newton's second law we have,

$$\begin{aligned} \Sigma F &= ma \\ 6.0\,\mathrm{N} &= (m_A + m_B)a_1 = (2.0\,\mathrm{kg} + 6.0\,\mathrm{kg})a_1, \\ a_1 &= 0.75\,\mathrm{m\,s^{-2}}. \end{aligned}$$

The force, F_1, by the block A on the block B causes block B to accelerate at 0.75 m s^{-2}. Thus,

$$F_1 = m_B a_1 = (6.0\,\mathrm{kg})\left(0.75\,\mathrm{m/s^2}\right) = 4.5\,\mathrm{N}.$$

• wxMaxima codes:

```
(%i4) fpprintprec:5; mA:2; mB:6; F:6;
(fpprintprec) 5
(mA) 2
```

```
(mB) 6
(F) 6
(%i6) solve([F=(mA+mB)*a1, F1=mB*a1], [a1,F1])$ float(%);
(%o6) [[a1=0.75,F1=4.5]]
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of m_A, m_B, and F.
(%i6) Solve $F = (m_A + m_B)a_1$ and $F_1 = m_B a_1$ for a_1 and F_1.

(b) Figure 5.12b shows the force of 6.0 N to the left acting on the blocks. The force causes the blocks to move with acceleration, a_2. One has,

$$\begin{aligned} \Sigma F &= ma, \\ 6.0\,\text{N} &= (m_A + m_B)a_2 = (2.0\,\text{kg} + 6.0\,\text{kg})a_2, \\ a_2 &= 0.75\,\text{m}\,\text{s}^{-2}. \end{aligned}$$

The force, F_2, of the block B on the block A causes the block A to accelerate at 0.75 m s^{-2}. This means that,

$$F_2 = m_A a_2 = (2.0\,\text{kg})\left(0.75\,\text{m/s}^2\right) = 1.5\,\text{N}.$$

• wxMaxima codes:

```
(%i4) fpprintprec:5; mA:2; mB:6; F:6;
(fpprintprec) 5
(mA) 2
(mB) 6
(F) 6
(%i6) solve([F=(mA+mB)*a2, F2=mA*a2], [a2,F2])$ float(%);
(%o6) [[a2=0.75,F2=1.5]]
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of m_A, m_B, and F.
(%i6) Solve $F = (m_A + m_B)a_2$ and $F_2 = m_A a_2$ for a_2 and F_2.

Problem 5.10 A skydiver falls at a constant speed while he extends his arms and legs. He then brings his arms and legs nearer to his body and accelerates to 2.0 m s^{-2}. His mass is 60 kg.

(a) What is the air friction when he extends his arms and legs?
(b) Calculate the air friction after he brings his arms and legs nearer to his body.

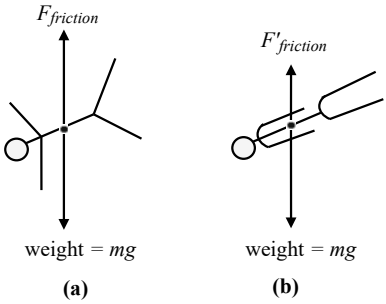

Fig. 5.13 A skydiver when he extends his hands and legs (**a**), and when he brings his hands and legs toward his body (**b**), Problem 5.10

Solution

(a) While he falls at constant speed, the net force on him is zero (Fig. 5.13a). The air friction $F_{friction}$ is of the same magnitude but in opposite direction as his weight mg. The skydiver's mass is m. The direction of air friction in opposite to the direction of motion. The air friction is,

$$F_{friction} = mg = (60\,\text{kg})\left(9.8\,\text{m/s}^2\right) = 588\,\text{N}.$$

(b) When he brings his arms and legs nearer to his body, the air friction decreases to $F'_{friction}$ (Fig. 5.13b). The downward net force becomes $mg - F'_{friction}$ and this force accelerates the falling skydiver. Therefore,

$$mg - F'_{friction} = ma.$$

The air friction during the acceleration or during his arms and legs are near his body is,

$$\begin{aligned} F_{friction} &= mg - ma = m(g - a) = (60\,\text{kg})\left(9.8\,\text{m/s}^2 - 2.0\,\text{m/s}^2\right) \\ &= 468\,\text{N}. \end{aligned}$$

• wxMaxima codes:

```
(%i4) fpprintprec:5; m:60; a:2; g:9.8;
(fpprintprec) 5
(m) 60
(a) 2
(g) 9.8
(%i5) Ffriction: m*g;
(Ffriction) 588.0
```

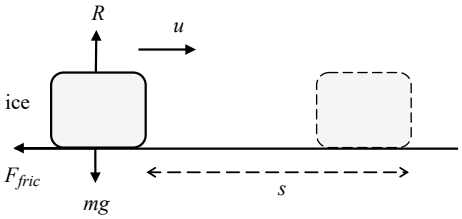

Fig. 5.14 A block of ice on a rough floor, Problem 5.11

```
(%i6) Ffrictionprime: m*(g-a);
(Ffrictionprime) 468.0
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of m, a, and g.
(%i5), (%i6) Calculate $F_{friction}$ and $F'_{friction}$.

Problem 5.11 A block of ice is pushed on a horizontal floor and released at a speed of 2.5 m s^{-1}. It moves a distance of 6.4 m before it comes to a complete stop. Calculate the coefficient of kinetic friction of the ice with the floor.

Solution
Figure 5.14 shows the ice block and the forces acting on it. There is no acceleration in the y direction, meaning that the net force in the y direction is zero i.e. $\Sigma F_y = 0$. The normal reaction, R, of the floor on the ice is the same in magnitude and opposite in direction to the weight, mg, of the ice. This means that, $R = mg$. In the x direction the net force is $\Sigma F_x = -F_{fric}$, where F_{fric} is the friction between the ice and the floor.

Using the definition of the coefficient of kinetic friction, we have,

$$\mu_k = \frac{F_{fric}}{R} = \frac{F_{fric}}{mg}.$$

The friction is $F_{fric} = \mu_k mg$. The friction decelerates the ice. Using the Newton's second law, we have,

$$\begin{aligned} \Sigma F &= ma, \\ -F_{fric} &= ma, \\ -\mu_k mg &= ma, \end{aligned}$$

$$\mu_k = -\frac{a}{g}. \tag{5.1}$$

The acceleration, a, is obtained from $v^2 - u^2 = 2as$, where the final speed of the ice block, v, is zero,

$$a = -\frac{u^2}{2s}. \tag{5.2}$$

The negative sign means this is a deceleration.
Thus, the coefficient of kinetic friction is,

$$\mu_k = -\frac{a}{g} = -\frac{\left(-u^2/2s\right)}{g} = \frac{u^2}{2sg} = \frac{(2.5\,\text{m/s})^2}{2(6.4\,\text{m})\left(9.8\,\text{m/s}^2\right)} = 0.05.$$

• wxMaxima codes:

```
(%i4) fpprintprec:5; u:2.5; s:6.4; g:9.8;
(fpprintprec) 5
(u) 2.5
(s) 6.4
(g) 9.8
(%i5) mu_k: u^2/(2*s*g);
(mu_k) 0.049825
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of u, s, and g.
(%i5) Calculate the coefficient of kinetic friction, μ_k.

• Alternative calculation by wxMaxima:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec) 5
(ratprint) false
(%i3) solve([mu_k=-a/g, a=-u^2/(2*s)], [mu_k,a]);
(%o3) [[mu_k=u^2/(2*g*s),a=-u^2/(2*s)]]
(%i4) subst([u=2.5, s=6.4, g=9.8], %);
(%o4) [[mu_k=0.049825,a=-0.48828]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3) Solve Eqs. (5.1) and (5.2) for μ_k and a.
(%o3) The solutions in symbols.
(%i4) Substitute values of u, s, and g into the solutions to get numerical values of μ_k and a.

Problem 5.12 Figure 5.15 shows a man pulling a box of mass $m = 45$ kg on a horizontal floor. The box is pulled with a constant speed and the angle between the rope and the horizontal is $\theta = 33°$. The coefficient of kinetic friction of the box with the floor is $\mu_k = 0.63$. What is the tension, T, in the rope?

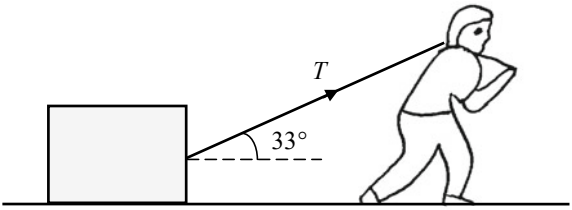

Fig. 5.15 A man pulling a box at a constant speed, Problem 5.12

Solution

Figure 5.16 shows the box and the forces acting on it. This is the free body diagram of the box. In the figure, F_{fric} is the kinetic frictional force of the box and the floor, R is the normal reaction of the floor on the box, mg is the weight of the box, and T is the tension in the rope. The tension, T, is resolved into the x component of $T\cos\theta$ and the y component of $T\sin\theta$.

The kinetic friction is,

$$F_{fric} = \mu_k R.$$

There is no acceleration in the x and y directions. This implies that the net force is zero, i.e. $\Sigma F_x = 0$ and $\Sigma F_y = 0$. This means that in the x direction,

$$T\cos\theta - F_{fric} = 0,$$

$$T\cos\theta - \mu_k R = 0,$$

and in the y direction,

$$R + T\sin\theta - mg = 0,$$

where the tension in the rope is resolved into x component of $T\cos\theta$ and y component of $T\cos\theta$. The last two equations are solved for the tension, T, in the rope,

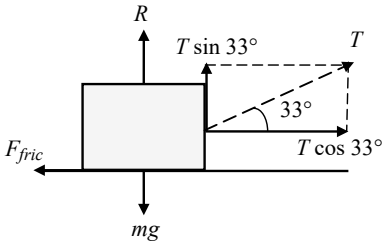

Fig. 5.16 Forces acting on the box, Problem 5.12

$$T\cos\theta - \mu_k(mg - T\sin\theta) = 0,$$
$$T\cos\theta - \mu_k mg + T\mu_k\sin\theta = 0,$$
$$T = \frac{\mu_k mg}{\cos\theta + \mu_k\sin\theta} = \frac{0.63(45\,\text{kg})\left(9.8\,\text{m/s}^2\right)}{\cos 33^\circ + 0.63\sin 33^\circ} = 235\,\text{N}.$$

• wxMaxima codes:

```
(%i6)       fpprintprec:5;       ratprint:false;       m:45;
theta:float(33*%pi/180); mu_k:0.63; g:9.8;
(fpprintprec) 5
(ratprint) false
(m) 45
(theta) 0.57596
(mu_k) 0.63
(g) 9.8
(%i7) T: mu_k*m*g/(cos(theta)+mu_k*sin(theta));
(T) 235.09
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m, θ, μ_k, and g.
(%i7) Calculate the tension in the rope, T.

• Alternative calculation:

```
(%i6) fpprintprec:5; ratprint:false; m:45; theta:float(33*%pi/
180); mu_k:0.63; g:9.8;
(fpprintprec) 5
(ratprint) false
(m) 45
(theta) 0.57596
(mu_k) 0.63
(g) 9.8
(%i8)       solve([T*cos(theta)-mu_k*R=0,       R+T*sin(theta)-
m*g=0],[T,R])$ float(%);
(%o8) [[T=235.09,R=312.96]]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m, θ, μ_k, and g.
(%i8) Solve $T\cos\theta - \mu_k R = 0$ and $R + T\sin\theta - mg = 0$ for T and R.

• Yet another alternative calculation:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec) 5
(ratprint) false
```

```
(%i3) sol: solve([T*cos(theta)-mu_k*R=0, R+T*sin(theta)-m*g=0],
[T,R]);
(sol) [[T=(g*m*mu_k)/(mu_k*sin(theta)+cos(theta)),
R=(g*m*cos(theta))/(mu_k*sin(theta)+cos(theta))]]
(%i4)    subst([m=45,    theta=float(33*%pi/180),    mu_k=0.63,
g=9.8],sol);
(%o4) [[T=235.09,R=312.96]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3) Solve $T\cos\theta - \mu_k R = 0$ and $R + T\sin\theta - mg = 0$ for T and R.
(sol) The solutions in symbols.
(%i4) Substitute values of m, θ, μ_k, and g into the solution to get numerical values of T and R.

Problem 5.13 A block of mass $m_1 = 6.0$ kg on a horizontal surface with coefficient of kinetic friction $\mu_k = 0.22$ is connected by a string passing over a light pulley to a block of mass $m_2 = 3.0$ kg (Fig. 5.17). The system is released from rest, moves by a distance $h = 0.50$ m, and the block 2 hits the floor.

(a) What are the acceleration, a, of the blocks and the tension, T, in the string?
(b) Calculate the speed, v, of the block 2 hitting the floor.

Solution

(a) Figure 5.18 shows the forces acting on both blocks. For block 1, the net force in the x direction is $T - \mu_k N$, where T is the tension in the string, $\mu_k N$ is the friction of the block and the surface, and N is the normal reaction of the table on the block.

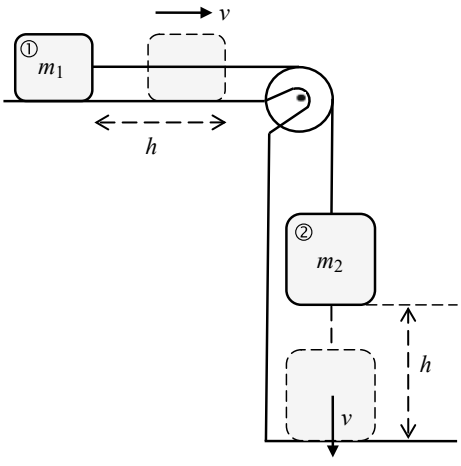

Fig. 5.17 Two-block system released from rest. Block 1 is on a rough surface. Problem 5.13

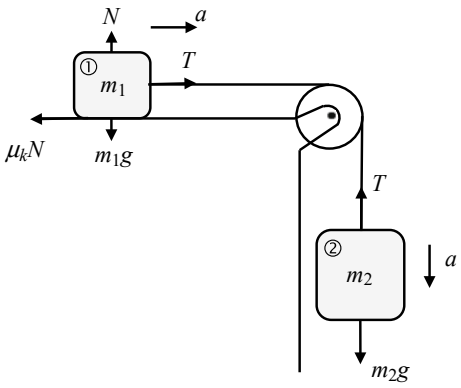

Fig. 5.18 Forces acting on the blocks, Problem 5.13

In the x direction, applying the Newton's second law to the block 1, we write,

$$\Sigma F_x = m_1 a,$$
$$T - \mu_k N = m_1 a,$$

where m_1 and a are the mass and the acceleration of the block. The normal reaction is the same in magnitude but opposite in direction to the weight of the block, i.e. $N = m_1 g$. The equation becomes,

$$T - \mu_k m_1 g = m_1 a. \tag{5.1}$$

The net force acting on the block 2 is $m_2 g - T$ downward, where $m_2 g$ is its weight and T is the tension in the string. Setting the y axis as positive downward and applying the Newton's second law to the block 2 gives,

$$\Sigma F_y = m_2 a,$$

$$m_2 g - T = m_2 a, \tag{5.2}$$

where a is the acceleration of the block 2. The magnitude of acceleration, a, is same for both blocks because they are connected by a string. Solving the two equations, one gets the acceleration, a, of the block as,

$$a = \frac{m_2 g - \mu_k m_1 g}{m_1 + m_2} = \frac{(3.0\,\text{kg})\left(9.8\,\text{m/s}^2\right) - 0.22(6.0\,\text{kg})\left(9.8\,\text{m/s}^2\right)}{6.0\,\text{kg} + 3.0\,\text{kg}} = 1.8\,\text{m s}^{-2}.$$

and the tension, T, in the string as,

$$T = \frac{m_1 m_2 g(1 + \mu_k)}{m_1 + m_2} = \frac{(6.0\,\text{kg})(3.0\,\text{kg})(9.8\,\text{N/kg})(1 + 0.22)}{6.0\,\text{kg} + 3.0\,\text{kg}} = 24\,\text{N}.$$

(b) The speed, v, at which the mass m_2 hits the floor is calculated as follows,

$$v^2 - u^2 = 2as, u = 0, s = h \Rightarrow v^2 = 2ah, \tag{5.3}$$

$$\begin{aligned} v = \sqrt{2ah} &= \sqrt{2\left(\frac{m_2 - \mu_k m_1}{m_1 + m_2}\right)gh} \\ &= \sqrt{2\left(\frac{3.0\text{ kg} - 0.22 \times 6.0\text{ kg}}{6.0\text{ kg} + 3.0\text{ kg}}\right)(9.8\text{ m/s}^2)(0.50\text{ m})} \\ &= 1.4\text{ m s}^{-1}. \end{aligned}$$

• wxMaxima codes:

```
(%i7) fpprintprec:5; ratprint:false; m1:6; m2:3; mu_k:0.22;
g:9.8; h:0.5;
(fpprintprec) 5
(ratprint) false
(m1) 6
(m2) 3
(mu_k) 0.22
(g) 9.8
(h) 0.5
(%i9)   solve([T-mu_k*m1*g=m1*a,   m2*g-T=m2*a],   [a,T])$
float(%);
(%o9) [[a=1.8293,T=23.912]]
(%i10) a: rhs(%[1][1]);
(a) 1.8293
(%i12) solve(v^2=2*a*h, v)$ float(%);
(%o12) [v=-1.3525,v=1.3525]
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of m_1, m_2, μ_k, g, and h.
(%i9) Solve $T - \mu_k m_1 g = m_1 a$ and $m_2 g - T = m_2 a$ for a and T. Part (a).
(%i10) Pick value of a.
(%i12) Solve $v^2 = 2ah$ for v. Part (b).

• Alternative calculation:

```
(%i7) fpprintprec:5; ratprint:false; m1:6; m2:3; mu_k:0.22; g:9.8;
h:0.5;
(fpprintprec) 5
(ratprint) false
(m1) 6
(m2) 3
(mu_k) 0.22
```

```
(g) 9.8
(h) 0.5
(%i8) a: (m2*g-mu_k*m1*g)/(m1+m2);
(a) 1.8293
(%i9) T: m1*m2*g*(1+mu_k)/(m1+m2);
(T) 23.912
(%i10) v: sqrt(2*g*h*(m2-mu_k*m1)/(m1+m2));
(v) 1.3525
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of m_1, m_2, μ_k, g, and h.
(%i8), (%i9), (%i10) Calculate acceleration, a, of the block, tension, T, in the string, and speed, v, of the block.

• Yet another alternative calculation by wxMaxima:

```
(%i2)fpprintprec:5; ratprint:false;
(fpprintprec) 5
(ratprint) false
(%i3) eq1: T-mu_k*m1*g=m1*a;
(eq1) T-g*m1*mu_k=a*m1
(%i4) eq2: m2*g-T=m2*a;
(eq2) g*m2-T=a*m2
(%i5) eq3: v^2=2*a*h;
(eq3) v^2=2*a*h
(%i6) solve([eq1,eq2,eq3], [a, T, v]);
(%o6)    [[a=-(g*m1*mu_k-g*m2)/(m2+m1),T=(g*m1*m2*mu_k+g*m1*m2)/
(m2+m1),
v=sqrt(2)*sqrt(-(g*h*m1*mu_k-g*h*m2)/(m2+m1))],
[a=-(g*m1*mu_k-g*m2)/(m2+m1),T=(g*m1*m2*mu_k+g*m1*m2)/(m2+m1),
v=-sqrt(2)*sqrt(-(g*h*m1*mu_k-g*h*m2)/(m2+m1))]]
(%i8)subst([m1=6, m2=3, mu_k=0.22, h=0.5, g=9.8], %)$ float(%);
(%o8)[[a=1.8293,T=23.912,v=1.3525],
[a=1.8293,T=23.912,v=1.3525]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3), (%i4), (%i5) Assign Eqs. (5.1), (5.2), and (5.3) as eq1, eq2, and eq3, respectively.
(%i6) Solve Eqs. (5.1), (5.2), and (5.3) for a, T, and v.
(%o6) Solutions in symbols.
(%i8) Substitute values of m_1, m_2, μ_k, h, and g into the solutions to get numerical values of a, T, and v.

Problem 5.14 Figure 5.19 shows a block of mass 2.0 kg placed on top of a 5.0 kg block on a horizontal floor. The coefficient of kinetic friction of the block with the floor is 0.05. Determine,

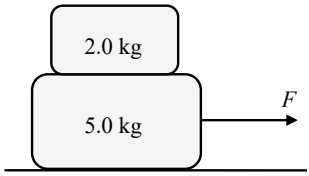

Fig. 5.19 Two blocks, one on top of the other, on a rough floor. A force F acts on the lower block, Problem 5.14

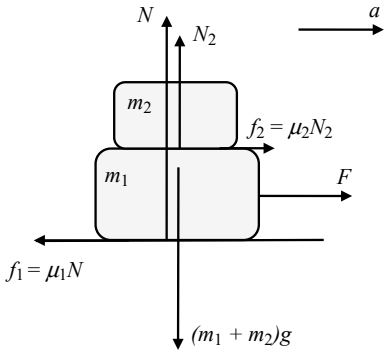

Fig. 5.20 Forces acting on the blocks, Problem 5.14

(a) the force, F, to accelerate both blocks at 3.0 m s^{-2},
(b) the minimum coefficient of friction between the two blocks so that the block of mass 2.0 kg does not slip by the 3.0 m s^{-2} acceleration.

Solution

(a) Figure 5.20 shows the forces acting on the blocks when the acceleration is $a =$ 3.0 m s^{-2}.

In the y direction, the normal reaction of the floor on the blocks N is balanced by the weight of the blocks $(m_1 + m_2)g$. The net force in the direction is zero. In the x direction, the net force to the right is $F - f_1$, where F is the pulling force and f_1 is the friction between the floor and the bottom block. Applying the Newton's second law in the y and x directions gives,

$$\Sigma F_y = N - (m_1 + m_2)g = 0,$$

$$\Sigma F_x = F - f_1 = (m_1 + m_2)a. \tag{5.1}$$

The friction between the floor and the bottom block is,

$$f_1 = \mu_1 N = \mu_1(m_1 + m_2)g.$$

This means that the pulling force is,

$$F = f_1 + (m_1 + m_2)a = \mu_1(m_1 + m_2)g + (m_1 + m_2)a$$

$$
\begin{aligned}
&= (m_1 + m_2)(\mu_1 g + a) \\
&= (5.0\,\text{kg} + 2.0\,\text{kg})\left[0.05\left(9.8\,\text{m/s}^2\right) + 3.0\,\text{m/s}^2\right] \\
&= 24\,\text{N}.
\end{aligned}
$$

• wxMaxima codes:

```
(%i7) fpprintprec:5; ratprint:false; m1:5; m2:2; mu1:0.05;
a:3; g:9.8;
(fpprintprec) 5
(ratprint) false
(m1) 5
(m2) 2
(mu1) 0.05
(a) 3
(g) 9.8
(%i8) f1: mu1*(m1+m2)*g;
(f1) 3.43
(%i10)   solve([N-(m1+m2)*g=0,   F-f1=(m1+m2)*a],   [F,N])$
float(%);
(%o10) [[F=24.43,N=68.6]]
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of m_1, m_2, μ_1, a, and g.
(%i8) Calculate the friction between the floor and the first block, f_1.
(%i10) Solve $N - (m_1 + m_2)g = 0$ and $F - f_1 = (m_1 + m_2)a$ for F and N.

(b) To have the top block accelerates at $a = 3.0$ m s^{-2}, there must be some net force acting on it. The force is the friction between the bottom and top block to the right with magnitude,

$$f_2 = m_2 a. \tag{5.2}$$

This frictional force accelerates the second block. The minimum coefficient of friction of the bottom with the top block is,

$$\mu_2 = \frac{\text{friction}}{\text{normal reaction}} = \frac{f_2}{N_2} = \frac{m_2 a}{m_2 g} = \frac{a}{g} = \frac{3.0\,\text{m/s}^2}{9.8\,\text{m/s}^2} = 0.31,$$

where N_2 is the normal reaction of the bottom on the top block with magnitude equals the weight of the top block, $m_2 g$.

• wxMaxima codes:

```
(%i6) fpprintprec:5; m1:5; m2:2; mu1:0.05; a:3; g:9.8;
(fpprintprec) 5
(m1) 5
(m2) 2
(mu1) 0.05
(a) 3
(g) 9.8
(%i7) mu2: a/g;
(mu2) 0.30612
```

Comments on the codes:

(%i6) Set floating point print precision to 5, assign values of m_1, m_2, μ_1, a, and g.
(%i8) Calculate μ_2.

• Alternative calculation:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec) 5
(ratprint) false
(%i6) N: (m1+m2)*g; N2:m2*g; f1:mu1*N; f2:mu2*N2;
(N) g*(m2+m1)
(N2) g*m2
(f1) g*(m2+m1)*mu1
(f2) g*m2*mu2
(%i8) eq1: F-f1=(m1+m2)*a; eq2: f2=m2*a;
(eq1) F-g*(m2+m1)*mu1=a*(m2+m1)
(eq2) g*m2*mu2=a*m2
(%i9) solve([eq1,eq2], [F,mu2]);
(%o9) [[F=(g*m2+g*m1)*mu1+a*m2+a*m1,mu2=a/g]]
(%i10) subst([m1=5, m2=2, mu1=0.05, a=3, g=9.8], %);
(%o10) [[F=24.43,mu2=0.30612]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i6) Assign $N = (m_1 + m_2)g$, $N_2 = m_2 g$, $f_1 = \mu_1 N$, and $f_2 = \mu_2 N_2$.
(%i8) Assign Eqs. (5.1) and (5.2) as eq1 and eq2, respectively.
(%i9) Solve Eqs. (5.1) and (5.2) for F and μ_2.
(%o9) The solutions in symbols.
(%i10) Substitute values of m_1, m_2, μ_1, a, and g into the solutions to get numerical values of F and μ_2.

Problem 5.15 A block of mass $m = 18$ kg is placed on an inclined plane of angle $\theta = 30°$ with the horizontal. Determine,

(a) the force needed to pull the block up along the plane at a constant speed and the friction between the block and the plane,

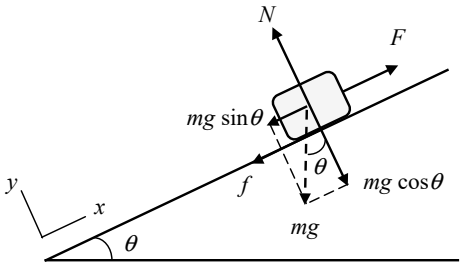

Fig. 5.21 A block is pulled up along a rough inclined plane at a constant speed

(b) the force needed to pull the block down along the plane at a constant speed if the coefficient of kinetic friction of the block with the plane is $\mu_k = 0.60$.

Solution

(a) Figure 5.21 shows the block, the inclined plane, and the forces acting on the block. The forces are the weight, mg, of the block pointing vertically downward, the normal reaction, N, of the plane on the block, the friction, f, of the block and the plane pointing down along the plane, and the pulling force, F, pointing up along the plane. The weight, mg, of the block is resolved into x component (parallel to the plane) of $mg\cos\theta$ and y component (perpendicular to the plane) of $mg\cos\theta$. These weight components replace the weight, mg, in the calculation.

The normal reaction, N, of the plane on the block is balanced by the weight component of $mg\cos\theta$. The net force in the y direction is zero and there is no motion in the direction,

$$\Sigma F_y = N - mg\cos\theta = 0. \tag{5.1}$$

Because the block moves up along the plane (along the positive x direction) at a constant speed (zero acceleration), the net force in the direction is zero as well,

$$\Sigma F_x = F - f - mg\ \sin\theta = 0. \tag{5.2}$$

The friction is $f = \mu_k N$ and $N = mg\cos\theta$. Therefore, the pulling force, F, is,

$$\begin{aligned} F &= f + mg\sin\theta = \mu_k N + mg\sin\theta = \mu_k mg\cos\theta + mg\sin\theta \\ &= 0.60(18\,\text{kg})\left(9.8\,\text{m/s}^2\right)\cos 30^\circ + (18\,\text{kg})\left(9.8\,\text{m/s}^2\right)\sin 30^\circ \\ &= 180\,\text{N}. \end{aligned}$$

The friction, f, is,

$$f = \mu_k N. \tag{5.3}$$

This means that,

$$\begin{aligned} f &= \mu_k mg\cos\theta \\ &= 0.60(18\,\text{kg})\left(9.8\,\text{m/s}^2\right)\cos 30^\circ \\ &= 92\,\text{N}. \end{aligned}$$

• wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; m:18; theta:float(30/
180*%pi); g:9.8; mu_k:0.6;
(fpprintprec) 5
(ratprint) false
(m) 18
(theta) 0.5236
(g) 9.8
(mu_k) 0.6
(%i8)                    solve([f/mu_k-m*g*cos(theta)=0,F-f-
m*g*sin(theta)=0],[F,f])$ float(%);
(%o8) [[F=179.86,f=91.66]]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m, θ, g, and μ_k.
(%i8) Solve $f/\mu_k - mg\cos\theta = 0$ and $F - f - mg\ \sin\theta = 0$ for F and f.

• Alternative calculation:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec) 5
(ratprint) false
(%i3) eq1: N-m*g*cos(theta)=0;
(eq1) N-g*m*cos(theta)=0
(%i4) eq2: F-f-m*g*sin(theta)=0;
(eq2) -g*m*sin(theta)-f+F=0
(%i5) eq3: f=mu_k*N;
(eq3) f=N*mu_k
(%i6) solve([eq1,eq2,eq3], [F,f,N]);
(%o6)[[F=g*m*sin(theta)+g*m*mu_k*cos(theta),f=g*m*mu_
k*cos(theta),
N=g*m*cos(theta)]]
(%i7) subst([m=18, theta=30/180*float(%pi), mu_k=0.6, g=9.8], %);
(%o7) [[F=179.86,f=91.66,N=152.77]]
```

Comments on the codes:

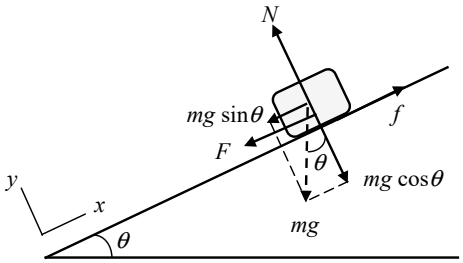

Fig. 5.22 A block is pulled down along a rough inclined plane at a constant speed

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3), (%i4), (%i5) Assign Eqs. (5.15.1), (5.15.2), and (5.15.3) as eq1, eq2, and eq3, respectively.
(%i6) Solve Eqs. (5.1), (5.2), and (5.3) for F, f, and N.
(%o6) The solutions in symbols.
(%i7) Substitute values of m, θ, μ_k, and g into the solutions to get numerical values of F, f, and N.

(b) Figure 5.22 shows the forces acting on the block. The forces are the weight, mg, of the block pointing vertically downward, the normal reaction, N, of the plane on the block, the friction, f, of the block and the plane pointing up along the plane, and the pulling force, F, pointing down along the plane. This time the block moves down on the plane at a constant speed. The weight, mg, of the block is resolved into x and y components of $mg \cos\theta$ and $mg \sin\theta$, respectively.

In the y direction, the normal reaction, N, of the plane on the block is balanced by the weight component of $mg \cos\theta$. We write,

$$\Sigma F_y = N - mg\ \cos\theta = 0,$$

$$N = mg\ \cos\theta. \tag{5.4}$$

In the x direction, the net force is zero as well because the block moves at a constant speed. Thus,

$$\Sigma F_x = f - F - mg\ \sin\theta = 0. \tag{5.5}$$

The friction, f, is,

$$f = \mu_k N. \tag{5.6}$$

This means that,

$$\begin{aligned} f &= \mu_k mg\cos\theta \\ &= 0.60(18\,\text{kg})\left(9.8\,\text{m/s}^2\right)\cos 30^\circ \\ &= 92\,\text{N}. \end{aligned}$$

The friction is the same as in part (a) that is $f = 92$ N.
The pulling force, F, is,

$$\begin{aligned} F &= f - mg\sin\theta = \mu_k mg\cos\theta - mg\sin\theta \\ &= 0.6(18\,\text{kg})\left(9.8\,\text{m/s}^2\right)\cos 30^\circ - (18\,\text{kg})\left(9.8\,\text{m/s}^2\right)\sin 30^\circ \\ &= 3.5\,\text{N}. \end{aligned}$$

• wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; m:18; theta:float(30/
180*%pi); g:9.8; mu_k:0.6;
(fpprintprec) 5
(ratprint) false
(m) 18
(theta) 0.5236
(g) 9.8
(mu_k) 0.6
(%i8)solve([f/mu_k-m*g*cos(theta)=0,f-F-
m*g*sin(theta)=0],[f,F])$ float(%);
(%o8) [[f=91.66,F=3.4601]]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m, θ, g, and μ_k.
(%i8) Solve $f/\mu_k - mg\cos\theta = 0$ and $f - F - mg\ \sin\ \theta = 0$ for f and F.

• Alternative calculaton:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec) 5
(ratprint) false
(%i3) eq4: N=m*g*cos(theta);
(eq4) N=g*m*cos(theta)
(%i4) eq5: f-F-m*g*sin(theta)=0;
(eq5)-g*m*sin(theta)+f-F=0
(%i5) eq6: f=mu_k*N;
(eq6) f=N*mu_k
(%i6) solve([eq4,eq5,eq6], [f,F,N]);
(%o6)            [[f=g*m*mu_k*cos(theta),F=g*m*mu_k*cos(theta)-
g*m*sin(theta),
N=g*m*cos(theta)]]
(%i7) subst([m=18, theta=30/180*float(%pi), mu_k=0.6, g=9.8], %);
```

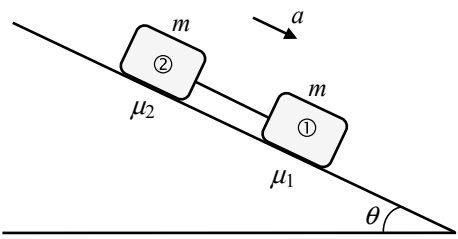

Fig. 5.23 Two blocks connected by a string on a rough inclined plane, Problem 5.16

```
(%o7) [[f=91.66,F=3.4601,N=152.77]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3), (%i4), (%i5) Assign Eqs. (5.4), (5.5), and (5.6) as eq4, eq5, and eq6, respectively.
(%i6) Solve Eqs. (5.4), (5.5), and (5.6) for f, F, and N.
(%o6) The solutions in symbols.
(%i7) Substitute values of m, θ, μ_k, and g into the solutions to get numerical values of f, F, and N.

Problem 5.16 Two blocks, each of mass m and made of different materials are connected by a string that accelerate down along an inclined plane (Fig. 5.23). The coefficients of kinetic friction of block 1 and block 2 with the plane are μ_1 and μ_2, respectively, and $\mu_2 > \mu_1$. Determine,

(a) the tension, T, in the string,
(b) the acceleration, a, of the blocks,
(c) the inclination angle, θ_m, such that the blocks move with a constant speed.
(d) numerical values of T, a, and θ_m, if $m = 3.0$ kg, $\mu_1 = 0.20$, $\mu_2 = 0.30$, and $\theta = 50°$.

Solution

(a) Figure 5.24 shows the two blocks, the inclined plane, and the relevant forces. For block 1, the forces are its weight, mg, kinetic friction, $\mu_1 N$, tension, T, in the string, and normal reaction, N, of the plane on the block. The weight, mg, is resolved into $mg \sin\theta$ along the x direction and $mg \cos\theta$ along the y direction. For block 2, the forces are its weight, mg, kinetic friction, $\mu_2 N$, tension, T, in the string, and normal reaction, N, of the plane on the block. The weight, mg, is resolved into $mg \sin\theta$ along the x direction and $mg \cos\theta$ along the y direction.

In the x direction, the net force on the block 2 is,

$$\Sigma F_x = T + mg \sin\ \theta - \mu_2 N,$$

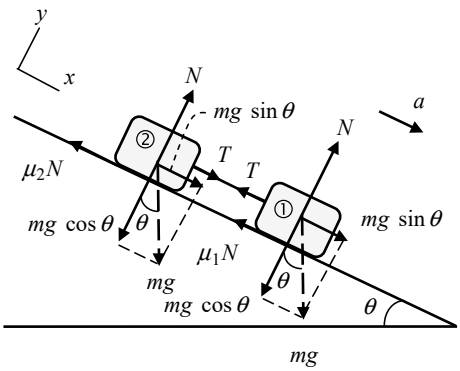

Fig. 5.24 Forces acting on the blocks, Problem 5.16

where T is the tension in the string, $mg \sin\theta$ is the weight component parallel to the incline, $\mu_2 N$ is the kinetic friction between the incline and the block, and N is the normal reaction of the incline on the block. This net force must be greater than zero because block 2 moves with the acceleration, a. Applying the Newton's second law to the block 2 and assuming the block slips with acceleration, a, we have,

$$\Sigma F_x = T + mg \sin\ \theta - \mu_2 N = ma,$$

$$T + mg \sin\ \theta - \mu_2 mg\ \cos\ \theta = ma, \tag{5.1}$$

because the normal reaction is $N = mg \cos\theta$. The fact that $N = mg \cos\theta$ is deduced from the knowledge that in the y direction the net force is zero because there is no motion in this direction. The normal reaction, N, is equal in magnitude and opposite in direction to $mg \cos\theta$.

In the x direction, the net force on the block 1 is,

$$\Sigma F_x = mg \sin\ \theta - \mu_1 N - T,$$

where $mg \sin\theta$ is the weight component of the block 1 parallel to the incline, $\mu_1 N$ is the kinetic friction between the block and the incline, and T is the tension in the string. The net force is greater than zero because the block moves down along the incline with acceleration, a. Applying the Newton's second law to the block 1 gives,

$$\Sigma F_x = mg \sin\ \theta - \mu_1 N - T = ma,$$

$$mg \sin\ \theta - \mu_1 mg\ \cos\ \theta - T = ma, \tag{5.2}$$

where the fact that $N = mg \cos\theta$ has been used.

Both blocks move with acceleration, a, and the string is always in tension, T. For block 2, T pulls it down along the incline. For block 1, T pulls it up along the incline.

Solving the two equations gives the tension, T, in the string,

$$T = \frac{1}{2}(\mu_2 - \mu_1)mg \cos\ \theta.$$

(b) The acceleration of any block, a, can be calculated by substituting the tension, T, in the string in the Newton's equation for block 2 given by,

$$\begin{aligned} ma &= T + mg \sin\ \theta - \mu_2 mg \cos\ \theta, \\ a &= \frac{1}{m}(T + mg \sin\ \theta - \mu_2 mg \cos\ \theta) \\ &= \frac{1}{m}[\frac{1}{2}(\mu_2 - \mu_1)mg \cos\ \theta + mg \sin\ \theta - \mu_2 mg \cos\ \theta] \\ &= g\ [\sin\ \theta - \frac{1}{2}(\mu_1 + \mu_2) \cos\ \theta]. \end{aligned}$$

The acceleration can also be obtained from the Newton's equation for block 1,

$$\begin{aligned} ma &= mg \sin\ \theta - \mu_1 mg \cos\ \theta - T, \\ a &= g \sin\ \theta - \mu_1 g \cos\ \theta - \frac{T}{m} \\ &= g \sin\ \theta - \mu_1 g \cos\ \theta - \frac{\frac{1}{2}(\mu_2 - \mu_1)mg \cos\ \theta}{m} \\ &= g\ [\sin\ \theta - \frac{1}{2}(\mu_1 + \mu_2) \cos\ \theta]. \end{aligned}$$

(c) Let the blocks move with a constant speed when the inclination angle is θ_m. This means that the acceleration, $a = 0$ when $\theta = \theta_m$. Substituting the values in the result of part (b),

$$a = g\ [\sin\ \theta - \frac{1}{2}(\mu_1 + \mu_2) \cos\ \theta],$$

$$0 = g\ [\sin\ \theta_m - \frac{1}{2}(\mu_1 + \mu_2) \cos\ \theta_m]. \tag{5.3}$$

The inclination angle, θ_m, is calculated as,

$$\tan\theta_m = \frac{1}{2}(\mu_1 + \mu_2), \quad \theta_m = \tan^{-1}[\frac{1}{2}(\mu_1 + \mu_2)].$$

(d) Using values of $m = 3.0$ kg, $\mu_1 = 0.20$, $\mu_2 = 0.30$, and $\theta = 50°$, we obtain,

$$T = \frac{1}{2}(\mu_2 - \mu_1)mg\cos\theta = \frac{1}{2}(0.30 - 0.20)(3.0\ \text{kg})\left(9.8\ \text{m s}^{-2}\right)\cos 50^{\circ}$$

$$= 0.94 \text{ N},$$

$$a = g\left[\sin\theta - \frac{1}{2}(\mu_1 + \mu_2)\cos\theta\right]$$

$$= \left(9.8 \text{ m s}^{-2}\right)\left[\sin 50^\circ - \frac{1}{2}(0.20 + 0.30)\cos 50^\circ\right]$$

$$= 5.9 \text{ m s}^{-2},$$

$$\theta_m = \tan^{-1}\left[\frac{1}{2}(\mu_1 + \mu_2)\right] = \tan^{-1}\left[\frac{1}{2}(0.20 + 0.30)\right] = 0.24 \text{ rad}$$

$$= 14^\circ.$$

• wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec) 5
(ratprint) false
(%i3) sol: solve([T+m*g*sin(theta)-mu2*m*g*cos(theta)=m*a,
m*g*sin(theta)-mu1*m*g*cos(theta)-T=m*a], [T,a]);
(sol) [[T=((g*m*mu2-g*m*mu1)*cos(theta))/2,
a=(2*g*sin(theta)+(-g*mu2-g*mu1)*cos(theta))/2]]
(%i4) a: rhs(sol[1][2]);
(a) (2*g*sin(theta)+(-g*mu2-g*mu1)*cos(theta))/2
(%i5) solve(a=0, sin(theta));
(%o5) [sin(theta)=((mu2+mu1)*cos(theta))/2]
(%i6) subst([m=3, mu1=0.2, mu2=0.3, theta=float(50/180*%pi),
g=9.8], sol);
(%o6) [[T=0.9449,a=5.9324]]
(%i8) mu1:0.2; mu2:0.3;
(mu1) 0.2
(mu2) 0.3
(%i10) theta_m: atan(0.5*(mu1+mu2)); deg: float(theta_m/
%pi*180);
(theta_m) 0.24498
(deg)14.036
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3) Solve $T + mg\sin\theta - \mu_2 mg\cos\theta = ma$ and $mg\sin\theta - \mu_1 mg\cos\theta - T = ma$ for T and a.
(sol) The solutions. Part (a) and (b).
(%i4) Assign $a = g\,[\sin\theta - \frac{1}{2}(\mu_1 + \mu_2)\cos\theta]$.
(%i5) Solve $g\,[\sin\theta - \frac{1}{2}(\mu_1 + \mu_2)\cos\theta] = 0$ for $\sin\theta$. Part (c).
(%o5) Solution is $\sin\theta = \frac{1}{2}(\mu_1 + \mu_2)\cos\theta \Rightarrow \tan\theta = \frac{1}{2}(\mu_1 + \mu_2) \Rightarrow \theta = \tan^{-1}[\frac{1}{2}(\mu_1 + \mu_2)]$. Part (c).

(%i6) Substitute values of m, μ_1, μ_2, θ, and g into (sol) to get numerical values of T and a. Part (d).
(%i8), (%i10) Assign values of μ_1 and μ_2, calculate θ_m and convert the angle to degree. Part (d).

• Alternative calculation:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec) 5
(ratprint) false
(%i3) eq1: T+m*g*sin(theta)-mu2*m*g*cos(theta)=m*a;
(eq1) g*m*sin(theta)-g*m*mu2*cos(theta)+T=a*m
(%i4) eq2: m*g*sin(theta)-mu1*m*g*cos(theta)-T=m*a;
(eq2) g*m*sin(theta)-g*m*mu1*cos(theta)-T=a*m
(%i5) solve([eq1,eq2], [T,a]);
(%o5) [[T=((g*m*mu2-g*m*mu1)*cos(theta))/2,a=
(2*g*sin(theta)+(-g*mu2-g*mu1)*cos(theta))/2]]
(%i6) eq3: 0=g*(sin(theta_m)-0.5*(mu1+mu2)*cos(theta_m));
(eq3) 0=g*(sin(theta_m)-0.5*(mu2+mu1)*cos(theta_m))
(%i7) solve(eq3, sin(theta_m));
(%o7) [sin(theta_m)=((mu2+mu1)*cos(theta_m))/2]
(%i8) tan_theta_m: rhs(%[1]/cos(theta_m));
(tan_theta_m) (mu2+mu1)/2
(%i9) theta_m: atan(tan_theta_m);
(theta_m) atan((mu2+mu1)/2)
(%i10)  subst([m=3,  mu1=0.2,  mu2=0.3,  theta=50/180*float(%pi),
g=9.8], %o5);
(%o10) [[T=0.9449,a=5.9324]]
(%i11) theta_m: subst([mu1=0.2,mu2=0.3], theta_m);
(theta_m) 0.24498
(%i12) theta_m_deg: theta_m/float(%pi)*180;
(theta_m_deg) 14.036
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3), (%i4) Assign Eqs. (5.1) and (5.2) as eq1 and eq2, respectively.
(%i5) Solve Eqs. (5.1) and (5.2) symbolicly for T and a.
(%i6) Assign Eq. (5.3) as eq3.
(%i7), (%i8), (%i9) Solve Eq. (5.3) symbolicly for θ_m.
(%i10) Substitute values of m, μ_1, μ_2, θ, and g into %o5 to get numerical values of T and a.
(%i11), (%i12) Substitute values of μ_1 and μ_2 to get numerical value of θ_m in rad and degree.

Problem 5.17 A bock of mass 3.0 kg on an inclined plane of angle 30° and another block of mass 4.0 kg are connected by a string passing over a light pulley (Fig. 5.25). The coefficient of dynamic friction of the 3.0 kg block with the incline is 0.20. The system is released from rest. Calculate,

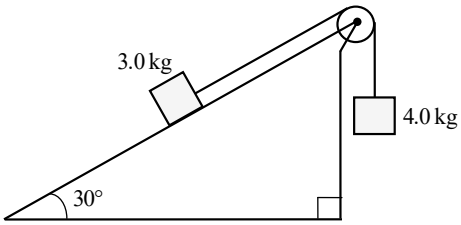

Fig. 5.25 Two blocks connected by a string of Problem 5.17

(a) the acceleration, a, of the 3.0 kg block,
(b) the tension, T, in the string.

Solution

In the y direction, the net force on the 3.0 kg block is zero,

$$\Sigma F_y = R - mg\ \cos\ \theta = 0,$$

and there is no motion in this direction. Here, R is the normal reaction of the incline on the block and $mg\cos\theta$ is the component of the weight perpendicular to the incline. In the x direction, the net force on the block is $T - mg\sin\theta - f$, where T is the tension in the string, $mg\ \sin\ \theta$ is the weight component parallel to the incline, and f is the kinetic friction between the block and the incline. The kinetic friction is $f = \mu R = \mu mg\cos\theta$. Applying Newton's second law to the 3.0 kg block, one gets,

$$\Sigma F_x = ma,$$

$$T - mg\ \sin\theta - f = ma,$$

$$T - mg\ \sin\theta - \mu mg\ \cos\ \theta = ma, \tag{5.1}$$

where a is the acceleration of the block.

For the 4.0 kg block (Fig. 5.26b), the net force on it is $Mg - T$ downward. Applying Newton's second law to this block gives,

$$\Sigma F_y = Ma,$$

$$Mg - T = Ma. \tag{5.2}$$

Solving the two equations gives the acceleration of the block as,

$$a = \frac{Mg - mg\sin\theta - \mu mg\cos\theta}{M + m}$$

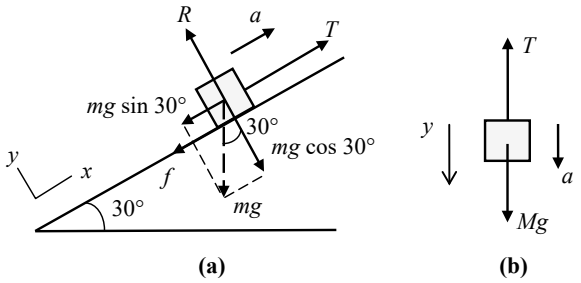

Fig. 5.26 Forces acting on the 3.0 kg (**a**) and 4.0 kg (**b**) blocks, Problem 5.17

$$= \frac{(4.0\,\text{kg})\left(9.8\,\text{m}\,\text{s}^{-2}\right) - (3.0\,\text{kg})\left(9.8\,\text{m}\,\text{s}^{-2}\right)\sin 30^{\circ} - 0.20(3.0\,\text{kg})\left(9.8\,\text{m}\,\text{s}^{-2}\right)\cos 30^{\circ}}{4.0\,\text{kg} + 3.0\,\text{kg}}$$

$$= 2.8\,\text{m}\,\text{s}^{-2}.$$

(b) The tension in the string is,

$$T = Mg - Ma = (4.0\,\text{kg})\left(9.8\,\text{m/s}^2\right) - (4.0\,\text{kg})\left(2.8\,\text{m/s}^2\right)$$
$$= 28\,\text{N}.$$

• wxMaxima codes:

```
(%i7)    fpprintprec:5;    ratprint:false;    m:3;    M:4;
theta:float(30/180*%pi); g:9.8; mu:0.2;
(fpprintprec) 5
(ratprint) false
(m) 3
(M) 4
(theta) 0.5236
(g) 9.8
(mu) 0.2
(%i9)       solve([T-m*g*sin(theta)-mu*m*g*cos(theta)=m*a,
M*g-T=M*a], [a,T])$ float(%);
(%o9) [[a=2.7725,T=28.11]]
(%i10) a: (M*g-m*g*sin(theta)-mu*m*g*cos(theta))/(M+m); /*
check */
(a) 2.7725
(%i11) T: M*g-M*a; /* check */
(T) 28.11
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of m, M, θ, g, and μ.
(%i9) Solve $T - mg\,\sin\theta - \mu mg\,\cos\,\theta = ma$ and $Mg - T = Ma$ for a and T.
(%o9) The solutions.

(%i10), (%i11) Recalculations of a and T as checks.

• Alternative calculation:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec) 5
(ratprint) false
(%i3) eq1: T-m*g*sin(theta)-mu*m*g*cos(theta)=m*a;
(eq1) -g*m*sin(theta)-g*m*mu*cos(theta)+T=a*m
(%i4) eq2: M*g-T=M*a;
(eq2) M*g-T=M*a
(%i5) solve([eq1,eq2], [a,T]);
(%o5) [[a=-(g*m*sin(theta)+g*m*mu*cos(theta)-M*g)/(m+M),
T=(M*g*m*sin(theta)+M*g*m*mu*cos(theta)+M*g*m)/(m+M)]]
(%i6) subst([m=3, M=4, mu=0.2, theta=30/180*float(%pi), g=9.8],
%);
(%o6) [[a=2.7725,T=28.11]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3), (%i4) Assign Eqs. (5.1) and (20 as eq1 and eq2, respectively.
(%i5) Solve Eqs. (5.1) and (5.2) symbolicly for a and T.
(%o5) The solutions in symbols.
(%i6) Substitute values of m, M, μ, θ, and g to get numerical values of a and T.

Problem 5.18 A block of mass 3.0 kg is placed on an inclined plane and connected by a light string passing over a light pulley to another block of mass 1.0 kg (Fig. 5.27). The inclination angle is 50°. The block of mass 3.0 kg slips down the incline with acceleration of $a = 1.0$ m s^{-2}. Determine,

(a) the tension, T, in the string,
(b) the coefficient of kinetic friction, μ, of the 3.0 kg block with the incline.

Solution

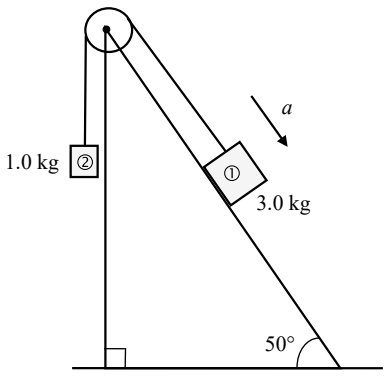

Fig. 5.27 Two blocks connected by a string of Problem 5.18

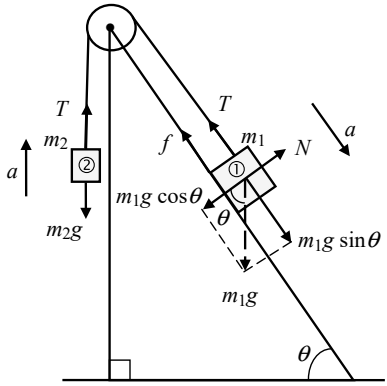

Fig. 5.28 Forces acting on the blocks, Problem 5.18

(a) Figure 5.28 shows the forces that act on the blocks. The net force on the 3.0 kg block is,

$$\begin{aligned} m_1 g \sin\theta - T - f &= m_1 g \sin\theta - T - \mu N \\ &= m_1 g \sin\theta - T - \mu m_1 g \cos\theta, \end{aligned}$$

where $m_1 g \sin\theta$ is the weight component of the block parallel to the incline, T is the tension in the string, and f is the kinetic friction between the block and the incline. The kinetic friction f is,

$$f = \mu N = \mu m_1 g \cos\theta,$$

where μ is the coefficient of kinetic friction of the block and the incline, and $N = m_1 g \cos\theta$ is the normal reaction of the incline on the block.

Applying the Newton's second law to the block of mass 3.0 kg, we have,

$$m_1 g \sin\theta - T - \mu m_1 g \cos\theta = m_1 a, \tag{5.1}$$

where a is the acceleration of the block.

The net force on the block of mass 1.0 kg is $T - m_2 g$ pointing upward, where $m_2 g$ is the weight of the block and T is the tension in the string. Applying the Newton's second law to the block of mass 1.0 kg, we have,

$$T - m_2 g = m_2 a, \tag{5.2}$$

where a is the acceleration of the block. The tension, T, in the sting is,

$$\begin{aligned} T &= m_2 g + m_2 a = m_2 (g + a) \\ &= (1.0\,\text{kg})\left(9.8\,\text{m/s}^2 + 1.0\,\text{m/s}^2\right) \end{aligned}$$

$$= 11\,\text{N}.$$

(b) The coefficient of kinetic friction, μ, is,

$$\mu = \frac{m_1 g\sin\theta - T - m_1 a}{m_1 g\cos\theta} = \frac{m_1 g\sin\theta - m_2(g+a) - m_1 a}{m_1 g\cos\theta}$$
$$= \frac{(3.0\,\text{kg})\left(9.8\,\text{m/s}^2\right)\sin 50^\circ - (1.0\,\text{kg})\left(9.8\,\text{m/s}^2 + 1.0\,\text{m/s}^2\right) - (3.0\,\text{kg})\left(1.0\,\text{m/s}^2\right)}{(3.0\,\text{kg})\left(9.8\,\text{m/s}^2\right)\cos 50^\circ}$$
$$= 0.46.$$

• wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec) 5
(%i2) solve([m1*g*sin(theta)-T-mu*m1*g*cos(theta)=m1*a, T-
m2*g=m2*a], [T,mu]);
(%o2)    [[T=g*m2+a*m2,mu=(g*m1*sin(theta)-g*m2-a*m2-a*m1)/
(g*m1*cos(theta))]]
(%i3)  subst([m1=3,  m2=1,  theta=float(50/180*%pi),  a=1,
g=9.8], %);
(%o3) [[T=10.8,mu=0.46152]]
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2) Solve Eqs. (5.1) and (5.2) for T and μ.
(%o2) The solutions in symbols.
(%i3) Substitute values of m_1, m_2, θ, a, and g into the solutions to get numerical values of T and μ.

• Direct calculation of T and μ.

```
(%i6) fpprintprec:5; m1:3; m2:1; theta:float(50/180*%pi); a:1;
g:9.8;
(fpprintprec) 5
(m1) 3
(m2) 1
(theta) 0.87266
(a) 1
(g) 9.8
(%i7) T: m2*(g+a);
(T) 10.8
(%i8) mu: (m1*g*sin(theta)-m2*(g+a)-m1*a)/(m1*g*cos(theta));
(mu) 0.46152
```

Comments on the codes:

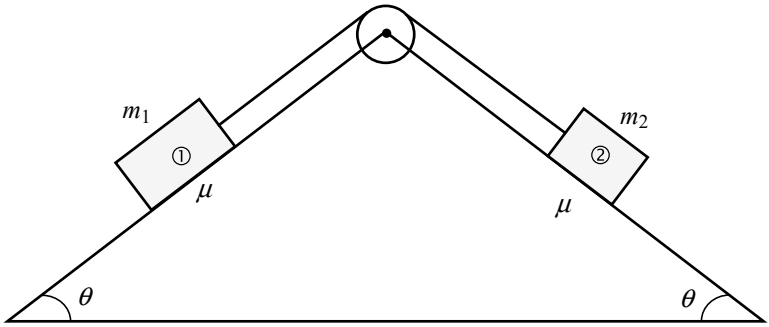

Fig. 5.29 Two blocks connected by a string of Problem 5.19

(%i6) Set floating point print precision to 5, assign values of m_1, m_2, θ, a, and g. (%i7), (%i8) Calculate T and μ.

Problem 5.19 Two blocks of masses m_1 and m_2 on two inclined planes connected by a string passing over a light pulley are released from rest (Fig. 5.29). If the angle of inclination is θ, the coefficient of kinetic friction of the blocks with the inclines is μ, and $m_1 > m_2$, determine,

(a) the acceleration, a, of the system and the tension, T, in the string,
(b) the numerical values of the acceleration and the tension if $m_1 = 9.0$ kg, $m_2 = 2.0$ kg, $\theta = 30°$, and $\mu = 0.20$.

Solution

(a) Figure 5.30 shows the two blocks, the inclined planes, and the forces relevant to the problem. Block 1 is more massive than block 2. So, intuitively the block 1 will slip down the incline to the left while block 2 will slip up the incline to the left. Both blocks will accelerate with acceleration of magnitude a.

Applying the Newton's second law to the block 1,

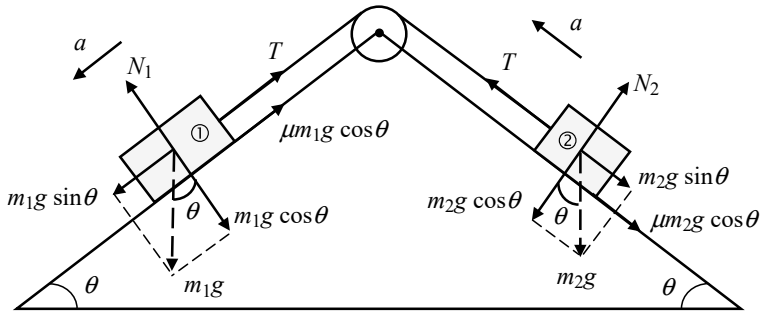

Fig. 5.30 Forces on the blocks, Problem 5.19

$$m_1 g \sin\theta - T - \mu m_1 g \cos\theta = m_1 a, \tag{5.1}$$

where $m_1 g \sin\theta - T - \mu m_1 g \cos\theta$ is the net force acting on the block 1 and a is the acceleration of the block. The forces acting on the block are the component of the weight parallel to the incline, $m_1 g \sin\theta$, the tension in the string, T, and the kinetic friction between the block and the incline, $\mu m_1 g \cos\theta$.

Applying the Newton's second law to the block 2,

$$T - m_2 g \sin\theta - \mu m_2 g \cos\theta = m_2 a, \tag{5.2}$$

where $T - m_2 g \sin\theta - \mu m_2 g \cos\theta$ is the net force on block 2 and a is the acceleration of the block. Here, T is the tension in the string, $m_2 g \sin\theta$ is the component of the weight parallel to the incline, and $\mu m_2 g \cos\theta$ is the friction between the block and the incline.

Solving the two equations gives the acceleration, a, as,

$$a = \frac{(m_1 - m_2) g \sin\theta - (m_1 + m_2)\mu g \cos\theta}{m_1 + m_2},$$

and the tension, T, in the string as,

$$T = \frac{2 m_1 m_2 g \sin\theta}{m_1 + m_2}.$$

(b) Substituting the given numerical values, the acceleration, a, of the block is,

$$\begin{aligned} a &= \frac{(m_1 - m_2) g \sin\theta - (m_1 + m_2)\mu g \cos\theta}{m_1 + m_2} \\ &= \frac{(9.0\,\text{kg} - 2.0\,\text{kg})(9.8\,\text{m s}^{-2}) \sin 30^\circ - (9.0\,\text{kg} + 2.0\,\text{kg})(0.20)(9.8\,\text{m s}^{-2}) \cos 30^\circ}{9.0\,\text{kg} + 2.0\,\text{kg}} \\ &= 1.4\,\text{m s}^{-2}. \end{aligned}$$

The tension, T, in the string is,

$$\begin{aligned} T &= \frac{2 m_1 m_2 g \sin\theta}{m_1 + m_2} = \frac{2(9.0\,\text{kg})(2.0\,\text{kg})(9.8\,\text{m s}^{-2}) \sin 30^\circ}{9.0\,\text{kg} + 2.0\,\text{kg}} \\ &= 16\,\text{N}. \end{aligned}$$

• wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec) 5
(%i2) solve([m1*g*sin(theta)-T-mu*m1*g*cos(theta)=m1*a,
```

```
T-m2*g*sin(theta)-mu*m2*g*cos(theta)=m2*a], [a,T]);
(%o2)                    [[a=((g*m1-g*m2)*sin(theta)+(-g*m2-
g*m1)*mu*cos(theta))/(m2+m1),
T=(2*g*m1*m2*sin(theta))/(m2+m1)]]
(%i3) subst([m1=9, m2=2, theta=float(30/180*%pi), mu=0.2,
g=9.8], %);
(%o3) [[a=1.4208,T=16.036]]
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2) Solve Eqs. (5.1) and (5.2) symbolicly for acceleration a and tension T in the string.
(%o2) The solutions in symbols.
(%i3) Substitute values of m_1, m_2, θ, μ, and g into the solutions to get numerical values of a and T.

• Direct numerical calculation of a and T:

```
(%i6) fpprintprec:5; m1:9; m2:2; theta:float(30/180*%pi); mu:0.2;
g:9.8;
(fpprintprec) 5
(m1) 9
(m2) 2
(theta) 0.5236
(mu) 0.2
(g) 9.8
(%i7) a: ((m1-m2)*g*sin(theta)-(m1+m2)*mu*g*cos(theta))/(m1+m2);
(a) 1.4208
(%i8) T: 2*m1*m2*g*sin(theta)/(m1+m2);
(T) 16.036
```

Comments on the codes:

(%i6) Set floating point print precision to 5, assign values of m_1, m_2, θ, μ, and g.
(%i7), (%i8) Calculate a and T.

Problem 5.20 Two blocks of masses $m_1 = 6.0$ kg and $m_2 = 1.0$ kg are connected by a string passing over a light pulley (Fig. 5.31). This is an Atwood machine with light pulley. The distance of block 1 to the floor is $h = 2.0$ m. The blocks are released from rest. Calculate,

(a) the acceleration, a, of the blocks and the tension, T, in the string,
(b) the speed, v, at which block 1 strikes the floor.

Solution

(a) Figure 5.32 shows the forces acting on the blocks and their acceleration a. The net force acting on the block 1 is $m_1 g - T$ toward the ground, where $m_1 g$ is the

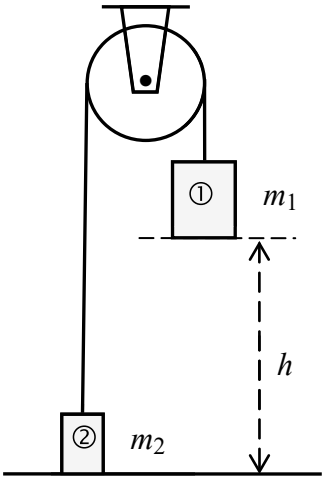

Fig. 5.31 Atwood machine with light pulley, Problem 5.20

weight of the block and T is the tension in the string. This net force accelerates the block 1. The net force on the block 2 is $T - m_2g$ pointing upward, where T is the tension in the string and m_2g is the weight of the block. This net force accelerates the block 2.

Applying the Newton's second law to the block 1 and block 2 gives,

$$\Sigma F = m_1 a \Rightarrow m_1 g - T = m_1 a, \tag{5.1}$$

$$\Sigma F = m_2 a \Rightarrow T - m_2 g = m_2 a. \tag{5.2}$$

Solving the two equations one gets the acceleration of the blocks and the tension in the string as,

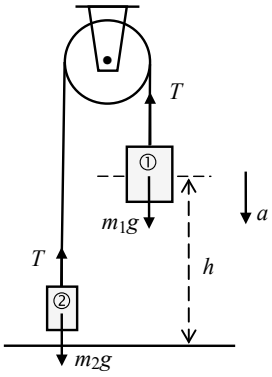

Fig. 5.32 Forces acting on the blocks, Problem 5.20

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g = \left(\frac{6.0\,\text{kg} - 1.0\,\text{kg}}{6.0\,\text{kg} + 1.0\,\text{kg}}\right) \times 9.8\,\text{m}\,\text{s}^{-2} = 7.0\,\text{m}\,\text{s}^{-2},$$

$$T = 2\left(\frac{m_1 m_2}{m_1 + m_2}\right) g = 2 \times \left(\frac{6.0\text{ kg} \times 1.0\text{ kg}}{6.0\text{ kg} + 1.0\text{ kg}}\right) \times 9.8\text{ N/kg} = 17\text{ N}.$$

(b) The speed at which block 1 strikes the floor is calculated as follows,

$$v^2 - u^2 = 2as \Rightarrow v^2 = 2ah, \tag{5.3}$$

$$v = \sqrt{2\left(\frac{m_1 - m_2}{m_1 + m_2}\right) gh} = \sqrt{2\left(\frac{6.0\,\text{kg} - 1.0\,\text{kg}}{6.0\,\text{kg} + 1.0\,\text{kg}}\right)(9.8\,\text{m/s}^2)(2.0\,\text{m})}$$
$$= 5.3\,\text{m}\,\text{s}^{-1}.$$

• wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; m1:6; m2:1; g:9.8; h:2;
(fpprintprec) 5
(ratprint) false
(m1) 6
(m2) 1
(g) 9.8
(h) 2
(%i8) solve([m1*g-T=m1*a, T-m2*g=m2*a], [a,T])$ float(%);
(%o8) [[a=7.0,T=16.8]]
(%i9) a: rhs(%[1][1]);
(a) 7.0
(%i11) solve(v^2=2*a*h, v)$ float(%);
(%o11) [v=-5.2915,v=5.2915]
(%i12) a: (m1-m2)/(m1+m2)*g;
(a) 7.0
(%i13) T: 2*m1*m2/(m1+m2)*g;
(T) 16.8
(%i14) v: sqrt(2*(m1-m2)/(m1+m2)*g*h);
(v) 5.2915
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m_1, m_2, g, and h.
(%i8) Solve $m_1 g - T = m_1 a$ and $T - m_2 g = m_2 a$ for a and T. Part (a).
(%i9) Pick the value of a.
(%i11) Solve $v^2 = 2ah$ for v. Part (b).
(%i12), (%i13), (%i14) Direct calculations of a, T, and v.

• Alternative calculation:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec) 5
(ratprint) false
(%i3) eq1: m1*g-T=m1*a;
(eq1) g*m1-T=a*m1
(%i4) eq2: T-m2*g=m2*a;
(eq2) T-g*m2=a*m2
(%i5) eq3: v^2=2*a*h;
(eq3) v^2=2*a*h
(%i6) solve([eq1,eq2,eq3], [a,T,v]);
(%o6) [[a=-(g*m2-g*m1)/(m2+m1),
T=(2*g*m1*m2)/(m2+m1),
v=sqrt(2)*sqrt(-(g*h*m2-g*h*m1)/(m2+m1))],
[a=-(g*m2-g*m1)/(m2+m1),
T=(2*g*m1*m2)/(m2+m1),
v=-sqrt(2)*sqrt(-(g*h*m2-g*h*m1)/(m2+m1))]]
(%i8) subst([m1=6, m2=1, h=2, g=9.8], %)$ float(%);
(%o8) [[a=7.0,T=16.8,v=5.2915],[a=7.0,T=16.8,v=-5.2915]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3), (%i4), (%i5) Assign Eqs. (5.1), (5.2), and (5.3) as eq1, eq2, and eq3, respectively.
(%i6) Solve Eqs. (5.1), (5.2), and (5.3) symbolicly for a, T, and v.
(%o6) The solutions in symbols.
(%i8) Substitute values of m_1, m_2, h, and g into the solutions to get numerical values of a, T, and v.

Problem 5.21 A system of three blocks is arranged with a string passing over a light pulley as in Fig. 5.33. The system accelerates with an acceleration of a and block 3 does not slip in any way on block 1. The coefficient of kinetic friction of block 1 with the table is μ_k, while the coefficient of static friction of the block 1 with the block 3 is μ_s. Then,

(a) calculate the acceleration, a, the tension, T, in the string, and the coefficient of static friction, μ_s, of block 1 with the block 3,
(b) given that $m_1 = 5.0$ kg, $m_2 = 10$ kg, $m_3 = 2.0$ kg, and $\mu_k = 0.05$, calculate a, T, and μ_s.

Solution

(a) Figure 5.34 shows the forces acting on the system. For block 1 and 3, their weight is $(m_1 + m_3)g$ directed downward and the normal reaction of the table on them is N_1 directed upward. The magnitude of N_1 is the same as the weight of the two blocks, that is, $N_1 = (m_1 + m_3)g$. The kinetic friction of the two blocks and the table is $f_k = \mu_k N_1 = \mu_k(m_1 + m_3)g$ to the left. The tension in the string is T. Applying Newton's second law to the block 1 and 3, one gets,

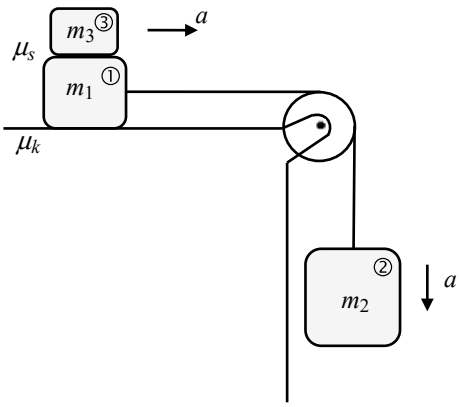

Fig. 5.33 System of three blocks of Problem 5.21

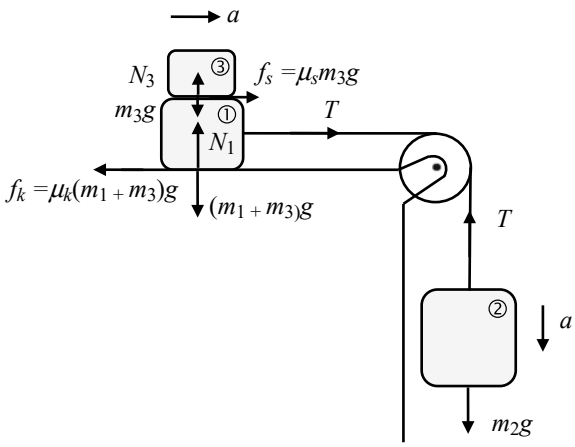

Fig. 5.34 Forces acting on the blocks, Problem 5.21

$$\Sigma F = ma,$$

$$T - \mu_k (m_1 + m_3) g = (m_1 + m_3) a.$$

Forces acting on block 2 are its weight, $m_2 g$, directed downward and the tension in the string, T, directed upward. Applying Newton's second law to the block 2, one gets,

$$\Sigma F = ma,$$

$$m_2 g - T = m_2 a.$$

Solving the two equations one obtains the acceleration, a, of the block and tension, T, in the string as,

$$a = \frac{m_2 g - (m_1 + m_3)\mu_k g}{m_1 + m_2 + m_3},$$

$$T = \frac{m_2(m_1 + m_3)(1 + \mu_k)g}{m_1 + m_2 + m_3}.$$

Forces acting on the block 3 are its weight, $m_3 g$, directed downward and the normal reaction, N_3, of the block 1 acting on it that is directed upward. The magnitude of N_3 is the same as the weight, that is, $N_3 = m_3 g$. The static friction between block 3 and 1 is $f_s = \mu_s N_3 = \mu_s m_3 g$ directed to the right. The friction that accelerates block 3 can be obtained by applying the Newton's second law as,

$$\Sigma F = ma,$$
$$\mu_s m_3 g = m_3 a.$$

Thus, the coefficient of static friction, μ_s, is,

$$\mu_s = \frac{m_3 a}{m_3 g} = \frac{a}{g} = \frac{m_2 - (m_1 + m_3)\mu_k}{m_1 + m_2 + m_3}.$$

(b) Substituting the given values into results of part (a) one gets the acceleration, a, of the blocks, tension, T, in the string, and coefficient of static friction, μ_s, as follows,

$$a = 5.6\,\mathrm{m\,s^{-2}},$$

$$T = 42\,\mathrm{N},$$

$$\mu_s = 0.57.$$

• wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec) 5
(%i2)       sol:       solve([T-uk*(m1+m3)*g=(m1+m3)*a,m2*g-
T=m2*a,us=a/g],[a,T,us]);
(sol) [[a=-((g*m3+g*m1)*uk-g*m2)/(m3+m2+m1),
T=((g*m2*m3+g*m1*m2)*uk+g*m2*m3+g*m1*m2)/(m3+m2+m1),
us=-((m3+m1)*uk-m2)/(m3+m2+m1)]]
(%i3) subst([m1=5, m2=10, m3=2, uk=0.05, g=9.8], sol);
(%o3) [[a=5.5629,T=42.371,us=0.56765]]
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2) Solve $T - \mu_k(m_1 + m_3)g = (m_1 + m_3)a$, $m_2 g - T = m_2 a$, and $\mu_s = a/g$ symbolically for a, T, and μ_s. Part (a).
(sol) The solutions in symbols. Part (a).
(%i3) Substitute values of m_1, m_2, m_3, μ_k, and g into the solutions to get numerical values of a, T, and μ_s. Part (b).

• Direct calculation of a, T, and μ_s:

```
(%i6) fpprintprec:5; m1:5; m2:10; m3:2; uk:0.05; g:9.8;
(fpprintprec) 5
(m1) 5
(m2) 10
(m3) 2
(uk) 0.05
(g) 9.8
(%i7) a: (m2*g-(m1+m3)*uk*g)/(m1+m2+m3);
(a) 5.5629
(%i8) T: m2*(m1+m3)*(1+uk)*g/(m1+m2+m3);
(T) 42.371
(%i9) us: (m2-(m1+m3)*uk)/(m1+m2+m3);
(us) 0.56765
```

Comments on the codes:

(%i6) Set floating point print precision to 5, assign values of m_1, m_2, m_3, μ_k, and g.
(%i7), (%i8), (%i9) Calculate a, T, and μ_s.

5.3 Summary

Newton's laws of motion are,

i. A body at rest remains at rest or a body moving uniformly in a straight line remains as such unless there is net force acting on the body.
ii. The time rate of momentum change of a body is the net force acting on the body.
iii. For two interacting bodies, the force on the second by the first is the same and opposite to the force on the first by the second.

5.4 Exercises

Exercise 5.1

A block of mass m is pulled on a smooth floor by a string at an angle θ to the horizontal (Fig. 5.35). The block moves with an acceleration of a. Calculate tension in the string and normal reaction of the floor on the block.

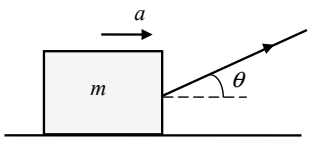

Fig. 5.35 A block being pulled on a smooth floor, Exercise 5.1

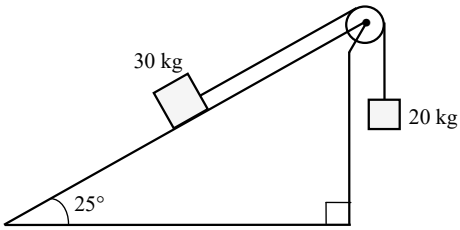

Fig. 5.36 Two blocks connected by a string of Exercise 5.2

(Answer: tension $= ma/\cos\theta$, normal reaction $= mg - ma\tan\theta$)

Exercise 5.2

A block of mass 30 kg on an inclined plane at 25° to the horizontal is connected to a hanging block of mass 20 kg by a string passing over a light pulley (Fig. 5.36). The friction between the 30 kg block and the incline is negligible. The system is released from rest. Calculate the acceleration of the 20 kg block and the distance it falls in 2.0 s.

(Answer: 1.4 m s^{-2}, 2.9 m)

Exercise 5.3

A block on a wooden inclined plane slides down at constant speed when the incline is tilted 25° to the horizontal. What is the coefficient of kinetic friction of the block with the plane?

(Answer: 0.47)

Exercise 5.4

A block of mass 50 kg on table is connected to another block by a string passing over a light pulley (Fig. 5.37). The coefficient of static friction of the 50 kg block with the table is 0.40. What is the smallest mass of the other block that will start the 50 kg block moving?

(Answer: 20 kg)

Exercise 5.5

A block of mass 10 kg on a 30° incline is connected to a 8.0 kg block by a string passing over a light pulley (Fig. 5.38). The coefficient of kinetic friction of the 10 kg

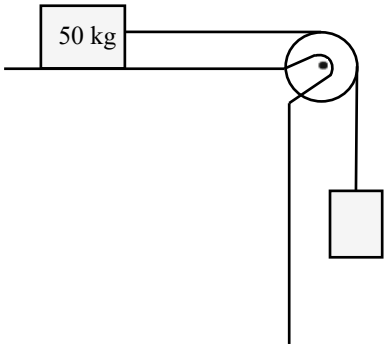

Fig. 5.37 Two blocks connected by a string of Exercise 5.4

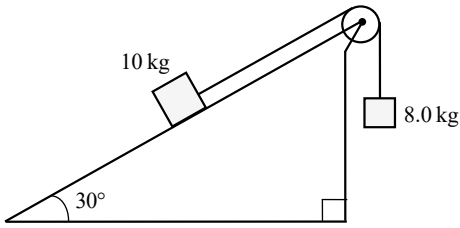

Fig. 5.38 Two blocks connected by a string of Exercise 5.5

block with the incline is 0.20. The system is released from rest. Calculate acceleration of the blocks and tension in the string.

(Answer: 0.69 m s^{-2}, 73 N)

Chapter 6
Uniform Circular Motion

6.1 Basic Concepts and Formulae

(1) A uniform circular motion is a motion in a circle with a constant angular speed of ω. For a particle in a uniform circular motion, its linear speed is $v = \omega r$, where r is the radius of the circle.
(2) A particle in a uniform circular motion has a centripetal acceleration of a_R,

$$a_R = \omega^2 r = \frac{v^2}{r}. \tag{6.1}$$

The centripetal acceleration points to the center of the circle.

(3) The centripetal force, F_R, acting on the particle, pointing to the centre of the circle is,

$$F_R = ma_R = m\omega^2 r = m\frac{v^2}{r}, \tag{6.2}$$

to sustain the circular motion. Here, m is the mass of the particle undergoing the circular motion.

6.2 Problems and Solutions

Problem 6.1 A stone of mass 0.20 kg is tied to a string and swirled in a horizontal circle of radius 0.50 m with a constant speed of 30 m s^{-1}. What is the tension, T, in the string?

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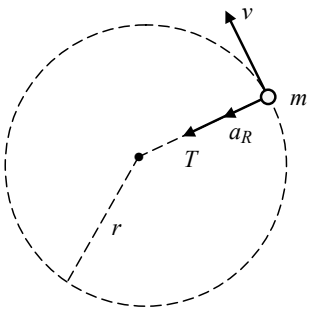


Fig. 6.1 A stone tied to a string and swirled in a horizontal circle, Problem 6.1

Solution

Figure 6.1 shows the stone, string, and circular path of the stone. Here, T is the tension in the string and a_R is the centripetal acceleration of the stone. The centripetal acceleration of the stone is (Eq. 6.1),

$$a_R = \frac{v^2}{r} = \frac{(30\,\text{m/s})^2}{0.50\,\text{m}} = 1800\,\text{m s}^{-2}.$$

The centripetal force is the tension, T, in the string (Eq. 6.2),

$$T = ma_R = m\frac{v^2}{r} = (0.20\,\text{kg})\frac{(30\,\text{m/s})^2}{(0.50\,\text{m})} = 360\,\text{N}.$$

• wxMaxima codes:

```
(%i3)  m:0.2; r:0.5; v:30;
(m)     0.2
(r)      0.5
(v)     30
(%i4)  aR: v^2/r;
(aR)    1800.0
(%i5)  T: m*aR;
(T)     360.0
```

Comments on the codes:

(%i3) Assign values of m, r, and v.
(%i4) Calculate centripetal acceleration, a_R.
(%i5) Calculate tension, T, in the string.

Problem 6.2 An object of mass 2.0 kg is tied to a string and swirled in a vertical circle of radius 0.20 m. The speed of the object at its lowest position is 3.0 m s^{-1}.

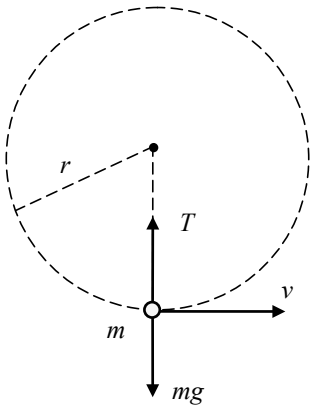


Fig. 6.2 An object tied to a string and swirled in a vertical circle, Problem 6.2

What is the centripetal force, F_R, on the object and the tension, T, in the string at the position?

Solution

Figure 6.2 shows the object at the lowest position in its circular path. The gravitational pull on the object or its weight, mg, is vertically downward. The centripetal acceleration of the object is $a_R = v^2/r$, where v is its linear speed and r is the radius of the circle.

The centripetal force, F_R, is,

$$F_R = ma_R = m\frac{v^2}{r} = (2.0\,\text{kg})\frac{(3.0\,\text{m/s})^2}{0.20\,\text{m}} = 90\,\text{N}.$$

This centripetal force is $F_R = T - mg$, where T is the tension in the string and mg is the weight of the object. The centripetal force causes the object to move in the circle. The tension, T, in the string is,

$$T = F_R + mg = 90\,\text{N} + (2.0\,\text{kg})(9.8\,\text{m/s}^2) = 110\,\text{N}.$$

• wxMaxima codes:

```
(%i5) fpprintprec:5; m:2; r:0.2; v:3; g:9.8;
(fpprintprec)      5
(m)    2
(r)     0.2
(v)    3
(g)    9.8
(%i6) FR: m*v^2/r;
(Fr)   90.0
(%i7) T: FR+m*g;
```

```
(T)     109.6
```

Comments on the codes:

(%i5)Set floating point print precision to 5, assign values of m, r, v, and g.
(%i6), (%i7) Calculate centripetal force, F_R, and tension, T, in the string.

Problem 6.3 An object of mass 1.0 kg is tied to a string and whirled in a horizontal circle of radius 2.0 m. The maximum tension, $T_{\max}$, that the string can support before it breaks is 800 N. What is the rotation rate limit so that the string does not break? Calculate the corresponding angular speed limit.

Solution

The centripetal force is the tension in the string. The tension T is,

$$T = \frac{mv^2}{r},$$

where m is the mass of the object, v is its linear speed, and r is the radius of the horizontal circle. The linear speed is $v = 2\pi rn$, where n is the number of rotations per second. Hence, the tension, T, is,

$$T = \frac{mv^2}{r} = \frac{m(2\pi rn)^2}{r} = 4\pi^2 mrn^2.$$

The tension in the string must be less than the maximum tension, $T_{\max}$, that the string can support. Thus,

$$T = 4\pi^2 mrn^2 < T_{\max}.$$

This gives,

$$n < \left(\frac{T_{\max}}{4\pi^2 mr}\right)^{1/2} = \left(\frac{800\,\text{N}}{4\pi^2(1.0\,\text{kg})(2.0\,\text{m})}\right)^{1/2} = 3.2.$$

The rotation rate must be less than 3.2 per second. The angular speed limit is,

$$\omega = 2\pi\, n = 2\pi(3.2) = 20\,\text{rad}\,\text{s}^{-1}.$$

The angular speed must be less than 20 rad s^{-1}.

• wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; m:1; r:2; Tmax:800;
(fpprintprec)      5
(ratprint)   false
(m)  1
(r)   2
(Tmax) 800
(%i6)  n: sqrt(Tmax/(4*float(%pi^2)*m*r));
(n)     3.1831
(%i7)  omega: 2*float(%pi)*n;
(omega) 20.0
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of m, r, and $T_{\max}$.
(%i6), (%i7) Calculate rotation rate, n, and angular speed, ω.

Problem 6.4 A circular bitumen road is designed for cars moving at speed of 40 km h^{-1}. The road radius is 100 m.

(a) What is the inclination angle θ of the road?
(b) If the road is flat instead of inclined, what is the minimum coefficient of friction of the tires and road surface so that there is no skidding?

Solution
(a) Figure 6.3a shows the car of mass m (represented by a dot) moving at speed of v on the inclined road at angle θ. The radius of the road is r. Two forces acting on the car are the normal reaction, N, of the road on the car and the gravitational pull, mg, on the car. The normal reaction, N, is resolved into the vertical component of N cos θ and horizontal component of N sin θ. In the vertical direction, the weight and vertical component of the normal reaction are equal in magnitude but opposite in directions,

$$N\cos\theta = mg.$$

Hence, there is no motion in this direction. The horizontal component of the normal reaction that points to the center of the circle is the centripetal force. So,

$$N\sin\theta = \frac{mv^2}{r},$$

where v is the speed of the car. The two equations give,

$$\tan\theta = \frac{v^2}{rg} = \frac{\left(\frac{40\times10^3\,\mathrm{m}}{60\times60\,\mathrm{s}}\right)^2}{(100\,\mathrm{m})(9.8\,\mathrm{m/s^2})} = 0.13.$$

This gives the angle of the incline of the road as $\theta = 7.2°$.

• wxMaxima codes:

```
(%i5)      fpprintprec:5;      ratprint:false;      v:40e3/
(60*60); r:100; g:9.8;
(fpprintprec)     5
(ratprint)   false
(v)   11.111
(r)    100
(g)   9.8
(%i7) solve(tan(theta)=v^2/(r*g), theta)$ float(%);
solve: using arc-trig functions to get a solution.
Some solutions will be lost.
(%o7)  [theta=0.12532]
(%i8)   theta: rhs(%[1]);
(theta)    0.12532
(%i9)  theta_deg: float(theta*180/%pi);
(theta_deg)        7.1801
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of v, r, and g.
(%i7) Solve $\tan\theta = v^2/(rg)$ for θ.
(%o7) The solution in radian.
(%i8), (%i9) Assign value of θ and convert the angle to degree.

(b) Figure 6.3b shows the flat road of radius, r, the car of mass, m, moving at a speed of v, and the forces involved. The three forces acting on the car are f, N, and mg; where f is the friction of the tires and the road, N is the normal reaction of the road on the car, and mg is the weight of the car. The centripetal force is the friction given by,

$$f = \frac{mv^2}{r}.$$

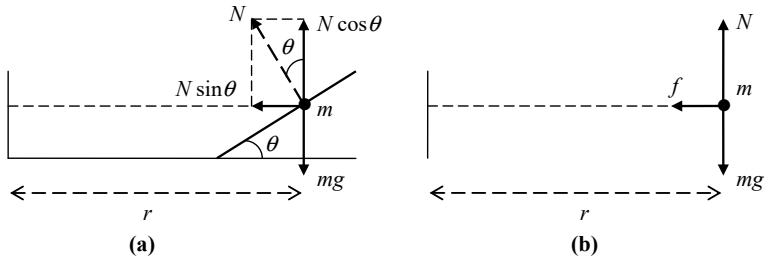


Fig. 6.3 A car on circular bitumen road, **a** inclined road, **b** flat road, Problem 6.4

But the friction is $f = \mu N = \mu mg$. This gives the coefficient of friction of the tires and the road as,

$$\mu = \frac{f}{mg} = \frac{mv^2}{rmg} = \frac{v^2}{rg} = \frac{\left(\frac{40\times10^3\,\text{m}}{60\times60\,\text{s}}\right)^2}{(100\,\text{m})(9.8\,\text{m/s}^2)} = 0.13.$$

The coefficient of friction must be greater than 0.13 to avoid skidding.

• wxMaxima codes:

```
(%i4) fpprintprec:5; v:40e3/(60*60); r:100; g:9.8;
(fpprintprec)     5
(v)    11.111
(r)     100
(g)    9.8
(%i5) mu: v^2/(r*g);
(mu)  0.12598
```

Comments on the codes:

(%i4) Set floating point print to 5, assign values of v, r, and g.
(%i5) Calculate the coefficient of friction, μ.

Problem 6.5 A car moves with a speed of 20 m s^{-1} around a circular bend of a road of radius 250 m.

(a) Calculate the minimum coefficient of friction, μ, of the tires and the road to avoid slipping.
(b) If the road is inclined and there is negligible friction between the tires and the road, what is the inclination angle, θ, so that there is no slipping?

Solution
(a) Figure 6.4a shows the car of mass m moving with speed, v, on a bend of radius, r. The forces acting on the car are N, mg, and f_{fric}, where N is the normal reaction of the road on the car, mg is the weight of the car, and f_{fric} is the friction between the tires and the surface of the road.

When the car takes the bend, the centripetal force is the frictional force,

$$f_{fric} = \frac{mv^2}{r}.$$

The normal reaction, N, is equal in magnitude and opposite in direction to the weight, mg, that is, $N = mg$. Thus, the coefficient of friction, μ, between the tires and the road is,

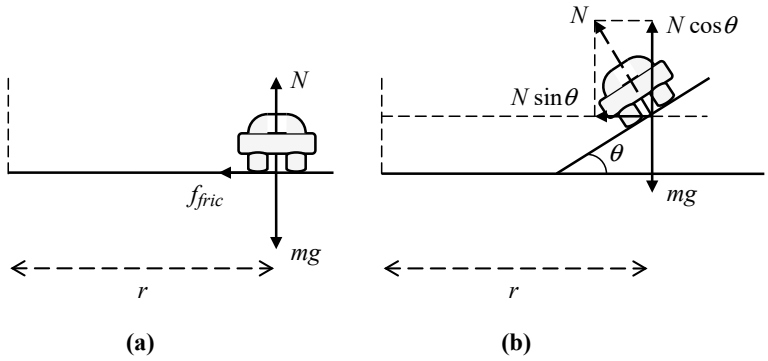


Fig. 6.4 A car on circular road, **a** flat road, **b** inclined road, Problem 6.5

$$\mu = \frac{f_{fric}}{N} = \frac{mv^2/r}{mg} = \frac{v^2}{rg} = \frac{(20\,\mathrm{m/s})^2}{(250\,\mathrm{m})(9.8\,\mathrm{m/s^2})} = 0.16.$$

To avoid slipping, μ must be larger than 0.16.

• wxMaxima codes:

```
(%i4) fpprintprec:5; r:250; v:20; g:9.8;
(fpprintprec)      5
(r)    250
(v)    20
(g)    9.8
(%i5) mu: v^2/(r*g);
(mu)  0.16327
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of r, v, and g.
(%i5) Calculate the coefficient of friction, μ.

(b) Figure 6.4b shows the car of mass m moving with speed of v on the inclined road. In the figure, θ is the inclination angle, N is the normal reaction of the road on the car, mg is the weight of the car, and r is the radius of the road. The normal reaction, N, is resolved into vertical component of $N\cos\theta$ and horizontal component of $N\sin\theta$. Assuming negligible friction between the tires and road, the centripetal force on the car is,

$$\frac{mv^2}{r} = N\sin\theta,$$

and the weight of the car is,

$$mg = N\cos\theta.$$

With these two equations, the tangent of the inclination angle and the angle itself can be calculated,

$$\tan\theta = \frac{mv^2}{rmg} = \frac{v^2}{rg} = \frac{(20\,\text{m/s})^2}{(250\,\text{m})(9.8\,\text{m/s}^2)} = 0.16,$$
$$\theta = 9.3^\circ.$$

• wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; r:250; v:20; g:9.8;
(fpprintprec)     5
(ratprint)   false
(r)   250
(v)   20
(g)   9.8
(%i7) solve(tan(theta)=v^2/(r*g), theta)$ float(%);
solve: using arc-trig functions to get a solution.
Some solutions will be lost.
(%o7)  [theta=0.16184]
(%i8)  theta: rhs(%[1]);
(theta)    0.16184
(%i9)  theta_deg: float(theta*180/%pi);
(theta_deg)        9.2726
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of r, v, and g.
(%i7) Solve $\tan\theta = v^2/(rg)$ for θ.
(%o7) The solutions in radian.
(%i8), (%i9) Assign value of θ and convert the angle to degree.

Problem 6.6 A bead can slide with negligible friction around a circular loop of radius r in a vertical plane (Fig. 6.5). The loop rotates at angular speed, ω, about its vertical diameter.

(a) In equilibrium, what is angle, θ?
(b) Is it possible that the bead is at equal height as the center of the loop?
(c) Letting the radius of the loop $r = 10$ cm, calculate the angle θ for the rotation rates of 3 per second and 6 per second.

Solution

(a) Figure 6.6 shows the bead of mass, m, at angle, θ, when the loop of radius, r, is rotating at angular speed, ω, about its vertical diameter. The forces acting on the

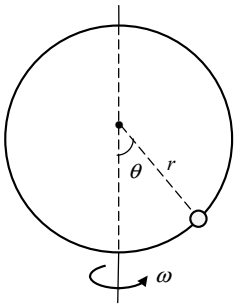


Fig. 6.5 A bead on a rotating circular loop, Problem 6.6

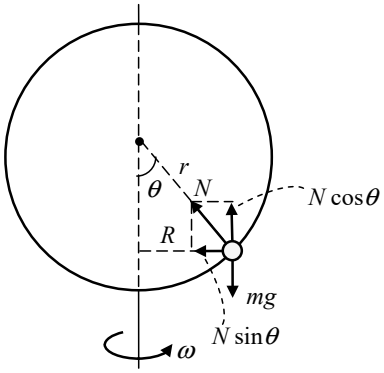


Fig. 6.6 Forces acting on the bead, Problem 6.6

bead are N and mg, where mg is the weight of the bead pointing to the ground and N is the normal reaction of the loop on the bead pointing to the center of the loop. The normal reaction, N, is resolved into vertical component of $N\cos\theta$ and horizontal component of $N\sin\theta$.

At equilibrium, in the vertical direction, the weight of the bead is balanced by the vertical component of the normal reaction,

$$mg = N\cos\theta.$$

The centripetal force is the horizontal component of the normal reaction,

$$\frac{mv^2}{R} = N\sin\theta,$$

where v is the linear speed of the bead and R is the radius of the bead rotation path, and r is the radius of the loop. These two equations give,

$$\tan\theta = \frac{v^2}{Rg}.$$

But $v = \omega R$ and $R = r \sin\theta$. It means that,

$$\tan\theta = \frac{\omega^2}{g} r \sin\theta.$$

This gives,

$$\cos\theta = \frac{g}{\omega^2 r},$$

because $\tan\theta = \sin\theta / \cos\theta$. Thus, in equilibrium, the angle is,

$$\theta = \cos^{-1}\left(\frac{g}{\omega^2 r}\right).$$

(b) The angular speed, ω, is finite. Hence,

$$\cos\theta = \frac{g}{\omega^2 r} > 0.$$

It is impossible that θ to be 90°, indicating that it is not possible for the bead to be at the same level as the center of the loop.

(c) For $r = 10$ cm and the loop is rotating at 3 rotations per second, the angle, θ, is,

$$\cos\theta = \frac{g}{\omega^2 r} = \frac{9.8\,\mathrm{m\,s^{-2}}}{(3 \times 2\pi\ \mathrm{s^{-1}})^2 (0.10\,\mathrm{m})} = 0.276,$$
$$\theta = \cos^{-1}(0.276) = 1.3\,\mathrm{rad} = 74°.$$

• wxMaxima codes:

```
(%i4)  fpprintprec:5; g:9.8; r:0.1; omega:float(3*2*%pi);
(fpprintprec)     5
(g)   9.8
(r)    0.1
(omega)   18.85
(%i5)  costheta: g/(omega^2*r);
(costheta)    0.27582
(%i6) theta: acos(costheta);
(theta) 1.2914
(%i7) theta_deg: float(theta*180/%pi);
(theta_deg)       73.989
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of g, r, and ω.
(%i5), (%i6), (%i7) Calculate θ and convert the angle to degree.

For $r = 10$ cm and the loop is rotating at 6 rotations per second, the angle, θ, is,

$$\cos\theta = \frac{g}{\omega^2 R} = \frac{9.8\,\mathrm{m\,s^{-2}}}{(6 \times 2\pi\ \mathrm{s^{-1}})^2(0.10\,\mathrm{m})} = 0.069,$$
$$\theta = \cos^{-1}(0.069) = 1.5\,\mathrm{rad} = 86^\circ.$$

• wxMaxima codes:

```
(%i4) fpprintprec:5; g:9.8; r:0.1; omega:float(6*2*%pi);
(fpprintprec)      5
(g)   9.8
(r)    0.1
(omega)       37.699
(%i5)costheta: g/(omega^2*r);
(costheta)      0.068955
(%i6) theta: acos(costheta);
(theta)   1.5018
(%i7) theta_deg: float(theta*180/%pi);
(theta_deg)       86.046
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of g, r, and ω.
(%i5), (%i6), (%i7) Calculate θ and convert the angle to degree.

• Alternative calculation:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)   false
(%i3) solve([m*g=N*cos(theta), m*v^2/R=N*sin(theta), v=omega*R,
R=r*sin(theta)], [N,v,cos(theta), R]);
(%o3) [[N=m*omega^2*r,v=omega*r*sin(theta),
          cos(theta)=g/(omega^2*r),R=r*sin(theta)]]
(%i4) theta: acos(rhs(%[1][3]));
(theta) acos(g/(omega^2*r))
(%i5) subst([r=0.1, omega=3*float(2*%pi), g=9.8], theta);
(%o5) 1.2914
(%i6)  deg: %/float(%pi)*180;
(deg)  73.989
(%i7)  subst([r=0.1, omega=6*float(2*%pi), g=9.8], theta);
(%o7) 1.5018
(%i8)  deg: %/float(%pi)*180;
(deg)   86.046
```

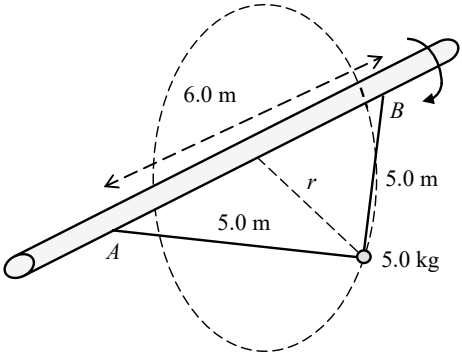


Fig. 6.7 An object tied to a horizontal rod by two string is rotated around the rod, Problem 6.7

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3) Solve $mg = N \cos\theta$, $mv^2/R = N \sin\theta$, $v = \omega R$, and $R = r \sin\theta$ for N, v, $\cos\theta$, and R in symbols.
(%i4) Assign the equilibrium angle θ. Part (a).
(%i5), (%i6) Calculate equilibrium angle θ in radian and degree. Part (c).
(%i7), (%i8) Calculate equilibrium angle θ in radian and degree. Part (d).

Problem 6.7 An object of mass 5.0 kg is tied to a horizontal light rod by two 5.0 m strings (Fig. 6.7). The object is rotated around the rod axis at a speed of 8.0 m s^{-1} and both strings are taut. Calculate the tension, T, in the string when the object is,

(a) at the lowest position,
(b) at the highest position,
(c) at the horizontal position the same level as the rod.

Solution

(a) Figure 6.8a shows the object at the lowest position. The vertical circular path traced by the object has a radius of,

$$r = \sqrt{(5.0\,\text{m})^2 - (3.0\,\text{m})^2} = 4.0\,\text{m}.$$

In the figure, T is the tension in each string, $F = 2T\cos\theta$ is the component of the tension perpendicular to the rod, and mg is the weight of the object. Also, $\cos\theta = 4/5$ and $\sin\theta = 3/5$.

The centripetal acceleration of the object is,

$$a_r = \frac{v^2}{r} = \frac{(8.0\,\text{m/s})^2}{4.0\,\text{m}} = 16\,\text{m}\,\text{s}^{-2}.$$

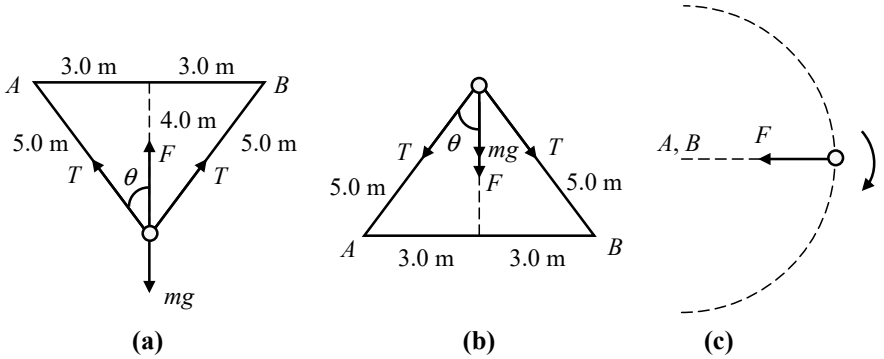


Fig. 6.8 Forces acting on the object when the object is at different positions, Problem 6.7

The centripetal force on the object is,

$$F - mg = ma_r.$$

The centripetal force is $F - mg$. This force causes the centripetal acceleration and the circular motion. Hence,

$$F = m(g + a_r) = (5.0\,\text{kg})(9.8\,\text{m/s}^2 + 16\,\text{m/s}^2) = 129\,\text{N}.$$

Because $2T\cos\theta = F$, the tension, T, in each string is,

$$T = \frac{F}{2\cos\theta} = \frac{129\,\text{N}}{2(4/5)} = 80.6\,\text{N}.$$

• wxMaxima codes:

```
(%i5) fpprintprec:5; m:5; v:8; r:4; g:9.8;
(fpprintprec)     5
(m)   5
(v)   8
(r)   4
(g)   9.8
(%i6) ar: v^2/r;
(ar)   16
(%i7) F: m*(g+ar);
(F)   129.0
(%i8) T: F/(2*4/5);
(T)    80.625
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of m, v, r, and g.
(%i6) Calculate centripetal acceleration, a_r.
(%i7), (%i8) Calculate F and the tension, T, in the string.

(b) When the object is at the highest position (Fig. 6.8b), the centripetal force acting on the object is,

$$F + mg = ma_r,$$

where F is the component of the tension of the string perpendicular to the rod. It means that,

$$F = m(a_r - g) = 5.0\,\text{kg}(16\,\text{m/s}^2 - 9.8\,\text{m/s}^2) = 31\,\text{N}.$$

From the figure, we get,

$$2T\cos\theta = F.$$

Hence, the tension, T, in the string is,

$$T = \frac{F}{2\cos\theta} = \frac{31\,\text{N}}{2(4/5)} = 19.4\,\text{N}.$$

• wxMaxima codes:

```
(%i5) fpprintprec:5; m:5; v:8; r:4; g:9.8;
(fpprintprec)      5
(m)   5
(v)    8
(r)    4
(g)    9.8
(%i6) ar: v^2/r;
(ar)    16
(%i7) F: m*(ar-g);
(F)    31.0
(%i8) T: F/(2*4/5);
(T)     19.375
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of m, v, r, and g.
(%i6) Calculate centripetal acceleration, a_r.
(%i7), (%i8) Calculate F and the tension, T, in the string.

(c) When the object is at horizontal position (Fig. 6.8c), the centripetal force acting on the object is F. We have,

$$F = ma_r = 5.0\,\text{kg} \times 16\,\text{m/s}^2 = 80\,\text{N}.$$

The tension, T, in the string is,

$$T = \frac{F}{2\cos\theta} = \frac{80\,\text{N}}{2(4/5)} = 50\,\text{N}.$$

• wxMaxima codes:

```
(%i5) fpprintprec:5; m:5; v:8; r:4; g:9.8;
(fpprintprec)     5
(m)   5
(v)    8
(r)    4
(g)    9.8
(%i6) ar: v^2/r;
(ar)    16
(%i7) F: m*ar;
(F)    80
(%i8) T: F/(2*4/5);
(T)    50
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of m, v, r, and g.
(%i6) Calculate centripetal acceleration, a_r.
(%i7), (%i8) Calculate F and the tension, T, in the string.

Problem 6.8 An object of mass 5.0 kg is tied by two 5.0 m strings to a vertical light rod (Fig. 6.9). The object is swung around the rod axis at a speed of 8.0 m s^{-1}. Determine the tensions, T_1, and, T_2, in the upper and lower strings.

Solution
Figure 6.10 shows the forces acting on the swung object. T_1 and T_2 are the tensions in the upper and lower strings, and mg is the weight of the object. The tension in the upper string is resolved as horizontal component of $T_1 \cos\theta$ and vertical component of $T_1 \sin\theta$, while the one in the lower string are resolved as $T_2 \cos\theta$ and $T_2 \sin\theta$.

The radius of the circular path traced by the object is,

$$r = \sqrt{(5.0\,\text{m})^2 - (3.0\,\text{m})^2} = 4.0\,\text{m}.$$

From the figure, $\cos\theta = 4/5$ and $\sin\theta = 3/5$.
The centripetal acceleration of the object is,

Fig. 6.9 Problem 6.8

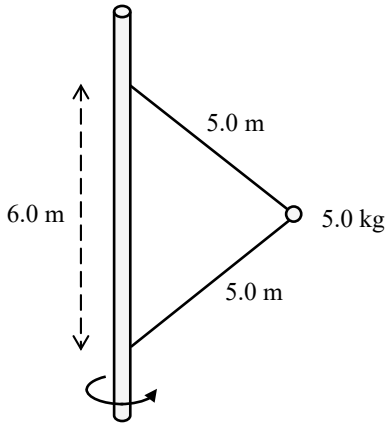
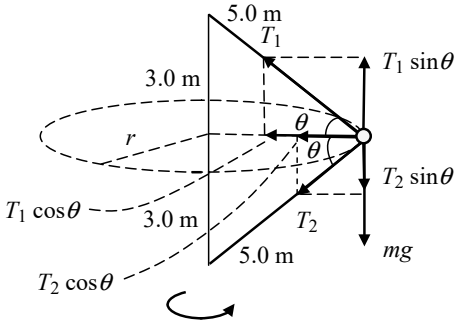


Fig. 6.10 Problem 6.8



$$a_r = \frac{v^2}{r} = \frac{(8.0\,\mathrm{m/s})^2}{4.0\,\mathrm{m}} = 16\,\mathrm{m\,s^{-2}}.$$

The centripetal force pointing to the center of the circle acting on the object is,

$$T_1 \cos\theta + T_2 \cos\theta = ma_r,$$

where $T_1 \cos\theta$ is the horizontal component of the tension in the upper string and $T_2 \cos\theta$ is the horizontal component of the tension in the lower one. This gives,

$$T_1 + T_2 = \frac{ma_r}{\cos\theta} = \frac{ma_r}{4/5}. \tag{6.1}$$

There is no motion in the vertical direction. This implies,

$$T_1 \sin\theta = mg + T_2 \sin\theta,$$

where $T_1 \sin\theta$ is the vertical component of the tension in the upper string, mg is the weight of the object, and $T_2 \sin\theta$ is the vertical component of the tension in the lower string. This gives,

$$T_1 - T_2 = \frac{mg}{\sin\theta} = \frac{mg}{3/5}. \tag{6.2}$$

Solving Eqs. (6.1) and (6.2), one gets the tensions in the upper and lower strings as,

$$T_1 = 91\,\text{N and } T_2 = 9.2\,\text{N}.$$

• wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; m:5; r:4; v:8; g:9.8;
(fpprintprec)      5
(ratprint)   false
(m)  5
(r)   4
(v)   8
(g)   9.8
(%i7) ar: v^2/r;
(ar)   16
(%i9)         solve([T1+T2=m*ar/(4/5),         T1-T2=m*g/(3/
5)], [T1,T2])$ float(%);
(%o9) [[T1=90.833,T2=9.1667]]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m, r, v, and g.
(%i7) Calculate centripetal acceleration, a_r.
(%i9) Solve $T_1 + T_2 = \frac{ma_r}{4/5}$ and $T_1 - T_2 = \frac{mg}{3/5}$ for T_1 and T_2.

Problem 6.9 Figure 6.11 shows a conical pendulum with 0.20 kg bob swinging at angle of 30° and radius of 0.50 m. Calculate,

(a) the tension in the string,
(b) the speed of the pendulum.

Solution

(a) Figure 6.12 shows the forces acting on the swinging pendulum. The forces are the tension, T, in the string and the weight, mg, of bob. The tension in the string is resolved into vertical component of T cos 30° and horizontal component of T sin 30°. The linear speed of swinging pendulum is v.

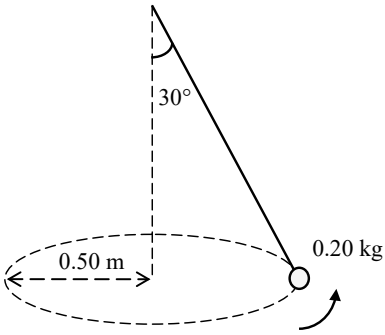


Fig. 6.11 A conical pendulum, Problem 6.9

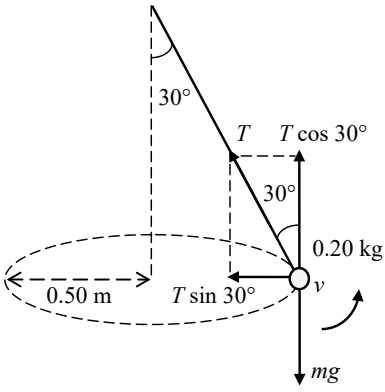


Fig. 6.12 Forces acting on the bob, Problem 6.9

In the vertical direction, the net force on the pendulum is zero. Hence, there is no motion in this direction. The vertical component of the tension is balanced by the weight of the pendulum, giving the tension, T, of the string as,

$$T\cos 30^\circ - mg = 0,$$
$$T = \frac{mg}{\cos 30^\circ} = \frac{(0.20\,\text{kg})(9.8\,\text{N kg}^{-1})}{\cos 30^\circ} = 2.3\,\text{N}.$$

(b) In the horizontal direction, the net force acting on the pendulum is $T \sin 30^\circ$. It is the centripetal force tht keeps the pendulum rotating in a circular path, having a centripetal acceleration with constant angular and linear speeds. The linear speed, v, of the pendulum is calculated as follows,

$$T\sin 30^\circ = m\frac{v^2}{r},$$
$$v = \sqrt{\frac{rT\sin 30^\circ}{m}} = \sqrt{\frac{(0.50\,\text{m})(2.3\,\text{N})\sin 30^\circ}{0.20\,\text{kg}}} = 1.7\,\text{m s}^{-1}.$$

• wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; m:0.2; r:0.5;
theta:float(30/180*%pi); g:9.8;
(fpprintprec)      5
(ratprint)   false
(m)  0.2
(r)   0.5
(theta)     0.5236
(g)   9.8
(%i7) T: m*g/cos(theta);
(T)    2.2632
(%i9)  solve(T*sin(theta)=m*v^2/r, v)$ float(%);
(%o9) [v=-1.682,v=1.682]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m, r, θ, and g.
(%i7), (%i9) Calculate tension, T, and solve $T\sin\theta = mv^2/r$ for v.

• Alternative calculation:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)      5
(ratprint)   false
(%i3)    solve([T*cos(theta)-m*g=0,    T*sin(theta)=m*v^2/
r], [T,v]);
(%o3)       [[T=(g*m)/cos(theta),v=-sqrt((g*r*sin(theta))/
cos(theta))],
             [T=(g*m)/cos(theta),v=sqrt((g*r*sin(theta))/
cos(theta))]]
(%i4)           subst([m=0.2,    r=0.5,    theta=float(30/
180*%pi), g=9.8], %);
(%o4)  [[T=2.2632,v=-1.682],[T=2.2632,v=1.682]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3) Solve $T\cos\theta - mg = 0$ and $T\sin\theta = mv^2/r$ for T and v.
(%o3) The solutions in symbols.
(%i4) Substitute values of m, r, θ, and g into the solutions to get numerical values of T and v.

Problem 6.10 A toy car is released from the rest at point A, height h, and slides on a track with negligible friction through the points B, C, and D (Fig. 6.13). The radius of the circular track is 0.20 m. Determine,

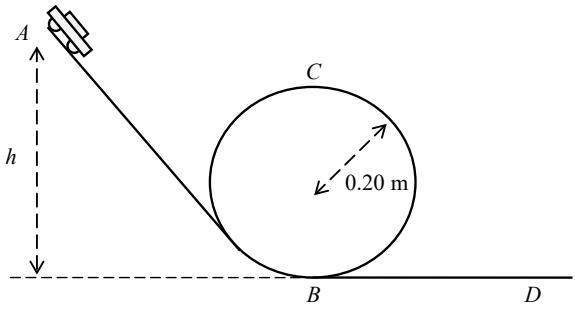


Fig. 6.13 A toy car on a tract of negligible friction, Problem 6.10

(a) the changes in the energy as the car moves from A to D,
(b) the minimum speed, v, of the car at C and the height, h, so that the car always touches the track.

Solution

(a) At point A the car has gravitational potential energy. At points B and D all energy of the car is kinetic energy. At point C the car has both potential and kinetic energies.
(b) Figure 6.14 shows the car of mass, m, at point C with the minimum speed, v, just enough for it to touch the track. The weight of the car is mg and radius of the circular track is R.

At this instant the centripetal force on the car is its weight. Thus, one gets,

$$\frac{mv^2}{R} = mg.$$

The minimum speed, v, of the car is,

$$v = \sqrt{Rg} = \sqrt{0.20 \text{ m} \times 9.8 \text{ m s}^{-2}} = 1.4 \text{ m s}^{-1}.$$

The potential energy of the car at A becomes kinetic and potential energies at C. This means that,

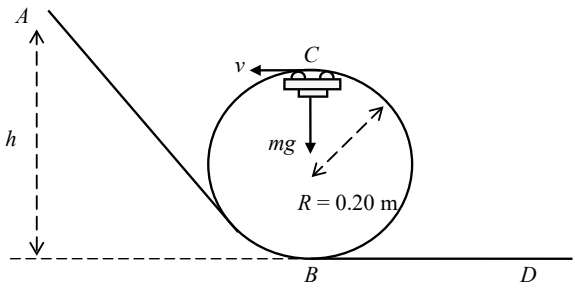


Fig. 6.14 The car just touching point C of the tract, Problem 6.10

$$mgh = \frac{1}{2}mv^2 + mg(2R) = \frac{1}{2}mRg + mg(2R) = \frac{5}{2}mgR.$$

This gives the height, h, as,

$$h = \frac{5}{2}R = \frac{5}{2}(0.20\,\text{m}) = 0.50\,\text{m}.$$

• wxMaxima codes:

```
(%i3) fpprintprec:5; R:0.2; g:9.8;
(fpprintprec)     5
I     0.2
(g)    9.8
(%i4) v: sqrt(R*g);
(v)    1.4
(%i5) h: 5/2*R;
(h)    0.5
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of R and g.
(%i4), (%i5) Calculate v and h.

Problem 6.11 A space station orbits the earth at an altitude of 400 km and constant speed of 7.6 km s^{-1}. The mass of the space station is 4.0×10^5 kg. The radius of earth is 6.37×10^6 m. Calculate the centripetal acceleration, a_r, and centripetal force, F, on the space station.

Solution

Figure 6.15 shows the space station orbiting the earth.

The centripetal acceleration, a_r, of the space station is (Eq. 6.1),

$$a_r = \frac{v^2}{r} = \frac{v^2}{r_E + h} = \frac{(7.6 \times 10^3\,\text{m/s})^2}{6.37 \times 10^6\,\text{m} + 400 \times 10^3\,\text{m}} = 8.5\,\text{m s}^{-2},$$

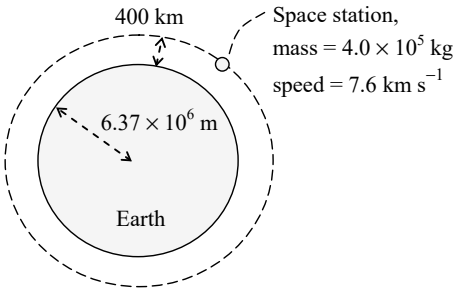


Fig. 6.15 A space station orbiting the earth, Problem 6.11

where r_E is the radius of the earth, h is the altitude of the space station, v is the station speed, and r is the distance from the station to the center of the earth. The centripetal force, F, on the station is (Eq. 6.2),

$$F = ma_r = (4.0 \times 10^5\,\text{kg})(8.5\,\text{m/s}^2) = 3.4 \times 10^6\,\text{N}.$$

• wxMaxima codes:

```
(%i5) fpprintprec:5; h:400e3; v:7.6e3; m:4e5; rE:6.37e6;
(fpprintprec)     5
(h)    4.0*10^5
(v)    7600.0
(m)    4.0*10^5
(rE)   6.37*10^6
(%i6) ar: v^2/(rE+h);
(a)     8.5318
(%i7) F: m*ar;
(F)    3.4127*10^6
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of h, v, m, and r_E.
(%i5), (%i7) Calculate centripetal acceleration, a_r, and centripetal force, F.

Problem 6.12 A small disk of mass 0.08 kg rests on a turntable rotating at an angular speed of 0.90 rad s^{-1}, at a distance of 0.20 m from the center of the turntable (Fig. 6.16). The coefficient of static friction between the disk and the turntable is 0.10.

(a) Sketch the velocity, acceleration, and the net force acting on the disk.
(b) If the angular speed of the turntable is increased gradually, then at what angular speed the disk gets slip out off the turntable?

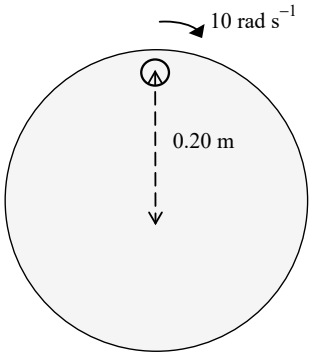


Fig. 6.16 A small disk on a rotating turntable, Problem 6.12

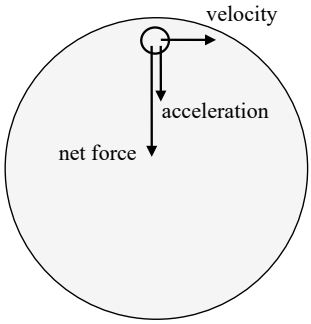


Fig. 6.17 The directions of velocity, acceleration, and net force, Problem 6.12

Solution

(a) The velocity of the disk points to the right and the acceleration points to the center of the turntable, so is the net force (Fig. 6.17).

(b) The centripetal force is the friction between the disk and the turntable. Thus, we have,

$$\begin{aligned} \text{centripetal force} &= \text{static friction}, \\ m\omega^2 r &= \mu_s mg, \\ \omega &= \sqrt{\frac{\mu_s g}{r}} = \sqrt{\frac{0.10\ (9.8\,\text{m/s}^2)}{0.20\,\text{m}}} = 2.2\,\text{rad s}^{-1}. \end{aligned}$$

where m is mass of the disk, ω is its angular speed, r is its distance from the center of the turntable, and μ_s is the coefficient of static friction between the disk and the turntable. The disk slips out off the turntable if the angular speed exceeds a value of 2.2 rad s^{-1}.

• wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; mu_s:0.1; g:9.8; r:0.2;
(fpprintprec)     5
(ratprint)   false
(mu_s)     0.1
(g)      9.8
(r)       0.2
(%i7)  solve(m*omega^2*r=mu_s*m*g, omega)$ float(%);
(%o7)  [omega=-2.2136,omega=2.2136]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of μ_s, g, and r.
(%i7) Solve $m\omega^2 r = \mu_s mg$ for ω.

6.3 Summary

An object of mass, m, undergoing a uniform circular motion in a circular path of radius, r, at angular speed of ω (or linear speed of $v = \omega r$) is accelerating toward the center of the circular path with centripetal acceleration of $a_R = \omega^2 r = v^2/r$. In such a motion, a centripetal force of $F_R = ma_R = m\omega^2 r = mv^2/r$ acts of the object.

6.4 Exercises

Exercise 6.1

A stone of mass 30 g tied to a string is swung with a constant speed in a horizontal circle with a radius of 1.5 m. It makes two revolutions in each second. Calculate the acceleration of the stone and tension in the string.

(Answer: 240 m s^{-2}, 7.1 N)

Exercise 6.2

Figure 6.18 shows a conical pendulum of length 0.24 m with 0.15 kg bob making an angle of 15° with the vertical in its swing. Calculate the speed of the pendulum bob and the tension in the string.

(Answer: 0.40 m s^{-1}, 1.5 N)

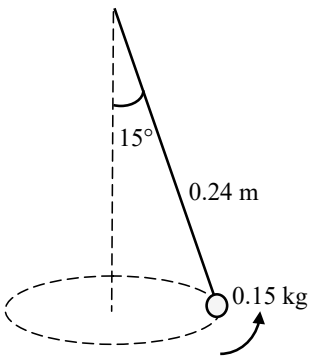


Fig. 6.18 A conical pendulum, Exercise 6.2

Exercise 6.3

A biological sample in a centrifuge of radius 1.0 m needs to have a centripetal acceleration of $25g$. Given that the acceleration of gravity is $g = 9.8$ m s^{-2}. What is the angular speed of the centrifuge?

(Answer: 16 rad s^{-1})

Exercise 6.4

A simple pendulum of length, l, and mass, m, is swinging as in Fig. 6.19. At an instant the angle of the string with the vertical is θ and the speed of the bob is v. What is the tension in the string?

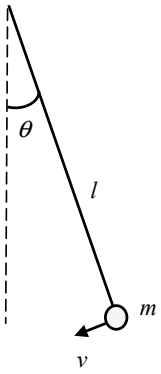


Fig. 6.19 A simple pendulum, Exercise 6.4

(Answer: tension $= mg \cos\theta + mv^2/l$)

Exercise 6.5

A bucket is filled with water and is whirled in a vertical circle of radius 0.50 m. What is the minimum speed at the top of the circular path so that the water does not fall out of the bucket?

(Answer: 2.3 m s^{-1})

Chapter 7
Work, Energy, and Power

7.1 Basic Concepts and Formulae

(1) A constant force, $\boldsymbol{F}$, acts on a particle and the displacement of the particle due to the force is $\boldsymbol{s}$. The work done by the force is defined as,

$$W = \boldsymbol{F} \cdot \boldsymbol{s} = Fs\cos\theta, \tag{7.1}$$

where θ is the angle between $\boldsymbol{F}$ and $\boldsymbol{s}$.

(2) If the force, $\boldsymbol{F}$, is not constant, the work done by the force in moving the particle from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ is,

$$W = \int_{r_1}^{r_2} \boldsymbol{F} \cdot \boldsymbol{dr}. \tag{7.2}$$

(3) The kinetic energy, K, of a particle of mass, m, moving at speed, v, is,

$$K = \frac{1}{2}mv^2, \tag{7.3}$$

where v is small compared to the speed of light, $c = 3 \times 10^8$ m s^{-1}.

(4) The work-energy theorem states that the work done by an external force on a particle is the change in the kinetic energy of the particle,

$$W = \Delta K = K - K_0 = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2. \tag{7.4}$$

(5) Power, P, is the rate of work done. An agent is applying a force, $\boldsymbol{F}$, on a particle and the particle moves with velocity, $\boldsymbol{v}$. The power, P, of the agent is,

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W. M. S. Wan Hassan, *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-42678-0_7

$$P = \frac{dW}{dt} = \boldsymbol{F} \cdot \boldsymbol{v}. \tag{7.5}$$

(6) A force is conservative if the work done by the force on a particle is independent of the path of the particle. A force is also conservative if the work done by the force is zero for an arbitrary closed path of the particle. Gravitational force and spring force are conservative forces. A force that does not satisfy these characteristics is non-conservative force. Frictional force is a non-conservative force.

(7) The potential energy, U, exists for a conservative force. A conservative force, $\boldsymbol{F}$, acts on a particle and it moves from x_{init} to x_{final} on the x axis, then the change in the potential energy is the negative of the work done by the force,

$$U_{final} - U_{init} = -W = -\int_{x_{init}}^{x_{final}} F_x \, dx. \tag{7.6}$$

(8) The gravitational potential energy, U, of a particle of mass, m, located at height, y, from the ground is,

$$U = mgy. \tag{7.7}$$

(9) The elastic potential energy, U, of a spring with a force constant, k, and stretched by x is,

$$U = \frac{1}{2}kx^2. \tag{7.8}$$

(10) The total mechanical energy, E, for a system is the sum of its kinetic energy, K, and potential energy, U,

$$E = K + U. \tag{7.9}$$

(11) The law of conservation of mechanical energy states that if a conservative force is acting on the system, then the total mechanical energy is conserved,

$$E_{init} = E_{final} \quad \text{or} \quad K_{init} + U_{init} = K_{final} + U_{final}. \tag{7.10}$$

(12) The work-energy theorem states that the work done by all non-conservative forces on a particle is the total change in the mechanical energy of the particle,

$$\begin{aligned} W_{non-conservative} &= E_{final} - E_{init} \\ &= (K_{final} + U_{final}) - (K_{init} + U_{init}). \end{aligned} \tag{7.11}$$

(13) An object of mass, m, is suspended from a spring of force constant, k. Then,

$$kx = mg, \tag{7.12}$$

where x is the extension of the spring. This means that the elastic force of the spring is equal to the weight of the object.

(14) The law of conservation of mechanical energy states that the total mechanical energy is always conserved for conservative forces,

$$E = K_1 + U_1 = K_2 + U_2, \tag{7.13}$$

where K_1 and U_1 are the kinetic and potential energies at an instant of time or a point in space, while K_2 and U_2 are the kinetic and potential energies at another instant or point.

7.2 Problems and Solutions

Problem 7.1 A particle is displaced by the displacement vector of $\boldsymbol{s} = 4\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ m when a few forces act on it. One of the forces is $\boldsymbol{F} = -3\mathbf{i} - 5\mathbf{j} + 9\mathbf{k}$ N. What is the work, W, done on the particle by the force?

Solution

From the definition of work, the work done by the force is (Eq. 7.1),

$$\begin{aligned} W &= \boldsymbol{F} \cdot \boldsymbol{s} = F_x s_x + F_y s_y + F_z s_z \\ &= (-3\text{ N})(4\text{ m}) + (-5\text{ N})(-5\text{ m}) + (9\text{ N})(6\text{ m}) = 67\text{ J}. \end{aligned}$$

• wxMaxima codes:

```
(%i1)   s: [4, -5, 6];
(d)      [4,-5,6]
(%i2)   F: [-3, -5, 9];
(F)      [-3,-5,9]
(%i3)   W: F.s;
(W)      67
```

Comments on the codes:

(%i1), (%i2) Assign vectors $\boldsymbol{s}$ and $\boldsymbol{F}$.
(%i3) Calculate the work, W, by calculating the dot product.

Problem 7.2 A force $F_x = -3x^3$ acts on a particle that can only move in the x axis. What is the work, W, done by the force on the particle if the particle moves from $x = -1$ to $x = 3$? (F_x is in N and x in m).

Solution

Applying the definition of work, $W = \int \boldsymbol{F} \cdot \boldsymbol{dr}$ (Eq. 7.2), the work done is,

$$W = \int_{-1}^{3} F_x \, dx = \int_{-1}^{3} (-3x^3) \, dx = \left[-\frac{3}{4} x^4 \right]_{-1}^{3} = -60 \text{ J}.$$

• wxMaxima codes:

```
(%i1)   W: integrate(-3*x^3, x, -1, 3);
(W)      -60
```

Comments on the codes:

(%i1) Calculate the integration $W = \int_{-1}^{3} (-3x^3) \, dx$.

The negative work means the force and the displacement are in opposite directions to each other. For example, the work done by frictional force is negative.

Problem 7.3 A body of mass 5.00 kg is moving with the velocity vector of $\boldsymbol{v} = 4\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$ m s^{-1}. What is its kinetic energy, K?

Solution

Using the definition of the kinetic energy as in Eq. (7.3), the kinetic energy, K, of the body is,

$$\begin{aligned} K &= \frac{1}{2} m v^2 = \frac{1}{2} m \boldsymbol{v} \cdot \boldsymbol{v} = \frac{1}{2} m (v_x^2 + v_y^2 + v_z^2) \\ &= \frac{1}{2} (5.00 \text{ kg}) [4^2 + (-3)^2 + (-2)^2] \text{ m}^2/\text{s}^2 \\ &= 72.5 \text{ J}. \end{aligned}$$

• wxMaxima codes:

```
(%i2)   m:5; v:[4,-3,-2];
(m)      5
(v)      [4,-3,-2]
```

```
(%i4)   K: 1/2*m*v.v; float(%);
(k)      145/2
(%o4)   72.5
```

Comments on the codes:

(%i2) Assign values of m and vector $\boldsymbol{v}$.
(%i4) Calculate kinetic energy, K, and its decimal value.

Problem 7.4 The kinetic energy, K_{init}, of a particle of mass 5.0 kg is 80 J. A force acting on the particle has done a work, W, of 56 N m. What is the final kinetic energy, K_{final}, of the particle?

Solution

Using the work-energy theorem (Eq. 7.11),

$$\text{work} = \text{ change in kinetic energy},$$

$$W = K_{final} - K_{init}.$$

The final kinetic energy of the particle is,

$$\begin{aligned} K_{final} &= W + K_{init} \\ &= 56 \text{ N m} + 80 \text{ J} \\ &= 136 \text{ J}. \end{aligned}$$

Problem 7.5 A varying position dependant force, $\boldsymbol{F} = (2y\mathbf{i} + 4x^2\mathbf{j})$ N, acts on a particle. The particle is displaced from (0, 0) to (2 m, 1 m). Calculate the work, W, done by the force, $\boldsymbol{F}$, if the particle is displaced on a path of (Fig. 7.1),

(a) OAC,

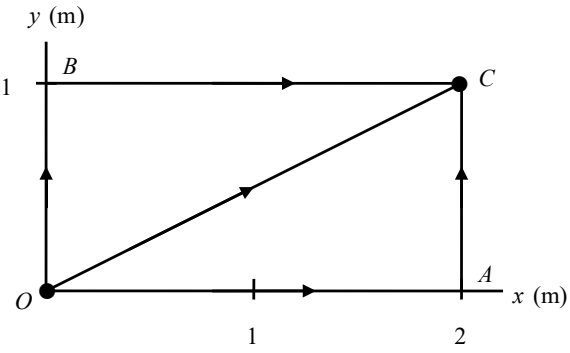


Fig. 7.1 A particle is displaced from O to C via three different paths, Problem 7.5

(b) *OBC*,
(c) *OC*.

Is the force, $\boldsymbol{F}$, conservative?

Solution

(a) This is a two-dimensional problem. The force is $\boldsymbol{F} = (2y\mathbf{i} + 4x^2\mathbf{j})$ and the elementary displacement is $\boldsymbol{dr} = dx\,\mathbf{i} + dy\,\mathbf{j}$. To calculate the work, the integration, $\int \boldsymbol{F} \cdot \boldsymbol{dr} = \int 2y\,dx + \int 4x^2\,dy$ has to be performed for each path of the particle.

For path *OAC*, the work is (Eq. 7.2),

$$W = \int \boldsymbol{F} \cdot \boldsymbol{dr} = \int 2y\,dx + \int 4x^2\,dy = \int_0^2 2(0)dx + \int_0^1 4(2)^2 dy = 0 + 16$$

$$= 16 \text{ J}.$$

(b) For path *OBC*, the work is (Eq. 7.2),

$$W = \int \boldsymbol{F} \cdot \boldsymbol{dr} = \int 2y\,dx + \int 4x^2\,dy = \int_0^2 2(1)dx + \int_0^1 4(0)^2 dy = 4 + 0$$

$$= 4 \text{ J}.$$

(c) For path *OC*, $y = x/2$ and $x^2 = 4y^2$. The work is (Eq. 7.2),

$$W = \int \boldsymbol{F} \cdot \boldsymbol{dr} = \int 2y\,dx + \int 4x^2\,dy = \int_0^2 2\left(\frac{x}{2}\right)dx + \int_0^1 4(4y^2)dy$$

$$= \int_0^2 x\,dx + \int_0^1 16y^2 dy = \left[\frac{x^2}{2}\right]_0^2 + \left[\frac{16y^3}{3}\right]_0^1 = \frac{22}{3} \text{ J}$$

$$= 7.3 \text{ J}.$$

The results show that the work done by the force depends on the path of the particle and the integrals are not equal. Thus, the force is non-conservative.

• wxMaxima codes:

```
(%i1)   fpprintprec:5;
(fpprintprec)   5
```

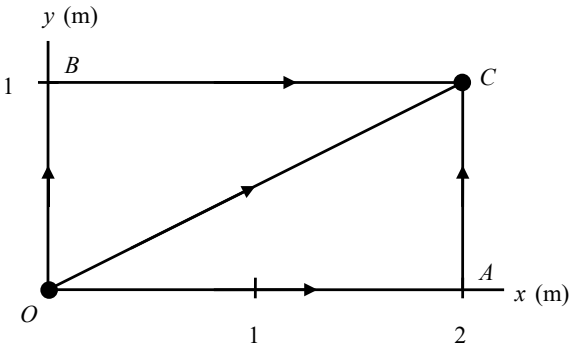


Fig. 7.2 A particle is displaced from O to C via three different paths, Problem 7.6

```
(%i3)       W:  integrate(x,x,0,2)+integrate(16*y^2,y,0,1);
float(%);
(W)      22/3
(%o3)   7.3333
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i3) Calculate the definite integration $W = \int_0^2 x\,dx + \int_0^1 16y^2 dy$ and get the decimal value.

Problem 7.6 A force acting of a particle varies with position, (x, y), according to $\boldsymbol{F} = (-x^2\mathbf{i} - y^2\mathbf{j})$ N. The particle is displaced from (0, 0) to (2 m, 1 m). Calculate the work, W, done by the force if the displacement occurs through the following paths (Fig. 7.2),

(a) OAC,
(b) OBC,
(c) OC.

Is the force, $\boldsymbol{F}$, conservative?

Solution

(a) This problem is similar to Problem 7.5. The force is $\boldsymbol{F} = (-x^2\mathbf{i} - y^2\mathbf{j})$ and the elementary displacement is $\boldsymbol{dr} = dx\,\mathbf{i} + dy\,\mathbf{j}$. To calculate the work, the integration $\int \boldsymbol{F} \cdot \boldsymbol{dr} = \int -x^2\,dx + \int -y^2\,dy$ has to be performed for each path of the particle.

For path OAC, the work is (Eq. 7.2),

$$W = \int \boldsymbol{F} \cdot \boldsymbol{dr} = \int_0^2 -x^2 dx + \int_0^1 -y^2 dy = \left[-\frac{x^3}{3}\right]_0^2 - \left[\frac{y^3}{3}\right]_0^1 = -\frac{8}{3} - \frac{1}{3}$$
$$= -3 \text{ J}.$$

(b) For path OBC, the work is (Eq. 7.2),

$$W = \int \boldsymbol{F} \cdot \boldsymbol{dr} = \int_0^2 -x^2 dx + \int_0^1 -y^2 dy = \left[-\frac{x^3}{3}\right]_0^2 - \left[\frac{y^3}{3}\right]_0^1 = -\frac{8}{3} - \frac{1}{3}$$
$$= -3 \text{ J}.$$

(c) For path OC, the work is (Eq. 7.2),

$$W = \int \boldsymbol{F} \cdot \boldsymbol{dr} = \int_{(0,0)}^{(2,1)} (-x^2\mathbf{i} - y^2\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j}) = \int_0^2 -x^2 dx + \int_0^1 -y^2 dy$$
$$= -3 \text{ J}.$$

All three integrations are the same, meaning that the work done is the same for each path. Thus, the work done depends on initial and final positions, not on the paths of the particle. This means that the force is conservative.

• wxMaxima codes:

```
(%i1)   fpprintprec:5;
(fpprintprec)   5
(%i2)   W:   integrate(-x^2,x,0,2) + integrate(-y^2,y,0,1);
(W)   -3
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2) Calculate the definite integration $W = \int_0^2 -x^2 dx + \int_0^1 -y^2 dy$.

Problem 7.7 A man is at a distance 20 m away from a 15 m tall building. He throws a ball of mass 0.20 kg to the building. The ball hits the building at a speed of 15 m s^{-1} (Fig. 7.3). What is the work, W, done by the man on the ball?

Solution
The work done by the man on the ball is converted to the potential and kinetic energies of the ball. The work, W, done is,

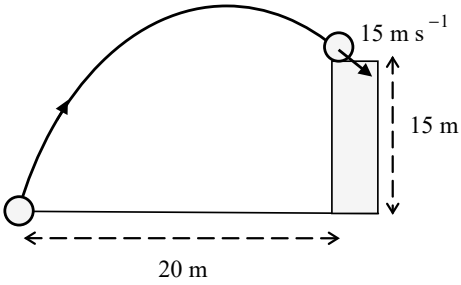


Fig. 7.3 A ball thrown to a building, Problem 7.7

$$\begin{aligned} W &= mgh + \frac{1}{2}mv^2 \\ &= (0.20\ \text{kg})(9.8\ \text{N/kg})(15\ \text{m}) + \frac{1}{2}(0.20\,\text{kg})(15\,\text{m/s})^2 \\ &= 52\ \text{J}. \end{aligned}$$

• wxMaxima codes:

```
(%i5) fpprintprec:   5;  m:0.2;  g:9.8;  h:15;  v:15;
(fpprintprec)   5
(m)       0.2
(g)       9.8
(h)       15
(v)       15
(%i6)   W: m*g*h + 1/2*m*v^2;
(W)       51.9
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of m, g, and h.
(%i6) Calculate the work done, W.

Problem 7.8 A projectile of mass 4.0 kg moves with the horizontal velocity of 25 m s^{-1} when it is at its maximum height of 20 m above the ground. Calculate,

(a) its potential energy, U, relative to the ground,
(b) its kinetic energy, K,
(c) its kinetic energy, K', when it hits the ground.

Solution

(a) Figure 7.4 shows the projectile at its maximum height.

The potential energy, U, of the projectile is (Eq. 7.7),

$$U = mgh = (4.0\ \text{kg})(9.8\ \text{N/kg})(20\ \text{m}) = 784\ \text{J}.$$

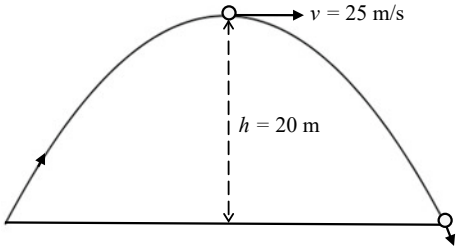


Fig. 7.4 A projectile at its highest position, Problem 7.8

(b) The kinetic energy, K, of the projectile is (Eq. 7.3),

$$K = \frac{1}{2}mv^2 = \frac{1}{2}(4.0\text{ kg})(25\text{ m/s})^2 = 1250\text{ J}.$$

(c) The energies of part (a) and (b) are converted to the kinetic energy as the projectile hits the ground. The kinetic energy, K', of the projectile as it hits the ground is,

$$K' = 784\text{ J} + 1250\text{ J} = 2034\text{ J}.$$

- wxMaxima codes:

```
(%i5)   fpprintprec:5; m:4; v:25; h:20; g:9.8;
(fpprintprec)   5
(m)      4
(v)      25
(h)      20
(g)      9.8
(%i6)   potential_energy: m*g*h;
(potential_energy)   784.0
(%i7)   kinetic_energy:   1/2*m*v^2;
(kinetic_energy)   1250
(%i8)   total_energy: potential_energy + kinetic_energy;
(total_energy)   2034.0
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of m, v, h, and g.
(%i6), (%i7), (%i8) Calculate potential, kinetic, and total energies.

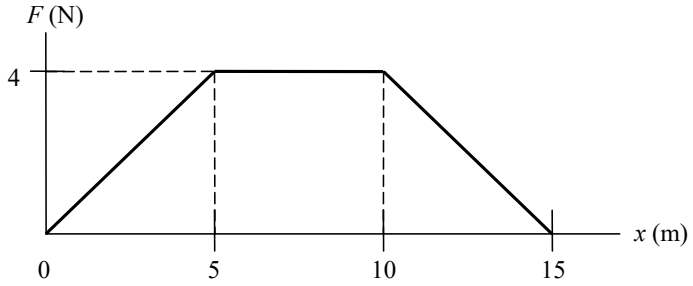


Fig. 7.5 Force against displacement curve of an object, Problem 7.9

Problem 7.9 An object of mass 2.0 kg is subjected to force, F, that varies with displacement, x, according to Fig. 7.5. The object is at rest at $x = 0$. What is the speed, v, of the object at $x = 5.0$, 10, and 15 m?

Solution

The work done by a force, $\boldsymbol{F}$, is $W = \int \boldsymbol{F} \cdot \boldsymbol{dr}$ (Eq. 7.2). In the F against x curve, the work is the area under the curve. From the work-energy theorem, $W = \Delta K = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$. For this problem, $v_0 = 0$ and the area under the curve is $\frac{1}{2}mv^2$. This means that the speed is,

$$v = \sqrt{2 \times area/m}.$$

At $x = 5.0$ m, the speed, v, is,

$$v = \sqrt{2[\frac{1}{2}(4.0\,\mathrm{N})(5.0\,\mathrm{m})]/2.0\,\mathrm{kg}} = 3.2\,\mathrm{m\,s^{-1}}.$$

At $x = 10$ m, the speed, v, is,

$$v = \sqrt{2[\frac{1}{2}(4.0\,\mathrm{N})(5.0\,\mathrm{m}) + (4.0\,\mathrm{N})(5.0\,\mathrm{m})]/2.0\,\mathrm{kg}} = 5.5\,\mathrm{m\,s^{-1}}.$$

At $x = 15$ m, the speed, v, is,

$v = \sqrt{2[\frac{1}{2}(4.0\,\mathrm{N})(5.0\,\mathrm{m}) + (4.0\,\mathrm{N})(5.0\,\mathrm{m}) + \frac{1}{2}(4.0\,\mathrm{N})(5.0\,\mathrm{m})]/2.0\,\mathrm{kg}} = 6.3\,\mathrm{m\,s^{-1}}$.

• wxMaxima codes:

```
(%i1)   fpprintprec:5;
(fpprintprec)   5
(%i2)   v: sqrt(2*0.5*4*5/2);
(v)      3.1623
(%i3)      v: sqrt(2*(0.5*4*5 + 4*5)/2);
(v)      5.4772
(%i4)   v: sqrt(2*(0.5*4*5 + 4*5 + 0.5*4*5)/2);
(v)      6.3246
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2), (%i3), (%i4) Calculate v at different displacements.

Problem 7.10 An object of mass, m, is subjected to a force, F, that varies with position, x, according to $F = A\sin(bx)$, where A and b are constants.

(a) Calculate the work done by the force, F, as the object is moved from $x = 0$ to $x = \pi/b$.
(b) If the object starts with speed, v_0, at $x = 0$, calculate its speed at $x = \pi/b$.

Solution

(a) The work, W, done by the force, F, is (Eq. 7.2),

$$W = \int F\,dx = \int_0^{\pi/b} A\sin(bx)\,dx = \left[-\frac{A}{b}\cos(bx)\right]_0^{\pi/b} = \frac{2A}{b}.$$

• wxMaxima codes:

```
(%i2)   assume (A>0); assume (b>0);
(%o1)   [A>0]
(%o2)   [b>0]
(%i3)   F: A*sin(b*x);
(F)   A*sin(b*x)
(%i4)   W: integrate(F,x,0,%pi/b);
(W)      (2*A)/b
```

Comments on the codes:

(%i2) Set the constants A and b positive.
(%i3), (%i4) Define F and calculate the definite integral, $W = \int_0^{\pi/b} F\,dx$.

(b) Using the work-energy theorem (Eq. 7.4),

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2,$$

one gets,

$$\frac{2A}{b} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$

From this equation, the speed, v, at $x = \pi/b$ is,

$$v = \sqrt{\frac{4A}{mb} + v_0^2}.$$

• wxMaxima codes:

```
(%i1)   solve(2*A/b=1/2*m*v^2-1/2*m*v0^2, v);
(%o1)         [v=-sqrt(v0^2+(4*A)/(b*m)),v=sqrt(v0^2+(4*A)/
(b*m))]
```

Comments on the codes:

(%i1) Solve $\frac{2A}{b} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$ for v.
(%o1) The solutions.

Problem 7.11 A spring is compressed by 0.15 m of its equilibrium length. The force constant of the spring is 5.0 N m^{-1}.

(a) What is the force, F, exerted by the spring?
(b) Calculate the elastic potential energy, U, of the compressed spring.

Solution

(a) According to the Hooke's law, the force, F, exerted by the compressed spring is,

$$F = kx = (5.0\ \text{N/m})(0.15\ \text{m}) = 0.75\ \text{N}.$$

(b) The elastic potential energy, U, of the compressed spring is (Eq. 7.8),

$$U = \frac{1}{2}kx^2 = \frac{1}{2}(5.0\ \text{N/m})(0.15\ \text{m})^2 = 5.6 \times 10^{-2}\ \text{J}.$$

• wxMaxima codes:

```
(%i3)   fpprintprec:5; k:5; x:0.15;
(fpprintprec)   5
(k)      5
(x)      0.15
(%i4)   F: k*x;
(F)      0.75
(%i5)   U: 0.5*k*x^2;
(U)      0.05625
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of k and x.
(%i4), (%i5) Calculate F and U.

Problem 7.12 An object of mass, m, is suspended to a spring with a force constant of k. What is the total potential energy, U_{total}, of the system with respect to the unstretched spring?

Solution

Figure 7.6 shows the spring before (a) and after (b) the object is suspended from it. The spring extends by x after the object is suspended, Fig. 7.6b. The spring elastic force points upward with magnitude of kx. This force is balanced by the weight of the object, mg. Therefore, using Eq. (7.12) one gets,

$$kx = mg,$$
$$x = \frac{mg}{k}.$$

This means that the extension, x, of the spring as the object is hung is mg/k.
The elastic potential energy in the stretched spring is (Eq. 7.8),

$$U_{elastic} = \frac{1}{2}kx^2 = \frac{1}{2}k\left(\frac{mg}{k}\right)^2 = \frac{1}{2}\frac{m^2g^2}{k}.$$

Taking gravitational potential energy at level O as zero, the gravitational potential energy of the object is (Eq. 7.7),

$$U_{gravity} = -mgx = -mg\left(\frac{mg}{k}\right) = -\frac{m^2g^2}{k}.$$

The total potential energy relative to the level O is,

$$U_{total} = U_{elastic} + U_{gravity}$$

$$= \frac{1}{2}\frac{m^2g^2}{k} - \frac{m^2g^2}{k}$$
$$= -\frac{1}{2}\frac{m^2g^2}{k}.$$

Table 7.1 summarizes the potential energies related to the mass and spring in a gravitational field.

Problem 7.13 A boy throws a ball with initial speed of 10 m s^{-1}. The point of throw is 6.0 m horizontally away and 1.0 m vertically down the basket (Fig. 7.7). Calculate the speed, v, when the ball hits the basket.

Solution

Applying the conservation of mechanical energy, the speed, v, of the ball when it hits the basket is calculated as follows (Eq. 7.10),

$$\begin{aligned} U_{init} + K_{init} &= U_{final} + K_{final}, \\ mgh_{init} + \frac{1}{2}mu^2 &= mgh + \frac{1}{2}mv^2, \\ 0 + \frac{1}{2}m(10\,\text{m/s})^2 &= m(9.8\,\text{m/s}^2)(1.0\,\text{m}) + \frac{1}{2}mv^2, \\ 50\,\text{m}^2/\text{s}^2 &= 9.8\,\text{m}^2/\text{s}^2 + 0.5v^2, \\ v &= 8.97\ \text{m s}^{-1}. \end{aligned}$$

where U_{init} and K_{init} are the initial potential and kinetic energies of the ball, while U_{final} and K_{final} are the potential and kinetic energies of the ball when it hits the basket. Here, $h_{init} = 0$ and $h = 1.0$ m are the initial and hit heights, $u = 10$ m s^{-1} and v are the initial and hit speeds, m is the mass of the ball, and g is the acceleration due to gravity.

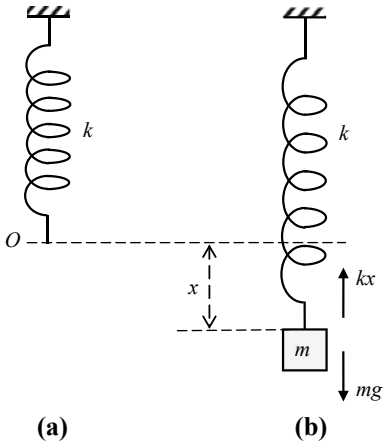
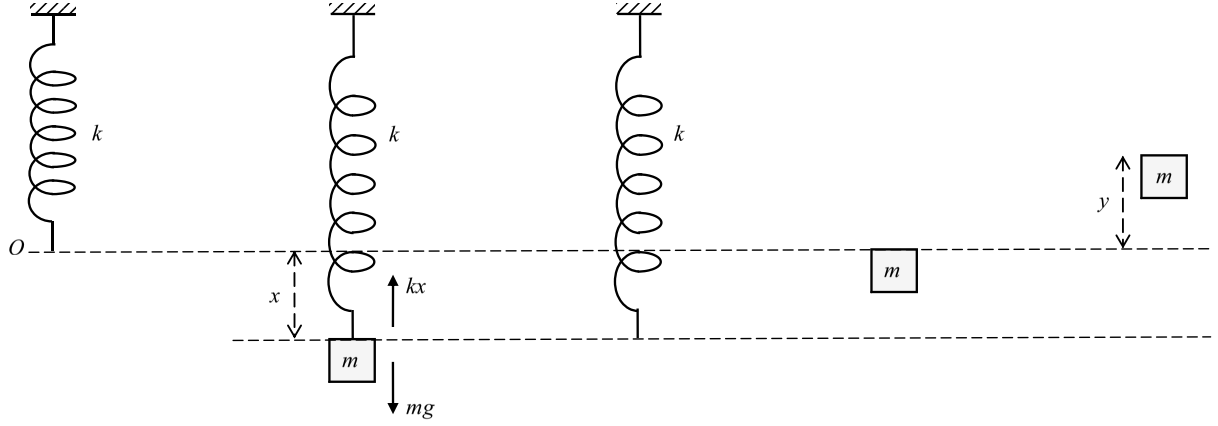


Fig. 7.6 A spring before (**a**) and after (**b**) a mass is suspended from it, Problem 7.12

Table 7.1 Potential energies of spring, mass, and mass-spring system

Unstretched spring	Spring stretched by a weight	Stretched spring	Object at level O	Object at height y
				
$U_{elastic} = 0$	$U_{elastic} = \frac{1}{2}\frac{m^2g^2}{k}$,	$U_{elastic} = \frac{1}{2}kx^2$	$U_{gravity} = 0$	$U_{gravity} = mgy$
	$U_{gravity} = -\frac{m^2g^2}{k}$,			
	$U_{total} = -\frac{1}{2}\frac{m^2g^2}{k}$			

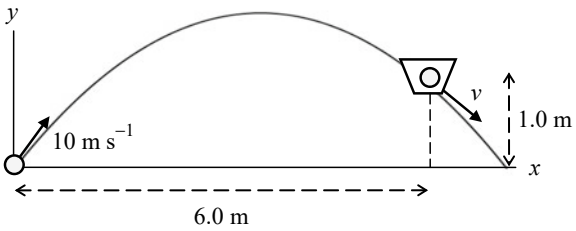


Fig. 7.7 A ball thrown into a basket, Problem 7.13

• wxMaxima codes:

```
(%i5)   fpprintprec:5; ratprint:false; u:10; h:1; g:9.8;
(fpprintprec)   5
(ratprint)   false
(u)      10
(h)      1
(g)      9.8
(%i7)   solve(1/2*m*u^2=m*g*h+1/2*m*v^2, v)$ float(%);
(%o7)   [v=-8.9666,v=8.9666]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of u, h, and g.
(%i7) Solve $\frac{1}{2}mu^2 = mgh + \frac{1}{2}mv^2$ for v.

Alternative solution: The same problem has been solved in Problem 4.11. There, the angle of throw, θ, is calculated from the equation of motion of the projectile (Eq. 4.8),

$$y = (\tan\theta)x - \left(\frac{g}{2u^2\cos^2\theta}\right)x^2.$$

That enables v_x, v_y, and v to be calculated. The alternative solution is lengthy and less straight forward compared to the solution of the Problem 7.13.

Problem 7.14 Figure 7.8 shows a block of mass, $m = 2.0$ kg, sliding down a track with negligible friction from rest at a height of $h = 1.0$ m. The block then compresses a spring with force constant of $k = 490$ N m^{-1}.

(a) Calculate the speed, v, of the block at point A
(b) How much is the spring compressed?

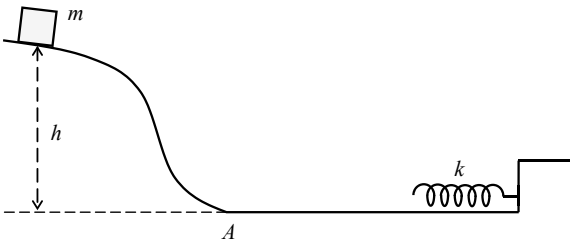


Fig. 7.8 A block slides down a track and hits a spring, Problem 7.14

Solution

(a) Using the conservation of the mechanical energy and setting point A as the reference of height (Eq. 7.10), we write,

$$U_{init} + K_{init} = U_A + K_A,$$
$$mgh + 0 = 0 + \frac{1}{2}mv_A^2,$$

where $U_{init} = mgh$ and $K_{init} = 0$ are the initial potential and kinetic energies of the block, while $U_A = 0$ and $K_A = mv_A{}^2/2$ are the potential and kinetic energies of the block at point A.

The speed, v_A, of the block at point A is,

$$v_A = \sqrt{2gh} = \sqrt{2(9.8\,\text{m/s}^2)(1.0\,\text{m})}$$
$$= 4.4\,\text{m}\,\text{s}^{-1}.$$

• wxMaxima codes:

```
(%i6)  fpprintprec:5; ratprint:false; m:2; h:1; k:490; g:9.8;
(fpprintprec)   5
(ratprint)   false
(m)      2
(h)      1
(k)      490
(g)      9.8
(%i8)   solve(m*g*h = 1/2*m*vA^2, vA)$ float(%);
(%o8)   [vA=-4.4272,vA=4.4272]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m, h, k, and g.
(%i8) Solve $mgh = \frac{1}{2}mv_A^2$ for v_A.

(b) Using the conservation of mechanical energy, at initial and final instances (Eq. 7.10), one gets,

$$U_{init} + K_{init} = U_{final} + K_{final},$$
$$mgh + 0 = \frac{1}{2}kx^2 + 0,$$

where $U_{init} = mgh$ and $K_{init} = 0$ are the initial potential and kinetic energies of the block, $U_{final} = kx^2/2$ and $K_{final} = 0$ are the final elastic potential energy of the spring and final kinetic energy of the block, while x is the amount of compression of the spring.

The amount of spring compression, x, is,

$$x = \sqrt{\frac{2mgh}{k}} = \sqrt{\frac{2(2.0\,\text{kg})(9.8\,\text{m/s}^2)(1.0\,\text{m})}{490\,\text{N/m}}}$$
$$= 0.28\,\text{m}.$$

• wxMaxima codes:

```
(%i6)  fpprintprec:5; ratprint:false; m:2; h:1; k:490; g:9.8;
(fpprintprec)   5
(ratprint)   false
(m)      2
(h)      1
(k)      490
(g)      9.8
(%i8)   solve(m*g*h = 1/2*k*x^2, x)$ float(%);
(%o8)   [x=-0.28284,x=0.28284]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m, h, k, and g.
(%i8) Solve $mgh = \frac{1}{2}kx^2$ for x.

Problem 7.15 Figure 7.9 shows a track consisting of two parts, part AB is a quadrant with the negligible friction of a circle of radius 1.0 m and the remaining part is a rough horizontal track. A block is released from rest at point A, it moves through the quadrant then along the straight track until it stops 3.0 m away from the point B.

(a) What is the speed, u, of the block at point B?
(b) Calculate the coefficient of kinetic friction, μ_k, of the block and the rough track.

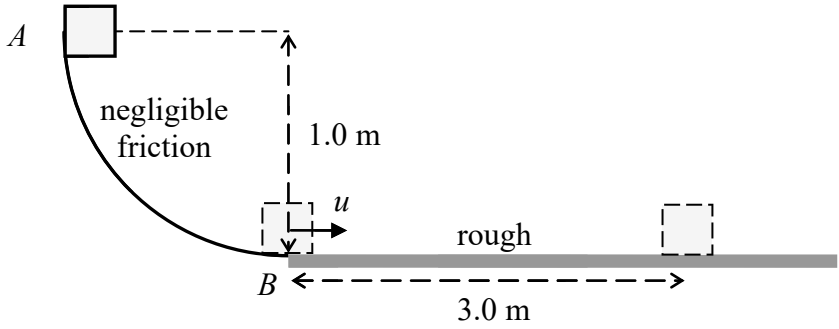


Fig. 7.9 A block sliding on a negligible friction and a rough tracks, Problem 7.15

Solution

(a) The block has gravitational potential energy at point A. This energy is converted into kinetic energy at point B. Beyond point B, the friction then decelerates the block until it stops. The gravitational potential energy of the block at point A is equal to the kinetic energy of the block at point B,

$$mgh = \frac{1}{2}mu^2.$$

The speed of the block at point B is,

$$u = \sqrt{2gh} = \sqrt{2(9.8\ \text{m/s}^2)(1.0\ \text{m})} = 4.4\ \text{m s}^{-1}.$$

• wxMaxima codes:

```
(%i5)   fpprintprec:5; ratprint:false; h:1; d:3; g:9.8;
(fpprintprec)   5
(ratprint)   false
(h)      1
(d)      3
(g)      9.8
(%i7)   solve(m*g*h = 1/2*m*u^2, u)$ float(%);
(%o7)   [u=-4.4272,u=4.4272]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of h, d, and g.
(%i7) Solve $mgh = \frac{1}{2}mu^2$ for u.

(b) The kinetic energy at point B is used to do work against friction,

$$\frac{1}{2}mu^2 = F_{friction}d = \mu_k mgd,$$

where $F_{friction} = \mu_k mg$ is friction of the block and track, μ_k is coefficient of kinetic friction, and d is distance travelled by the block on the track. The kinetic energy of the block at point B is equal to the gravitational potential energy of the block at point A as well,

$$\frac{1}{2}mu^2 = mgh.$$

Thus, the coefficient of kinetic friction, μ_k, of the block and the rough track is calculated as follows,

$$\mu_k mgd = mgh,$$
$$\mu_k = \frac{h}{d} = \frac{1.0\text{ m}}{3.0\text{ m}} = 0.33.$$

• wxMaxima codes:

```
(%i5)   fpprintprec:5; ratprint:false; h:1; d:3; g:9.8;
(fpprintprec)   5
(ratprint)   false
(h)      1
(d)      3
(g)      9.8
(%i7)   solve(mu_k*m*g*d = m*g*h, mu_k)$ float(%);
(%o7)   [mu_k=0.33333]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of h, d, and g.
(%i7) Solve $\mu_k mgd = mgh$ for μ_k.

Problem 7.16

(a) A bob of mass 5.0 kg of a simple pendulum is pulled aside so that its height is 6.0 cm higher than the lowest point of the pendulum (Fig. 7.10a). The bob is released from rest from that height. What is the speed, v_2, of the bob at the lowest point?
(b) From the same height, the bob is pushed and released at speed of $v_1 = 1.2\text{ m s}^{-1}$ (Fig. 7.10b). What is the speed, v_2, of the bob at the lowest point? What is the maximum height, h_3, reached by the bob?

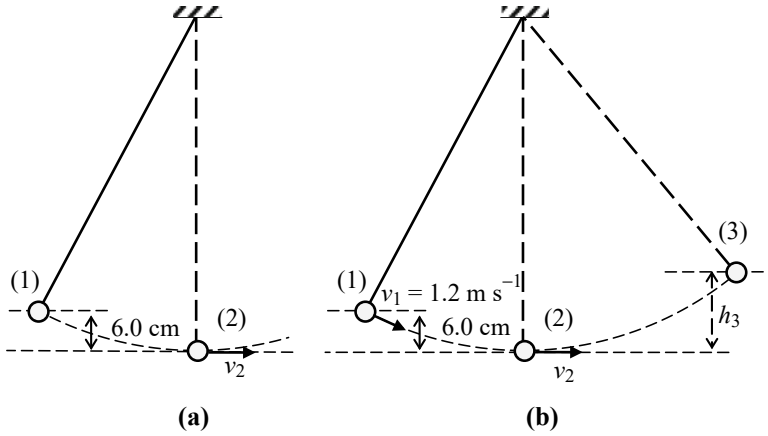


Fig. 7.10 Bob of a simple pendulum starts with two different initial conditions, Problem 7.16

Solution

(a) We take the reference line as the horizontal line that touches the lowest point of the pendulum path (Fig. 7.10a). Applying the conservation of energy at positions (1) and (2), i.e. total mechanical energy at (1) is equal to total mechanical energy at (2), using Eq. (7.13), we have,

$$U_1 + K_1 = U_2 + K_2,$$
$$mgh_1 + 0 = 0 + \frac{1}{2}mv_2^2,$$

where $U_1 = mgh_1$ is the potential energy and $K_1 = 0$ is the kinetic energy of the bob at point (1), while $U_2 = 0$ is the potential energy and $K_2 = mv_2{}^2/2$ is the kinetic energy of the bob at point (2), the lowest point. The speed, v_2, of the bob at its lowest point is,

$$v_2 = \sqrt{2gh_1} = \sqrt{2(9.8\ \text{m/s}^2)(6.0 \times 10^{-2}\ \text{m})} = 1.1\ \text{m s}^{-1}.$$

• wxMaxima codes:

```
(%i4)   fpprintprec:5; ratprint:false; h1:0.06; g:9.8;
(fpprintprec)   5
(ratprint)   false
(h1)      0.06
(g)       9.8
(%i6)   solve(m*g*h1 =1/2*m*v2^2, v2)$ float(%);
(%o6)   [v2=-1.0844,v2=1.0844]
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of h_1 and g.
(%i6) Solve $mgh_1 = \frac{1}{2}mv_2^2$ for v_2.

(b) The bob is pushed with initial speed of $v_1 = 1.2\ \text{m s}^{-1}$ (Fig. 7.10b). Applying the conservation of energy (Eq. 7.13), we write,

$$U_1 + K_1 = U_2 + K_2 = U_3 + K_3,$$
$$mgh_1 + \frac{1}{2}mv_1^2 = 0 + \frac{1}{2}mv_2^2 = mgh_3 + 0,$$

where the subscript 1 refers to the initial situation, 2 is the situation in which the bob is at the lowest position, and 3 is the situation in which the bob is at the highest position.

At the bob's lowest position, the potential energy is zero, $U_2 = 0$. It means that,

$$mgh_1 + \frac{1}{2}mv_1^2 = \frac{1}{2}mv_2^2.$$

The speed, v_2, of the bob at its lowest position is,

$$v_2 = \sqrt{2gh_1 + v_1^2} = \sqrt{2(9.8\ \text{m/s}^2)(6.0 \times 10^{-2}\ \text{m}) + (1.2\ \text{m/s})^2}$$
$$= 1.6\ \text{m s}^{-1}.$$

At the bob highest position the kinetic energy is $K_3 = 0$ and potential energy is $U_3 = mgh_3$. We have,

$$mgh_1 + \frac{1}{2}mv_1^2 = mgh_3.$$

Thus, the maximum height, h_3, of the bob is,

$$h_3 = h_1 + \frac{1}{2}\frac{v_1^2}{g} = 6.0 \times 10^{-2}\ \text{m} + \frac{1}{2}\frac{(1.2\,\text{m/s})^2}{9.8\,\text{m/s}^2}$$
$$= 0.13\,\text{m}.$$

• wxMaxima codes:

```
(%i5)      fpprintprec:5; ratprint:false; h1:0.06; v1:1.2;
g:9.8;
(fpprintprec)   5
(ratprint)   false
(h)       0.06
(v1)       1.2
(g)       9.8
(%i7)  solve(m*g*h1 + 1/2*m*v1^2 = 1/2*m*v2^2, v2)$ float(%);
(%o7)   [v2=-1.6174,v2=1.6174]
(%i9)   solve(m*g*h1+1/2*m*v1^2=m*g*h3, h3)$ float(%);
(%o9)   [h3=0.13347]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of h_1, v_1, and g.
(%i7) Solve $mgh_1 + \frac{1}{2}mv_1^2 = \frac{1}{2}mv_2^2$ for v_2.
(%i9) Solve $mgh_1 + \frac{1}{2}mv_1^2 = mgh_3$ for h_3.

Problem 7.17 The length of a light spring is $l = 0.15$ m. An object of mass $m = 3.0$ kg is suspended to the spring. The object is raised such that the spring is neither stretched nor compressed. The object is suddenly released, falls by a distance of $h = 0.10$ m, and oscillates until it stops. Calculate the force constant, k, of the spring and the position of the object when it stops.

Solution
Figure 7.11 shows the three situations of the spring and the object. Let the reference line be the line at the level of the unstretched and uncompressed spring. The line is also the level where the gravitational potential energy is zero**.**

Applying the conservation of energy in the initial and final situations (Eq. 7.10), we write,

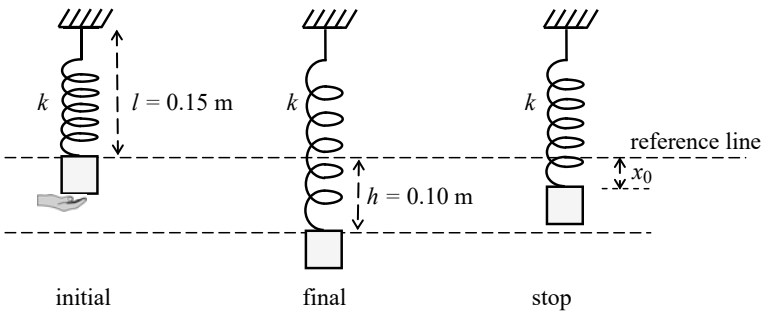


Fig. 7.11 Three situations of the mass-spring system, Problem 7.17

$$U_{init} + K_{init} = U_{final} + K_{final},$$
$$U_{init,elastic} + U_{init,gravity} + K_{init} = U_{final,elastic} + U_{final,gravity} + K_{final},$$

because the potential energy of a mass-spring system is the elastic potential energy of the spring plus the gravitational potential energy of the object.

One gets,

$$U_{init,elastic} + U_{init,gravity} + K_{init} = U_{final,elastic} + U_{final,gravity} + K_{final},$$
$$0 + 0 + 0 = \frac{1}{2}kh^2 - mgh + 0.$$

In the initial situation, $U_{init,elastic} = 0$ because the spring is neither stretched nor compressed, $U_{init,gravity} = 0$ because the object is on the reference line, and $K_{init} = 0$ because the object is at rest. In the final situation, $U_{final,elastic} = kh^2/2$ because the spring is stretched by a distance of h, $U_{final,gravity} = -mgh$ because the object is a distance of h below the reference line, and $K_{final} = 0$ because the object stops momentarily.

From the last equation, the force constant, k, of the spring is,

$$k = \frac{2mg}{h} = \frac{2(3.0\,\text{kg})(9.8\,\text{N/kg})}{0.10\,\text{m}} = 588\,\text{N m}^{-1}.$$

The extension, x_0, of the spring due to suspension of 3.0 kg object is calculated by the Hooke's law, as as shown in the "stop" situation (Fig. 7.11),

$$kx_0 = mg,$$
$$x_0 = \frac{mg}{k} = \frac{(3.0\,\text{kg})(9.8\,\text{N/kg})}{588\,\text{N/m}} = 0.05\,\text{m}.$$

The equilibrium stretched length of the spring when the object stops oscillating in "stop" situation of Fig. 7.11 is,

$$\text{unstretched length } l + \text{extension } x_0 = 0.15\,\text{m} + 0.05\,\text{m} = 0.20\,\text{m}.$$

• wxMaxima codes:

```
(%i6)    fpprintprec:5; ratprint:false; l:0.15; m:3; h:0.1;
g:9.8;
(fpprintprec)   5
(ratprint)   false
(l)      0.15
(m)      3
(h)      0.1
(g)      9.8
(%i8)   solve(0 = 1/2*k*h^2 - m*g*h, k)$ float(%);
```

```
(%o8)    [k=588.0]
(%i9)    k: rhs(%[1]);
(k)       588
(%i10)   x0: m*g/588;
(x0)    0.05
(%i11)   l + x0;
(%o11)   0.2
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of l, m, h, and g.
(%i7) Solve $0 = \frac{1}{2}kh^2 - mgh$ for k.
(%i9) Assign value of k.
(%i10) Calculate extension, x_0.
(%i11) Calculate the stretched length.

Problem 7.18 A block of mass, m, is suspended to a spring with a force constant of k. The block is raised such that the spring is neither stretched nor compressed. Then, abruptly the block is released. Determine the lowest position of the block after the release.

Solution
Figure 7.12 shows the situation before (1) and after (2) the block is released. Let P be the reference line where the gravitational potential energy is zero.

Before (1) the block is released, we have,

K_1 = kinetic energy = 0, the block is at rest
$U_{elastic1}$ = elastic potential energy = 0, the spring is neither stretched nor compressed
$U_{gravity1}$ = gravitational potential energy = 0, the block is at zero gravitational potential energy level

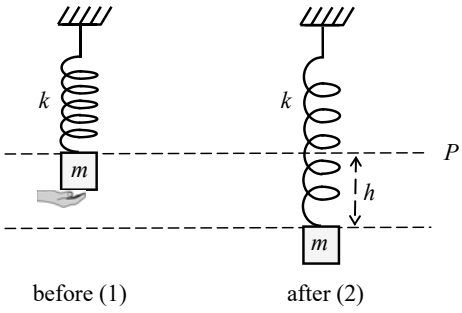


Fig. 7.12 Two situations of a mass-spring system, Problem 7.18

The block is abruptly released and falls to the lowest position, and the spring is stretched. So after (2) the block is released,

K_2 = kinetic energy = 0, the block stops momentarily
$U_{elastic2}$ = elastic potential energy = $\frac{1}{2}kh^2$, the spring is stretched by a distance of h
$U_{gravity2}$ = gravitational potential energy = $-mgh$, the block is a distance of h down the zero gravitational potential energy level

Applying the conservation of energy, we have,

$K_1 + U_{elastic1} + U_{gravity1} = K_2 + U_{elastic2} + U_{gravity2}$,
$0 + 0 + 0 = 0 + \frac{1}{2}kh^2 - mgh$.

Solving the last equation gives the lowest position, h, of the block as,

$$h = \frac{2mg}{k}.$$

Note: The equilibrium extension of the spring, x_0, is calculated by the Hooke's law,

$$kx_0 = mg,$$
$$x_0 = \frac{mg}{k} = \frac{mg}{(2mg/h)} = h/2 = \frac{mg}{k}.$$

This means that the equilibrium extension, x_0, is one half of h.

Problem 7.19 An object slides from the rest on a track with negligible friction (Fig. 7.13). The track consists of a steep section, followed by a circular section of radius $r = 0.50$ m and ending with a horizontal flat one. What is the height, h, so that,

(a) the normal reaction of the track on the object at point A is 6 times the weight of the object?
(b) the normal reaction of the track on the object at point B is 4 times the weight of the object?
(c) the object just traverses the circular track?

Solution

(a) The speed of the object at point A, v_A, is obtained by applying the conservation of mechanical energy (Eq. 7.13),

$$U_{init} + K_{init} = U_A + K_A,$$
$$mgh + 0 = mgr + \frac{1}{2}mv_A^2,$$

where U_{init} and K_{init} are the initial potential and kinetic energies of the object, while U_A and K_A are the potential and kinetic energies of the object at point A.

Solving the last equation one gets the speed of the object at point A as,

$$v_A = \sqrt{2g(h-r)}. \tag{7.14}$$

At point A, the normal reaction of the track on the object is the centripetal force on the object and it is 6 times the weight of the object. This means that,

$$F_{centripetal} = \frac{mv_A^2}{r} = 6mg. \tag{7.15}$$

Substituting Eq. (7.14) into (7.15) one gets,

$$\frac{m \cdot 2g(h-r)}{r} = 6mg,$$
$$2h - 2r = 6r,$$
$$h = 4r = 4(0.50\,\text{m}) = 2.0\,\text{m}.$$

The object should be released from the height of 2.0 m.

(b) The speed, v_B, of the object at point B is obtained by applying the conservation of mechanical energy (Eq. 7.13),

$$U_{init} + K_{init} = U_B + K_B,$$

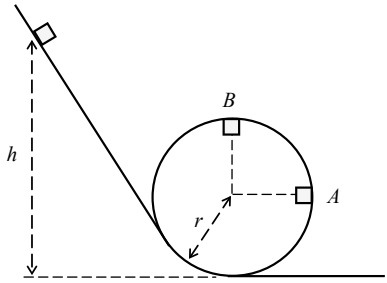


Fig. 7.13 An object slides on a tract of negligible friction, Problem 7.19

$$mgh + 0 = mg(2r) + \frac{1}{2}mv_B^2,$$

where U_{init} and K_{init} are the initial potential and kinetic energies of the object, while U_B and K_B are the potential and kinetic energies of the object at point B.

Solving the last equation one gets the speed of the object at point B as,

$$v_B = \sqrt{2g(h - 2r)}. \tag{7.16}$$

At point B, the centripetal force on the object is the normal reaction of the track on the object (the magnitude is $4mg$) and the weight of the object (the magnitude is mg) given by,

$$F_{centripetal} = \frac{mv_B^2}{r} = 4mg + mg = 5mg. \tag{7.17}$$

Substituting Eq. (7.16) into (7.17) one gets,

$$\begin{aligned} \frac{m \cdot 2g(h - 2r)}{r} &= 5mg, \\ 2h - 4r &= 5r, \\ h = \frac{9r}{2} &= \frac{9(0.50\,\text{m})}{2} = 2.25\,\text{m}. \end{aligned}$$

The object should be released from the height of 2.25 m.

(c) The speed, v_B, of the object at point B is obtained by applying the conservation of mechanical energy (Eq. 7.13),

$$\begin{aligned} U_{init} + K_{init} &= U_B + K_B, \\ mgh + 0 &= mg(2r) + \frac{1}{2}mv_B^2, \end{aligned}$$

where U_{init} and K_{init} are the initial potential and kinetic energies of the object, while U_B and K_B are the potential and kinetic energies of the object at point B.

Solving the last equation we get the speed of the object at point B as,

$$v_B = \sqrt{2g(h - 2r)}. \tag{7.18}$$

If the object just traverses the circular section of the track at point B, then the centripetal force on the object is the weight of the object and the normal reaction of the track on the object is zero. Thus,

$$F_{centripetal} = \frac{mv_B^2}{r} = mg. \tag{7.19}$$

Substituting Eq. (7.18) into (7.19) we get,

$$\frac{m \cdot 2g(h-2r)}{r} = mg,$$
$$2h - 4r = r,$$
$$h = \frac{5r}{2} = \frac{5(0.50\,\text{m})}{2} = 1.25\,\text{m}.$$

The object should be released from the height of 1.25 m.
• wxMaxima codes:

```
(%i4)   fpprintprec:5; ratprint:false; r:0.5; g:9.8;
(fpprintprec)   5
(ratprint)   false
(r)      0.5
(g)      9.8
(%i6)   solve(m*2*g*(h-r)/r=6*m*g, h)$ float(%);
(%o6)   [h=2.0]
(%i8)   solve(m*2*g*(h-2*r)/r=5*m*g, h)$ float(%);
(%o8)   [h=2.25]
(%i10)  solve(m*2*g*(h-2*r)/r=m*g, h)$ float(%);
(%o10)  [h=1.25]
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of r and g.
(%i6) Solve $m \cdot 2g(h-r)/r = 6mg$ for h. Part (a).
(%i8) Solve $m \cdot 2g(h-2r)/r = 5mg$ for h. Part (b).
(%i10) Solve $m \cdot 2g(h-2r)/r = mg$ for h. Part (c).

Problem 7.20 A block of mass, $m = 2.0$ kg, on a rough inclined plane at 37° is connected to a light spring with a force constant of $k = 100$ N m^{-1} (Fig. 7.14). The block is positioned so that the spring is neither stretched nor compressed. The block is released from rest and it stretches the spring and slides down by 20 cm. Calculate the coefficient of dynamic friction, μ, of the block and the inclined plane.

Solution
Figure 7.15 shows the spring and the block in the initial and final situations. The weight of the block, mg, is resolved into the components of $mg\cos\theta$ and $mg\sin\theta$. The normal reaction of the incline on the block is $R = mg\cos\theta$, the friction is $\mu mg\cos\theta$, and the block moves a distance, x, while stretching the spring.

As the block slides down the incline and stretches the spring, the change in gravitational potential energy of the block is used to stretch the spring and to do work against the friction. Then,

change in potential energy of the block = energy stored in the stretched spring + work against friction
$mgh = \frac{1}{2}kx^2 + x\mu mg \cos\theta.$

But $h = x \sin\theta$, the equation becomes,

$$mgx \sin\theta = \frac{1}{2}kx^2 + x\mu mg \cos\theta.$$

The coefficient of dynamic friction, μ, is obtained from this equation as,

$$\begin{aligned}\mu &= \frac{mg\sin\theta - \frac{1}{2}kx}{mg\cos\theta} \\ &= \frac{(2.0\,\text{kg})(9.8\,\text{m/s}^2)\sin 37^\circ - \frac{1}{2}(100\,\text{N/m})(0.20\,\text{m})}{(2.0\,\text{kg})(9.8\,\text{m/s}^2)\cos 37^\circ} \\ &= 0.11.\end{aligned}$$

• wxMaxima codes:

```
(%i7)    fpprintprec:5; ratprint:false; m:2; theta:float(37/
180*%pi); k:100; x:0.2; g:9.8;
(fpprintprec)    5
```

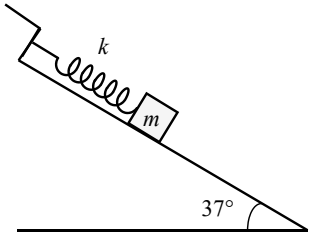


Fig. 7.14 An object attached to a spring on a rough inclined plane, Problem 7.20

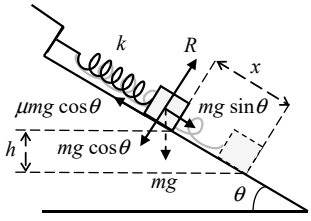


Fig. 7.15 Forces acting on the object, Problem 7.20

```
(ratprint)    false
(m)    2
(theta)       0.64577
(k)       100
(x)       0.2
(g)       9.8
(%i9)              solve(m*g*x*sin(theta)   =   1/2*k*x^2   +
x*mu*m*g*cos(theta), mu)$ float(%);
(%o9)    [mu=0.11471]
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of m, θ, k, x, and g.
(%i9) Solve $mgx \sin\theta = \frac{1}{2}kx^2 + x\mu mg \cos\theta$ for μ.

Problem 7.21 A particle of mass, $m = 0.20$ kg, is fixed to a light rod of length, $l = 1.0$ m. The other end of the rod is pivoted to a wall, and the system is free to move in vertical circle (Fig. 7.16). The system is released from rest at angle, $\theta = 53°$. Calculate the speed, v, of the particle and the tension, T, in the rod when the particle is at the lowest position.

Solution
Figure 7.17 shows the particle and the rod in the initial and final instances. Let the horizontal line through the lowest position of the particle be the reference line.

Applying the conservation of energy in the initial and final instances, we have,

$$K_{init} + U_{init} = K_{final} + U_{final},$$
$$0 + mg(l\sin\theta + l) = \frac{1}{2}mv^2 + 0,$$

where K_{init} and U_{init} are the initial kinetic and potential energies of the particle, while K_{final} and U_{final} are the final kinetic and potential energies of the particle. K_{init} is zero

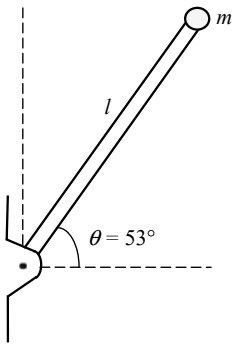


Fig. 7.16 A particle attached to a light rod free to swing around a pivot, Problem 7.21

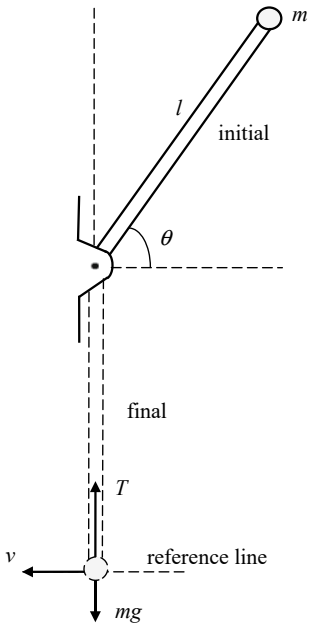


Fig. 7.17 The particle-rod system in the initial and final situations, Problem 7.21

because the system starts from rest, $l \sin\theta + l$ is the height of the particle relative to the reference line at the beginning, and v is the speed of the particle at its lowest position.

The last equation gives the speed, v, of the particle at the lowest position as,

$$v = \sqrt{2gl(1+\sin\theta)} = \sqrt{2(9.8\,\mathrm{m/s^2})(1.0\,\mathrm{m})(1+\sin 53^\circ)}$$
$$= 5.9\,\mathrm{m\,s^{-1}}.$$

At the final instance, the centripetal force on the particle is,

$$\frac{mv^2}{r} = T - mg.$$

Therefore, the tension, T, in the rod is,

$$T = \frac{mv^2}{r} + mg = \frac{m \cdot 2gl(1+\sin\theta)}{l} + mg = mg(3+2\sin\theta)$$
$$= (0.20\mathrm{kg})(9.8\mathrm{m/s^2})(3+2\sin 53^\circ)$$
$$= 9.0\ \mathrm{N}.$$

• wxMaxima codes:

```
(%i6)          fpprintprec:5;  ratprint:false;  m:0.2;  l:1;
theta:float(53/180*%pi); g:9.8;
(fpprintprec)    5
(ratprint)    false
(m)       0.2
(l)       1
(theta)    0.92502
(g)       9.8
(%i8)   solve(m*g*(l*sin(theta)+l) = 1/2*m*v^2, v)$ float(%);
(%o8)    [v=-5.9374,v=5.9374]
(%i9)    T: m*g*(3+2*sin(theta));
(T)        9.0107
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m, l, θ, and g.
(%i8) Solve $mg(l\sin\theta + l) = \frac{1}{2}mv^2$ for v.
(%i9) Calculate tension, T, in the rod.

Problem 7.22 A rope of length, L, is placed on a slippery table (Fig. 7.18). A partial length, y_0, of the rope is hanging at the edge of the table. The rope is let go from rest so that it falls to the floor.

(a) Calculate the speed, v_{final}, of the rope when all of it leaves the table
(b) What is the time, τ, taken by the rope to reach the speed?

Solution

(a) Figure 7.19 shows the rope at its initial and final instances. Let the horizontal line of the table surface be the reference level and ρ be the mass per unit length of the rope.

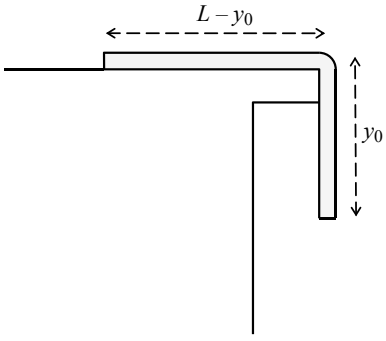


Fig. 7.18 A rope on a slippery table, Problem 7.22

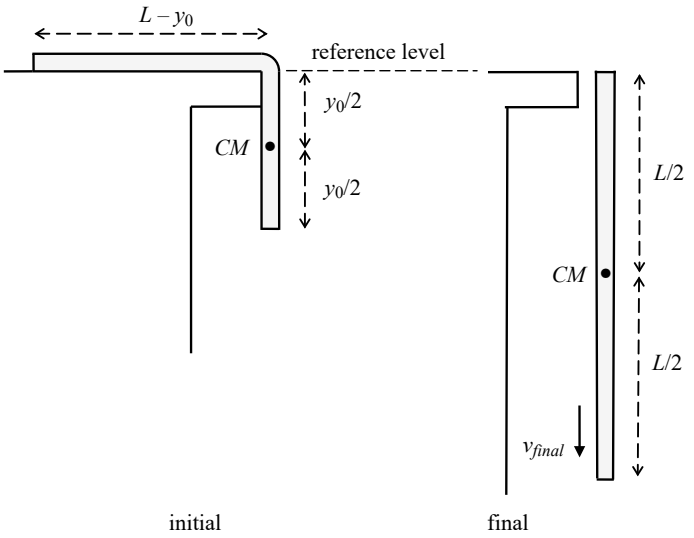


Fig. 7.19 The rope in initial and final situations, Problem 7.22

Applying the conservation of energy to initial and final instances gives,

$$U_{init} + K_{init} = U_{final} + K_{final},$$
$$-(\rho y_0)g\left(\frac{y_0}{2}\right) + 0 = -(\rho L)g\left(\frac{L}{2}\right) + \frac{1}{2}(\rho L)v_{final}^2.$$

where U_{init} and K_{init} are the initial potential and kinetic energies, while U_{final} and K_{final} are the final potential and kinetic energies of the rope.

At the initial instance, ρy_0 is the mass of hanging part of the rope and $y_0/2$ is the distance of the center of mass of hanging part of the rope to the reference level. Thus, $U_{init} = -(\rho y_0)g(y_0/2)$. $K_{init} = 0$ because the rope is released from rest.

At the final instance, ρL is the mass of the whole rope, $L/2$ is the distance of the center of mass of the whole rope to the reference level. Thus, $U_{final} = -(\rho L)g(L/2)$. $K_{final} = \frac{1}{2}(\rho L)v_{final}^2$, where v_{final} is the speed as the whole rope leaves the table.

Solving the equation gives the speed, v_{final}, of the whole rope leaving the table,

$$v_{final} = \sqrt{\frac{g}{L}(L^2 - y_0^2)}.$$

(b) Let at a particular instance, the length of the hanging part of the rope be y. The conservation of energy at initial and particular instances becomes,

$$U_{init} + K_{init} = U + K,$$

$$-(\rho y_0)g\left(\frac{y_0}{2}\right)+0=-(\rho y)g\left(\frac{y}{2}\right)+\frac{1}{2}(\rho L)v^2,$$

where U_{init} and K_{init} are the initial potential and kinetic energies, while U and K are the potential and kinetic energies of the rope at the particular instance.

At the initial instance, ρy_0 is the mass of hanging part of the rope and $y_0/2$ is the distance of the center of mass of hanging part of the rope to the reference level. Thus, $U_{init} = -(\rho y_0)g(y_0/2)$. $K_{init} = 0$ because the rope is released from rest.

At the particular instance, ρy is the mass of hanging part of the rope and $y/2$ is the distance of the center of mass of hanging part of the rope to the reference level. Thus, $U = -(\rho y)g(y/2)$.

The mass of the whole rope is ρL, $K = \frac{1}{2}(\rho L)v^2$, and v is the speed of the rope at that particular instance.

Solving the equation one gets,

$$v=\frac{dy}{dt}=\sqrt{\frac{g}{L}}\sqrt{y^2-y_0^2}.$$

The time, τ, for the rope to get to final speed, v_{final}, is obtained by definite integration as follows (Appendix D),

$$\sqrt{\frac{g}{L}}dt=\frac{dy}{\sqrt{y^2-y_0^2}}$$

$$\int_0^{\tau}\sqrt{\frac{g}{L}}dt=\int_{y_0}^{L}\frac{dy}{\sqrt{y^2-y_0^2}}$$

$$\sqrt{\frac{g}{L}}\tau=\left[\ln\left(y+\sqrt{y^2-y_0^2}\right)\right]_{y_0}^{L}$$

$$\tau=\sqrt{\frac{L}{g}}\ln\left(\frac{L+\sqrt{L^2-y_0^2}}{y_0}\right).$$

• wxMaxima codes:

```
(%i1)   assume(y0>0,L>0,L>y0);
(%o1)   [y0>0,L>0,L>y0]
(%i2)   left: integrate( sqrt(g/L), t, 0, tau);
(left)    (sqrt(g)*tau)/sqrt(L).
(%i3)   right: integrate( 1/sqrt(y^2 - y0^2), y, y0, L);
(right)   log(8*L*sqrt(L^2-y0^2)-4*y0^2+8*L^2)/2-log(2*y0)
(%i5)   solve(left = right, tau) $ radcan(%);
(%o5)         [tau=(sqrt(L)*(log(2*L*sqrt(L-y0)*sqrt(y0+L)-
y0^2+2*L^2)
```

```
-2*log(y0)))/(2*sqrt(g))]
```

Comments on the codes:

(%i1) Assume $y_0 > 0$, $L > 0$, and $L > y_0$.
(%i2), (i3) Calculate $\int_0^\tau \sqrt{\frac{g}{L}}dt$ and assign it as *left*, calculate $\int_{y_0}^{L} \frac{dy}{\sqrt{y^2-y_0^2}}$ and assign it as *right*.
(%i5) Solve *left* = *right* for τ.
(%o5) The result.

The codes say that, (%o5),

$$\tau = \frac{\sqrt{L}[\ln(2L\sqrt{L-y_0}\sqrt{L+y_0} - y_0^2 + 2L^2) - 2\ln y_0]}{2\sqrt{g}}.$$

We simplify this as,

$$\begin{aligned}
\tau &= \frac{\sqrt{L}}{\sqrt{g}}[\frac{\ln(2L\sqrt{L-y_0}\sqrt{L+y_0} - y_0^2 + 2L^2)}{2} - \ln y_0] \\
&= \frac{\sqrt{L}}{\sqrt{g}}[\frac{\ln((L+\sqrt{L^2-y_0^2})^2)}{2} - \ln y_0] \\
&= \frac{\sqrt{L}}{\sqrt{g}}[\ln(L+\sqrt{L^2-y_0^2}) - \ln y_0] \\
&= \sqrt{\frac{L}{g}}\ln\left(\frac{L+\sqrt{L^2-y_0^2}}{y_0}\right).
\end{aligned}$$

Problem 7.23 Two blocks of masses $m_1 = 6.0$ kg and $m_2 = 1.0$ kg connected by a string through a light pulley are released from rest (Fig. 7.20). This is an Atwood machine with a light pulley. The distance of the block 1 from the floor is $h = 2.0$ m. Calculate the speed at which the block 1 hits the floor.

Solution
Figure 7.21 shows the two blocks in the initial and final instances. The reference level is as indicated and the speed at which the block 1 hits the floor is v_{final}.

Applying the conservation of energy to the initial and final instances (Eq. 7.10), we obtain,

$$\begin{aligned}
U_{init} + K_{init} &= U_{final} + K_{final}, \\
m_1 gh + 0 &= m_2 gh + \frac{1}{2}(m_1 + m_2)v_{final}^2.
\end{aligned}$$

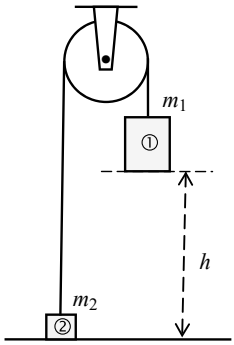


Fig. 7.20 An Atwood machine with a light pulley, Problem 7.23

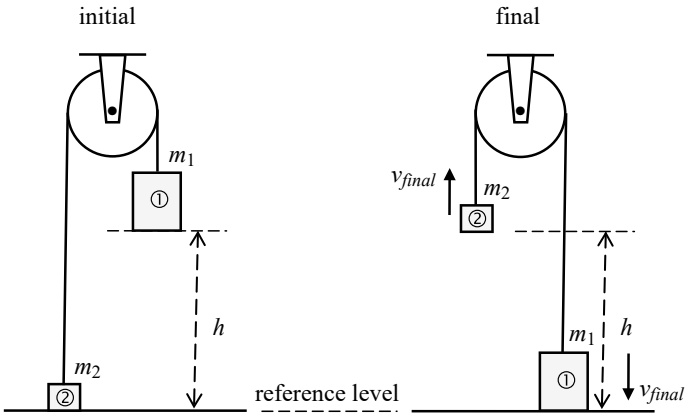


Fig. 7.21 Initial and final instances of the Atwood machine, Problem 7.23

where U_{init} and K_{init} are the initial potential and kinetic energies of both blocks, while U_{final} and K_{final} are their final potential and kinetic energies.

At the initial instance, $U_{init} = m_1 gh$ because the block 1 is at a distance h above the reference level and block 2 is at zero potential energy reference level, and $K_{init} = 0$ because both blocks are at rest.

At the final instance, $U_{final} = m_2 gh$ because the block 2 is at a distance h above the reference level and block 1 is at zero potential energy reference level. $K_{final} = \frac{1}{2}(m_1 + m_2)v_{final}^2$ because both blocks are moving with speed v_{final}.

Solving the last equation gives the speed, v_{final}, at which the block 1 hits the floor as,

$$v_{final} = \sqrt{2\left(\frac{m_1 - m_2}{m_1 + m_2}\right)gh}$$

$$= \sqrt{2\left(\frac{6.0\,\text{kg} - 1.0\,\text{kg}}{6.0\,\text{kg} + 1.0\,\text{kg}}\right)(9.8\,\text{m/s}^2)(2.0\,\text{m})}$$
$$= 5.3\,\text{m}\,\text{s}^{-1}.$$

• wxMaxima codes:

```
(%i6)  fpprintprec:5; ratprint:false; m1:6; m2:1; g:9.8; h:2;
(fpprintprec)   5
(ratprint)   false
(m1)      6
(m2)      1
(g)      9.8
(h)      2
(%i8)   solve(m1*g*h=m2*g*h+0.5*(m1+m2)*vfinal^2, vfinal)$
float(%);
(%o8)   [vfinal=-5.2915,vfinal=5.2915]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m_1, m_2, g, and h.
(%i8) Solve $m_1gh = m_2gh + \frac{1}{2}(m_1 + m_2)v_{final}^2$ for v_{final}.

• Alternative calculation:

```
(%i2)   fpprintprec:5; ratprint:false;
(fpprintprec)   5
(ratprint)   false
(%i3)   solve(m1*g*h=m2*g*h+0.5*(m1+m2)*vfinal^2, vfinal);
(%o3)      [vfinal=-sqrt(2)*sqrt((g*h*m1)/(m2+m1)-(g*h*m2)/
(m2+m1)),
             vfinal=sqrt(2)*sqrt((g*h*m1)/(m2+m1)-(g*h*m2)/
(m2+m1))]
(%i5)   subst([m1=6, m2=1, g=9.8, h=2], %)$ float(%);
(%o5)   [vfinal=-5.2915,vfinal=5.2915]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i3) Solve $m_1gh = m_2gh + \frac{1}{2}(m_1 + m_2)v_{final}^2$ in symbols for v_{final}.

(%o3) The solutions in symbols.

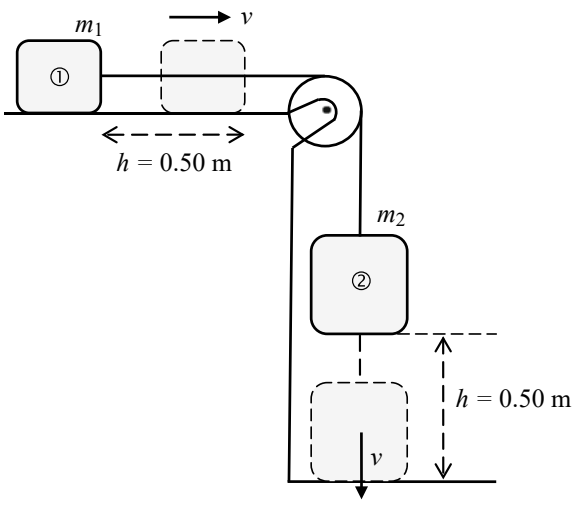


Fig. 7.22 Two-block system released from rest. Block 1 is on a rough surface. Problem 7.24

(%i5) Substitute values of m_1, m_2, g, and h into the solutions to get numerical value of v_{final}.

Alternative solution: A similar problem has been solved in Problem 5.20 using the Newton's second law. It needs consideration of all the forces acting on both masses, setting two Newton's equations for both masses, and solving for acceleration. The acceleration then is used to calculate the final speed. One should appreciate the power and beauty of the energy consideration used to solve Problem 7.23.

Problem 7.24 A block 1 of mass $m_1 = 6.0$ kg on a horizontal surface with coefficient of kinetic friction $\mu_k = 0.22$ is connected by a string and a light pulley to the block 2 of mass $m_2 = 3.0$ kg (Fig. 7.22). The system is released from rest, moves by a distance of $h = 0.50$ m, and the block 2 hits the floor. Calculate the speed, v, of the block 2 hitting the floor.

Solution
Let the initial state be the rest state before the system is released and the final state be the instance at which block 2 hits the floor with speed v. Applying the conservation of energy, we have,

change in gravitational potential energy of block 2 = Kinetic energy of both blocks + work against friction by block 1
$m_2gh = \frac{1}{2}(m_1 + m_2)v^2 + \mu_k m_1 gh.$

The work against friction by the block 1 is $W = f_{friction}h = \mu_k m_1 gh$.
The speed, v, at which the block 2 hits the floor is,

$$
\begin{aligned}
v &= \sqrt{2\left(\frac{m_2 - \mu_k m_1}{m_1 + m_2}\right)gh} \\
&= \sqrt{2\left(\frac{3.0\,\mathrm{kg} - 0.22 \times 6.0\,\mathrm{kg}}{6.0\,\mathrm{kg} + 3.0\,\mathrm{kg}}\right)(9.8\,\mathrm{m/s^2})(0.50\,\mathrm{m})} \\
&= 1.4\,\mathrm{m\,s^{-1}}.
\end{aligned}
$$

• wxMaxima codes:

```
(%i7)   fpprintprec:5; ratprint:false; m1:6; m2:3; mu_k:0.22;
g:9.8; h:0.5;
(fpprintprec)   5
(ratprint)   false
(m1)      6
(m2)      3
(mu_k)    0.22
(g)       9.8
(h)       0.5
(%i9)         solve(m2*g*h=0.5*(m1+m2)*v^2+mu_k*m1*g*h, v)$
float(%);
(%o9)   [v=-1.3525,v=1.3525]
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of m_1, m_2, μ_k, g, and h.
(%i9) Solve $m_2gh = \frac{1}{2}(m_1 + m_2)v^2 + \mu_k m_1 gh$ for v.

• Alternative calculation:

```
(%i2)   fpprintprec:5; ratprint:false;
(fpprintprec)   5
(ratprint)   false
(%i3)   solve(m2*g*h=0.5*(m1+m2)*v^2 + mu_k*m1*g*h, v);
(%o3)      [v=-sqrt(2)*sqrt((g*h*m2)/(m2+m1)-(g*h*m1*mu_k)/
(m2+m1)),
            v=sqrt(2)*sqrt((g*h*m2)/(m2+m1)\-(g*h*m1*mu_k)/
(m2+m1))]
(%i5)      subst([m1=6, m2=3, mu_k=0.22, g=9.8, h=0.5], %)$
float(%);
(%o5)   [v=-1.3525,v=1.3525]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i3) Solve $m_2gh = \frac{1}{2}(m_1 + m_2)v^2 + \mu_k m_1 gh$ in symbols for v.
(%o3) The solutions in symbols.
(%i5) Substitute values of m_1, m_2, μ_k, g, and h into the solutions to get numerical value of v.

Alternative solution: An alternative solution is Problem 5.13 where the Newton's second law has been used. Here, in Problem 7.24 the conservation of energy is evoked instead of the Newton's second law.

Problem 7.25 A stone is dropped from a height of h and hits the ground with speed of v. At what height, h_1, must the stone be dropped so that it hits the ground with speed of $2v$?

Solution
Potential energy of the stone at height h ($U = mgh$) is converted to kinetic energy of the stone $\left(K = \frac{1}{2}mv^2\right)$. This means that h is directly proportional to v^2. We have,

$$\frac{h_1}{h} = \left(\frac{v_1}{v}\right)^2 = \left(\frac{2v}{v}\right)^2 = 4,$$
$$h_1 = 4h.$$

The stone is dropped from a height $h_1 = 4h$ so that it hits the ground with the speed of $2v$.

Alternative solution: The speed is directly proportional to the square root of the height. Thus,

$$\frac{v_1}{v} = \sqrt{\frac{h_1}{h}}$$
$$\frac{2v}{v} = \sqrt{\frac{h_1}{h}}$$
$$h_1 = 4h.$$

7.3 Summary

- Work done by a constant force, $W = \boldsymbol{F}\cdot\boldsymbol{s} = Fs\cos\theta$.
- Work done by a variable force, $W = \int_i^f \boldsymbol{F}\cdot\boldsymbol{dr}$.
- Kinetic energy of a particle, $K = \frac{1}{2}mv^2$.
- Work-energy theorem, $W = \Delta K = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$.
- Elastic potential energy stored in a spring, $U_{elastic} = \frac{1}{2}kx^2$.
- Gravitational potential energy, $U_{gravity} = mgy$.
- Conservation of mechanical energy, $E = K_1 + U_1 = K_2 + U_2 = \text{constant}$.

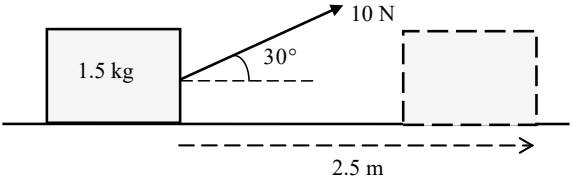


Fig. 7.23 A force applied on a block displaces the block by 2.5 m, Exercise 7.1

7.4 Exercises

Exercise 7.1

A force of 10 N at 30° above the horizontal is applied to a block of mass 1.5 kg on a table (Fig. 7.23). The block moves a distance of 2.5 m. The coefficient of kinetic friction of the block and the table is 0.20. Calculate the work done by the force and the friction.

(Answer: 22 J, − 4.8 J)

Exercise 7.2

A block of mass 3.0 kg is pushed up an incline plane at 30° above the horizontal a distance of 2.0 m by a horizontal force of 40 N (Fig. 7.24). The coefficient of kinetic friction of the block with the incline is 0.10. Calculate the work done by the force of 40 N, the gravitational force, and the friction.

(Answer: 69 J, − 29 J, − 9.1 J)

Exercise 7.3

A projectile of mass 0.50 kg is launched with an initial speed of 10 m s^{-1} at an angle 60° above the horizontal. What is the potential energy and kinetic energy of the projectile at its highest point?

(Answer: 19 J, 6 J)

Exercise 7.4

A particle is to climb at a height of 0.5 m on a track with negligible friction (Fig. 7.25). What is the minimum speed required to reach this height?

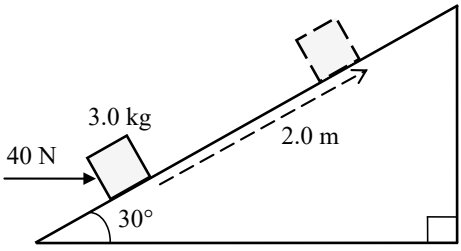


Fig. 7.24 A force pushes a block up a rough inclined plane, Exercise 7.2

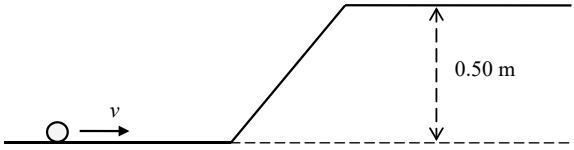


Fig. 7.25 A particle climbs a track of 0.50 m high, Exercise 7.4

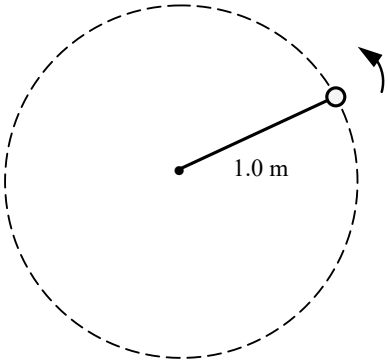


Fig. 7.26 An object tied to a sting swung in a vertical circle, Exercise 7.5

(Answer: 3.1 m s^{-1})

Exercise 7.5

An object tied to a string of length 1.0 m is swung in a vertical circle just fast enough to prevent the string from going slack at the top of the circle (Fig. 7.26). What are the speeds of the object at the bottom and top of the circle?

(Answer: 7.0 m s^{-1}, 3.1 m s^{-1})

Chapter 8
Linear Momentum and Collision

8.1 Basic Concepts and Formulae

(1) A particle of mass, m, moving with velocity, $\boldsymbol{v}$, has a linear momentum of,

$$\boldsymbol{p} = m\boldsymbol{v}. \tag{8.1}$$

(2) The impulse of a force, $\boldsymbol{F}$, on a particle is the change in momentum of the particle,

$$I = \Delta \boldsymbol{p} = \int_{t_{init}}^{t_{final}} \boldsymbol{F}\, dt. \tag{8.2}$$

(3) The conservation of linear momentum states that for an isolated system in which the bodies are interacting, the total momentum is always conserved. For example, for a system of two particles in collision, the total linear momentum before and after collision are the same,

$$\boldsymbol{p}_{1,init} + \boldsymbol{p}_{2,init} = \boldsymbol{p}_{1,final} + \boldsymbol{p}_{2,final}, \tag{8.3}$$

or,

$$m_1\boldsymbol{v}_{1,init} + m_2\boldsymbol{v}_{2,init} = m_1\boldsymbol{v}_{1,final} + m_2\boldsymbol{v}_{2,final},$$

where $\boldsymbol{p}_{1,init}$ and $\boldsymbol{p}_{2,init}$ are the momenta of particle 1 and 2 before the collision, while $\boldsymbol{p}_{1,final}$ and $\boldsymbol{p}_{2,final}$ are their momenta after the collision.

(4) In an elastic collision, the linear momentum and the kinetic energy are conserved. The total linear momenta of the bodies before and after the collision are the same. So is the kinetic energy, the total kinetic energies of the bodies before and after the collision are equal.

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https://doi.org/10.1007/978-3-031-42678-0_8

(5) In an inelastic collision, the linear momentum is conserved but not the kinetic energy. This means that the total linear momenta of the bodies before and after the collision are the same. However, the total kinetic energies of the bodies before and after the collision are not equal.
(6) In a completely inelastic collision, the particles that collide stick together after the collision.
(7) The coefficient of restitution, e, is defined as,

$$e = \frac{\text{relative speed after the collision}}{\text{relative speed before the collision}} = \frac{v_2 - v_1}{u_1 - u_2}, \tag{8.4}$$

where u_1 and u_2 are speeds of particle 1 and 2 before the collision, while v_1 and v_2 are their speeds after the collision. For an elastic collision, $e = 1$.
(8) The position vector of the center of mass of the system of point particles is,

$$\boldsymbol{r}_{CM} = \frac{\sum_i m_i \boldsymbol{r}_i}{\sum_i m_i}, \tag{8.5}$$

where m_i and $\boldsymbol{r}_i$ are the mass and the position vector of the ith particle. This means that,

$$\boldsymbol{r}_{CM} = x_{CM}\mathbf{i} + y_{CM}\mathbf{j} + z_{CM}\mathbf{k},$$
$$x_{CM} = \frac{\sum_i m_i x_i}{\sum_i m_i}, \quad y_{CM} = \frac{\sum_i m_i y_i}{\sum_i m_i}, \quad z_{CM} = \frac{\sum_i m_i z_i}{\sum_i m_i}.$$

(9) For a rigid body, the position vector of the center of mass is,

$$\boldsymbol{r}_{CM} = \frac{1}{M}\int \boldsymbol{r}\, dm, \tag{8.6}$$

where M is the mass of the body.

8.2 Problems and Solutions

Problem 8.1 A particle of mass 0.40 kg moving with the velocity of 3.0 m s^{-1} collides head on with a particle of mass 0.60 kg at rest. The collision is perfectly elastic and occurred in one dimension. Calculate the velocity of each particle after the collision.

Solution

Figure 8.1 shows the states of two colliding particles, before and after the collision. Here, m_1 and m_2 are the masses of particle 1 (0.40 kg) and particle 2 (0.60 kg), u_1

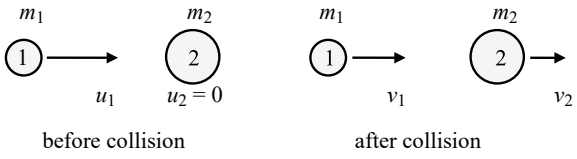

Fig. 8.1 The states of two colliding particles, before and after the collision, Problem 8.1

and u_2 are the velocities of particle 1 and 2 before the collision, while v_1 and v_2 are their velocities after the collision, respectively.

Application of conservation of linear momentum (the total momentum is the same before and after the collision) as in Eq. (8.3) gives,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2,$$

$$(0.4)(3) + (0.6)(0) = 0.4v_1 + 0.6v_2, \tag{8.1}$$

where $m_1u_1 + m_2u_2$ is the total momentum before the collision and $m_1v_1 + m_2v_2$ is the one after.

For an elastic collision the kinetic energy is conserved, this means that the total kinetic energy of the particles before the collision is equal to the one after. We write,

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2,$$

$$\frac{1}{2}(0.4)(3)^2 + \frac{1}{2}(0.6)(0)^2 = \frac{1}{2}(0.4)v_1^2 + \frac{1}{2}(0.6)v_2^2, \tag{8.2}$$

where the two terms on the left of the equations are the total kinetic energy of the particles before the collision, while the two terms on the right are the ones after the collision.

Using the conservation of linear momentum and kinetic energy equations, Eqs. (8.1) and (8.2) are solved to get the solutions given by,

$$v_1 = 3.0\,\mathrm{m\,s^{-1}} \text{ and } v_2 = 0,$$
$$v_1 = -0.60\,\mathrm{m\,s^{-1}} \text{ and } v_2 = 2.4\,\mathrm{m\,s^{-1}}.$$

The first solution is unacceptable because it implies nothing changed after the collision. The second is the required solution. It means that after the collision, the velocity of particle 1 is -0.60 m s^{-1} to the left, while the velocity of particle 2 is 2.4 m s^{-1} to the right.

- wxMaxima codes:

```
(%i6)   fpprintprec:5; ratprint:false; m1:0.4; m2:0.6; u1:3;
u2:0;
(fpprintprec)       5
(ratprint)   false
(m1)     0.4
(m2)     0.6
(u1)     3
(u2)     0
(%i8)      solve([m1*u1+m2*u2=m1*v1+m2*v2, 0.5*m1*u1^2+0.5
*m2*u2^2=0.5*m1*v1^2+0.5*m2*v2^2], [v1,v2])$ float(%);
(%o8)    [[v1=-0.6,v2=2.4],[v1=3.0,v2=0.0]]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m_1, m_2, u_1, and u_2.

(%i8) Solve $m_1u_1+m_2u_2 = m_1v_1+m_2v_2$ and $\frac{1}{2}m_1u_1^2+\frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2$ for v_1 and v_2.

Alternative solution: Another way of solving the problem is by using the coefficient of restitution, e, Eq. (8.4) given by,

$$e = \frac{\text{relative speed after collision}}{\text{relative speed before collision}} = \frac{v_2 - v_1}{u_1 - u_2}.$$

For an elastic collision, $e = 1$, meaning that,

$$v_2 - v_1 = u_1 - u_2.$$

For the problem at hand,

$$v_2 - v_1 = u_1 - u_2 = 3. \tag{8.3}$$

Equation (8.3) and the conservation of linear momentum Eq. (8.1) are solved to get the solutions given by,

$$v_1 = -0.60\,\mathrm{m\,s^{-1}} \text{ and } v_2 = 2.4\,\mathrm{m\,s^{-1}}.$$

- wxMaxima codes:

```
(%i6)  fpprintprec:5; ratprint:false; m1:0.4; m2:0.6; u1:3;
u2:0;
(fpprintprec)  5
(ratprint)     false
(m1)  0.4
(m2)  0.6
(u1)  3
(u2)  0
(%i8)    solve([m1*u1+m2*u2=m1*v1+m2*v2,1=(v2-v1)/(u1-u2)],
[v1, v2])$ float(%);
(%o8) [[v1=-0.6,v2=2.4]]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m_1, m_2, u_1, and u_2.
(%i8) Solve $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ and $1 = (v_2 - v_1)/(u_1 - u_2)$ for v_1 and v_2.

Problem 8.2 A proton of mass 1.66×10^{-27} kg collides head on with a helium atom of mass 6.64×10^{-27} kg at rest. The helium atom recoils with a velocity of 5.00×10^5 m s^{-1}. The collision is elastic.

(a) Calculate the initial and final velocities of the proton,
(b) What fraction of the proton energy is transferred to the helium atom?

Solution

(a) Figure 8.2 shows the colliding particles before and after the collision. Here, m_p and m_{He} are masses of proton and helium, u_p and u_{He} are velocities of proton and helium before the collision, while v_p and v_{He} are their velocities after, respectively.

Applying the conservation of linear momentum (Eq. 8.3), one obtains,

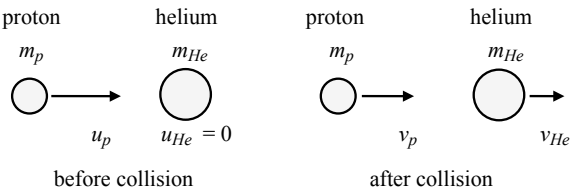

Fig. 8.2 The states of proton and helium, before and after the collision, Problem 8.2

$$\begin{aligned}
&\text{momentum before collision} = \text{momentum after collision}\\
&m_p u_p + m_{He} u_{He} = m_p v_p + m_{He} v_{He}\\
&1.66 \times 10^{-27} u_p = 1.66 \times 10^{-27} v_p + (6.64 \times 10^{-27})(5.00 \times 10^5).
\end{aligned}$$

The collision is elastic, indicating that the coefficient of restitution is 1. Thus,

$$\begin{aligned}
e &= \frac{v_{He} - v_p}{u_p - u_{He}} = 1,\\
\frac{5.00 \times 10^5 - v_p}{u_p} &= 1.
\end{aligned}$$

Solving this equation and the conservation of linear momentum equation gives the initial and final velocities of the proton as,

$$u_p = 1.25 \times 10^6\,\mathrm{m\,s^{-1}}, \quad v_p = -7.50 \times 10^5\,\mathrm{m\,s^{-1}}.$$

- wxMaxima codes:

```
(%i6)    fpprintprec:5; ratprint:false; mp: 1.66e-27; mHe:
6.64e-27; uHe: 0; vHe:5e5;
(fpprintprec)        5
(ratprint)     false
(mp)    1.66*10^-27
(mHe)   6.64*10^-27
(uHe)   0
(vHe)   5.0*10^5
(%i8)    solve([mp*up+mHe*uHe=mp*vp+mHe*vHe, (vHe-vp)/ (up-
uHe)=1], [up,vp])$ float(%);
(%o8)   [[up=1.25*10^6,vp=-7.5*10^5]]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m_p, m_{He}, u_{He}, and v_{He}.
(%i8) Solve $m_p u_p + m_{He} u_{He} = m_p v_p + m_{He} v_{He}$ and $(v_{He} - v_p)/(u_p - u_{He}) = 1$ for u_p and v_p.

(b) The fraction of proton kinetic energy transferred to helium is,

$$\frac{\text{kinetic energy of helium atom after collision}}{\text{kinetic energy of proton before collision}} = \frac{\frac{1}{2} m_{He} v_{He}^2}{\frac{1}{2} m_p u_p^2}$$

$$= \frac{\frac{1}{2}\left(6.64 \times 10^{-27}\,\text{kg}\right)\left(5.00 \times 10^{5}\,\text{m/s}\right)^2}{\frac{1}{2}\left(1.66 \times 10^{-27}\,\text{kg}\right)\left(1.25 \times 10^{6}\,\text{m/s}\right)^2}$$
$$= 0.64.$$

This means that 64% of the proton kinetic energy is transferred to the helium atom.

- wxMaxima codes:

```
(%i6)     fpprintprec:5; ratprint:false; mp: 1.66e-27; mHe:
6.64e-27; uHe: 0; vHe: 5e5;
(fpprintprec)       5
(ratprint)   false
(mp)    1.66*10^-27
(mHe)   6.64*10^-27
(uHe)   0
(vHe)   5.0*10^5
(%i8)    solve([mp*up+mHe*uHe=mp*vp+mHe*vHe, (vHe-vp)/ (up-
uHe)=1], [up,vp])$ float(%);
(%o8)   [[up=1.25*10^6,vp=-7.5*10^5]]
(%i9)   up: 1.25e6;
(up)    1.25*10^6
(%i10)  fraction: (1/2*mHe*vHe^2)/(1/2*mp*up^2);
(fraction)  0.64
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m_p, m_{He}, u_{He}, and v_{He}.
(%i8) Solve $m_p u_p + m_{He} u_{He} = m_p v_p + m_{He} v_{He}$ and $(v_{He} - v_p)/(u_p - u_{He}) = 1$ for u_p and v_p.
(%i9) Assign value of u_p.
(%i10) Calculate the fraction of transferred kinetic energy.

Problem 8.3 Two identical particles collide head on in one dimension elastically. The initial velocities of the particle 1 and particle 2 are 0.75 m s^{-1} and -0.43 m s^{-1}, respectively. Determine the velocities of the particles after the collision.

Solution

Figure 8.3 shows the state of the particles before and after the collision. Here, m is the mass of each particle, $u_1 = 0.75$ m s^{-1} and $u_2 = -0.43$ m s^{-1} are the velocities of particle 1 and 2 before the collision, while v_1 and v_2 are the ones after, respectively.

Applying the conservation of linear momentum (Eq. 8.3), one gets,

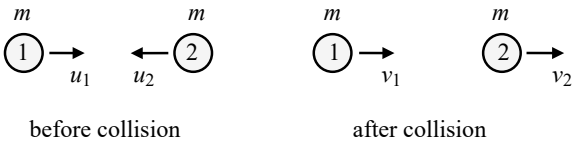

Fig. 8.3 The states of two identical particles, before and after a head on one-dimensional collision, Problem 8.3

$$\begin{array}{ll} \text{momentum before collision} & = \text{momentum after collision} \\ mu_1 + mu_2 & = mv_1 + mv_2 \\ 0.75m - 0.43m & = mv_1 + mv_2 \\ 0.32 & = v_1 + v_2. \end{array}$$

The collision is perfectly elastic and the coefficient of restitution is 1, that is,

$$e = \frac{v_2 - v_1}{u_1 - u_2} = 1,$$
$$\frac{v_2 - v_1}{0.75 - (-0.43)} = 1.$$

This equation and the conservation of linear momentum equation are solved and the velocities of the particles after the collision are obtained,

$$v_1 = -0.43\,\mathrm{m\,s^{-1}}, \quad v_2 = 0.75\,\mathrm{m\,s^{-1}}.$$

This shows that two particles of the same mass exchange their velocities in a one-dimensional perfectly elastic collision. The velocity of particle 1 is transferred to particle 2 and the velocity of particle 2 is transferred to particle 1.

- wxMaxima codes:

```
(%i6)   fpprintprec:5; ratprint:false; u1:0.75; u2:-0.43; m1:
m; m2:m;
(fpprintprec)      5
(ratprint)     false
(u1)     0.75
(u2)     -0.43
(m1)     m
(m2)     m
(%i8)    solve([m1*u1+m2*u2=m1*v1+m2*v2,(v2-v1)/(u1-u2)=1],
[v1,v2])$ float(%);
(%o8)    [[v1=-0.43,v2=0.75]]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of u_1, u_2, m_1, and m_2.
(%i8) Solve $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ and $(v_2 - v_1)/(u_1 - u_2) = 1$ for v_1 and v_2.

Further question: Two identical particles collide head on in a one-dimensional perfectly elastic collision. The initial velocity of particle 1 is 0.75 m s^{-1} and particle 2 is at rest. What are the velocities of the particles after the collision?

Answer: Applying the conservation of linear momentum (Eq. 8.3), we have,

$$\begin{aligned} \text{momentum before collision} &= \text{momentum after collision} \\ mu_1 &= mv_1 + mv_2 \\ 0.75m &= mv_1 + mv_2 \\ 0.75 &= v_1 + v_2. \end{aligned}$$

The collision is perfectly elastic and the coefficient of restitution is 1. Thus,

$$\begin{aligned} e = \frac{v_2 - v_1}{u_1 - u_2} &= 1, \\ \frac{v_2 - v_1}{0.75} &= 1. \end{aligned}$$

This equation and the conservation of linear momentum equation are solved to give the velocities of the particles after the collision as,

$$v_1 = 0\,\text{m s}^{-1}, \quad v_2 = 0.75\,\text{m s}^{-1}.$$

Again, the particles of the same mass exchange their velocities in a perfectly elastic collision. Therefore, in a one-dimensional perfectly elastic collision, if a particle collides with a stationary particle of the same mass, the former will stop and the later will recoil with the speed of the former.

- wxMaxima codes:

```
(%i6)    fpprintprec:5; ratprint:false; u1:0.75; u2:0; m1:m;
m2:m;
(fpprintprec)        5
(ratprint)     false
(u1)     0.75
(u2)     0
(m1)     m
(m2)     m
```

```
(%i8)    solve([m1*u1+m2*u2=m1*v1+m2*v2,(v2-v1)/(u1-u2)=1],
[v1,v2])$ float(%);
(%o8)    [[v1=0.0,v2=0.75]]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of u_1, u_2, m_1, and m_2.
(%i8) Solve $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ and $(v_2 - v_1)/(u_1 - u_2) = 1$ for v_1 and v_2.

Problem 8.4 Particle A moving at velocity 30 m s^{-1} collides with the particle B at rest. Both particles have the same mass. After the collision particle A is deflected 30° from its original direction, while particle B recoils. The collision is elastic. Calculate the velocities of the particles after the collision.

Solution

The collision is shown in Fig. 8.4a. This is not a head on collision and it is a collision in two dimensions. Here, u_A is the velocity of particle A before the collision, while v_A and v_B are the velocities of particle A and B after the collision, respectively. The mass of each particle is m and particle B recoils at an angle, θ, after being hit.

Applying the conservation of linear momentum in the x and y directions gives,

$$mu_A = mv_A \cos 30^\circ + mv_B \cos\theta, \tag{8.1}$$

$$0 = mv_A \sin 30^\circ - mv_B \sin\theta. \tag{8.2}$$

Applying the conservation of kinetic energy gives,

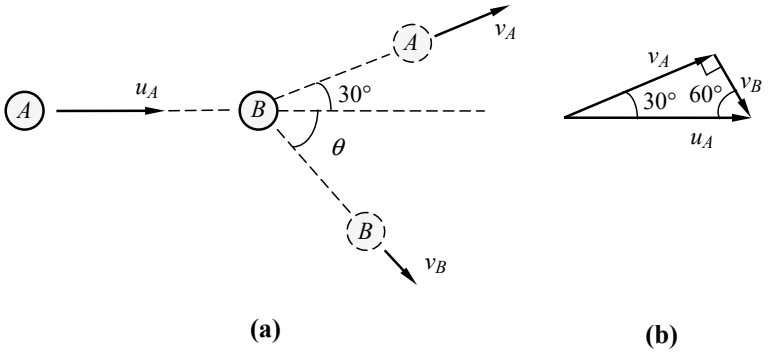

Fig. 8.4 A two-dimensional elastic collision between two identical particles (**a**), velocity vectors of the collision (**b**), Problem 8.4

$$\frac{1}{2}mu_A^2 = \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2. \tag{8.3}$$

Cancelling m and squaring Eq. (8.1) and rearranging, it gives,

$$u_A^2 - 2u_A v_A \cos 30^\circ + v_A^2 \cos^2 30^\circ = v_B^2 \cos^2 \theta. \tag{8.4}$$

Cancelling m and squaring Eq. (8.2) and rearranging, it gives,

$$v_A^2 \sin^2 30^\circ = v_B^2 \sin^2 \theta. \tag{8.5}$$

Equations (8.4) and (8.5) give,

$$u_A^2 - 2u_A v_A \cos 30^\circ + v_A^2 = v_B^2. \tag{8.6}$$

Cancelling $m/2$ in Eq. (8.3) gives,

$$u_A^2 - v_A^2 = v_B^2. \tag{8.7}$$

Equations (8.6) and (8.7) give,

$$\begin{aligned} -2u_A v_A \cos 30^\circ + 2v_A^2 &= 0, \\ v_A(v_A - u_A \cos 30^\circ) &= 0. \end{aligned}$$

The speed of the particle A after the collision is,

$$v_A = u_A \cos 30^\circ = (30\,\text{m/s}) \cos 30^\circ = 26\,\text{m s}^{-1}.$$

The speed of the particle B after the collision is obtained from Eq. (8.3),

$$v_B = \sqrt{u_A^2 - v_A^2} = \sqrt{(30\,\text{m/s})^2 - (26\,\text{m/s})^2} = 15\,\text{m s}^{-1}.$$

The direction of recoil of particle B is calculated from Eq. (8.2),

$$\begin{aligned} \sin\theta &= \frac{v_A}{v_B} \sin 30^\circ = \frac{26}{15} \sin 30^\circ = 0.87, \\ \theta &= 60^\circ. \end{aligned}$$

This indicates that the angle between the directions of the particle A and B after the collision is $60^\text{o} + 30^\text{o} = 90^\circ$. Thus, in a 2D elastic collision, when a particle collides

with a stationary particle of the same mass, both will come out of the collision at right angle to each other.

- wxMaxima codes:

```
(%i2)   fpprintprec:5; uA:30;
(fpprintprec)       5
(uA)    30
(%i3)   vA: uA*cos(float(30/180*%pi));
(vA)    25.981
(%i4)   vB: sqrt(uA^2 - vA^2);
(vB)    15.0
(%i5)   theta: asin(vA*sin(float(30/180*%pi))/vB);
(theta)  1.0472
(%i6)   theta_deg: float((theta*180/%pi));
(theta_deg)    60.0
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and assign u_A.
(%i3), (%i4) Calculate v_A and v_B.
(%i5), (%i6) Calculate θ and convert the angle to degree.

Alternative solution: The conservation of kinetic energy Eq. (8.3) is written as,

$$u_A^2 = v_A^2 + v_B^2.$$

Here, u_A, v_A, and v_B form a right-angled triangle with u_A as the hypotenuse, while v_A and v_B are the two sides that are at right angle to each other (Fig. 8.4b). From the triangle, using Pythagoras theorem one gets,

$$\begin{aligned}
&\theta = 60^\circ, \\
&v_A = u_A \cos 30^\circ = (30\,\text{m/s}) \cos 30^\circ = 26\,\text{m s}^{-1}, \\
&v_B = u_A \sin 30^\circ = (30\,\text{m/s}) \sin 30^\circ = 15\,\text{m s}^{-1}.
\end{aligned}$$

This means that in two dimensions, when a moving particle collides elastically with a stationary particle of the same mass, both will come out of the collision at right angle (90°) to each other.

Problem 8.5 An iron sphere of mass 0.10 kg is placed on a floor with negligible friction. A bullet of mass 2.0 g is shot at a speed of 400 m s^{-1} to the sphere. The bullet is deflected by 90° from its original direction with a speed of 300 m s^{-1}, while the iron sphere recoils.

(a) Calculate the speed and direction of the iron sphere recoil.

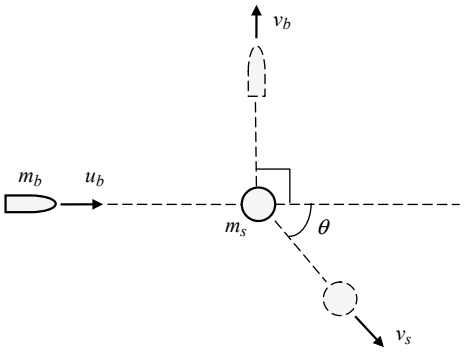

Fig. 8.5 A bullet hits an iron sphere, Problem 8.5

(b) Is the collision elastic?
(c) What is the fraction of energy transfer from the bullet to the iron sphere?

Solution

(a) Figure 8.5 shows the collision of the bullet and sphere. In the figure, m_b and m_s are the masses of the bullet and iron sphere, u_b and v_b are the speeds of the bullet before and after the collision, while v_s and θ are the speed and direction of the sphere after being shot, respectively.

Applying the conservation of linear momentum, in the x and y directions, one gets,

$$m_b u_b = m_s v_s \cos\theta, \tag{8.1}$$

$$0 = m_b v_b - m_s v_s \sin\theta. \tag{8.2}$$

The direction of recoil of the iron sphere can be calculated from the two equations,

$$\frac{m_s v_s \sin\theta}{m_s v_s \cos\theta} = \frac{m_b v_b}{m_b u_b},$$
$$\tan\theta = \frac{v_b}{u_b} = \frac{300\,\text{m/s}}{400\,\text{m/s}} = 0.75,$$
$$\theta = 37^\circ.$$

From Eq. (8.2), the speed, v_s, of the iron sphere after the collision is,

$$v_s = \frac{m_b u_b}{m_s \cos\theta} = \frac{\left(2.0 \times 10^{-3}\,\text{kg}\right)(400\,\text{m/s})}{(0.10\,\text{kg})\cos 37^\circ} = 10\,\text{m}\,\text{s}^{-1}.$$

(b) In an elastic collision, the kinetic energies before and after the collision are equal. We calculate the kinetic energies for both instances. The kinetic energy

before the collision is given by,

$$K_{before} = \frac{1}{2}m_b u_b^2 = \frac{1}{2}\left(2.0 \times 10^{-3}\,\text{kg}\right)(400\,\text{m/s})^2$$
$$= 160\,\text{J}.$$

The kinetic energy after the collision is,

$$K_{after} = \frac{1}{2}m_b v_b^2 + \frac{1}{2}m_s v_s^2$$
$$= \frac{1}{2}\left(2.0 \times 10^{-3}\,\text{kg}\right)(300\,\text{m/s})^2 + \frac{1}{2}(0.10\,\text{kg})(10\,\text{m/s})^2$$
$$= 95\,\text{J}.$$

The kinetic energies are not the same and this means that the collision is inelastic. The kinetic energies of the bullet and sphere after the collision are less than the ones before the collision. An amount of $160\,\text{J} - 95\,\text{J} = 65\,\text{J}$ of kinetic energy is lost or converted to other forms of energy.

(c) The fraction of energy transfer from the bullet to the iron sphere is the kinetic energy of the sphere after the collision divided by the kinetic energy of the bullet before the collision,

$$\text{fraction} = \frac{\frac{1}{2}m_s v_s^2}{\frac{1}{2}m_b u_b^2} = \frac{\frac{1}{2}(0.10\,\text{kg})(10\,\text{m/s})^2}{\frac{1}{2}\left(2.0 \times 10^{-3}\,\text{kg}\right)(400\,\text{m/s})^2}$$
$$= 0.031.$$

This means that 3% of the bullet kinetic energy is transferred to the iron sphere.

- wxMaxima codes:

```
(%i5)   fpprintprec:5; mb:2e-3; ub:400; vb:300; ms:0.1;
(fpprintprec)       5
(mb)    0.002
(ub)    400
(vb)    300
(ms)    0.1
(%i6)   theta: float(atan(vb/ub));
(theta)  0.6435
(%i7)   degree: float(theta*180/%pi);
(degree)  36.87
(%i8)   vs: mb*ub/(ms*cos(theta));
(vs)    10.0
(%i9)   Kbefore: 1/2*mb*ub^2;
```

```
(Kbefore)  160.0
(%i10)  Kafter: 1/2*mb*vb^2 + 1/2*ms*vs^2;
(Kafter)  95.0
(%i11)  Fraction: (0.5*ms*vs^2 )/(0.5*mb*ub^2);
(Fraction)  0.03125
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of m_b, u_b, and m_s.
(%i6), (%i7) Calculate θ and convert the angle to degree.
(%i8) Calculate the speed of the sphere after the collision, v_s.
(%i9), (%i10) Calculate K_{before} and K_{after}.
(%i11) Calculate the fraction of energy transfer.

- Alternative calculation:

```
(%i2)    fpprintprec:5; ratprint:false;
(fpprintprec)         5
(ratprint)   false
(%i3)    eq1: mb*ub=ms*vs*cos(theta);
(eq1)    mb*ub=ms*cos(theta)*vs
(%i4)    eq2: 0=mb*vb-ms*vs*sin(theta);
(eq2)    0=mb*vb-ms*sin(theta)*vs
(%i5)    solve([eq1,eq2], [vs,sin(theta)]);
(%o5)   [[vs=(mb*ub)/(ms*cos(theta)),sin(theta)=(cos(theta)
*vb)/ub]]
(%i11)    ms: 0.1; mb: 0.002; ub: 400; vb: 300; theta: float
(atan(vb/ub)); theta_deg: float(theta/%pi*180);
(ms)    0.1
(mb)    0.002
(ub)    400
(vb)    300
(theta)    0.6435
(theta_deg)    36.87
(%i12)  vs: mb*ub/(ms*cos(theta));
(vs)    10.0
(%i13)  Kbefore: 0.5*mb*ub^2;
(Kbefore)    160.0
(%i14)  Kafter: 0.5*mb*vb^2 + 0.5*ms*vs^2;
(Kafter)      95.0
(%i15)  fraction: 0.5*ms*vs^2/(0.5*mb*ub^2);
(fraction)    0.03125
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3), (%i4) Assign Eqs. (8.1) and (8.2) as eq1 and eq2, respectively.
(%i5) Solve Eqs. (8.1) and (8.2) in symbols for v_s and sin θ.
(%i11) Assign values of m_s, m_b, u_b, v_b, θ (rad), and θ (deg).
(%i12), (%i13), (%i14), (%i15) Calculate v_s, K_{before}, K_{after}, and fraction.

Problem 8.6 Disk A moving with speed $u_A = 30.0$ m s^{-1} collides with disk B at rest on a surface with negligible friction. Disk A is deflected by 30° from its original direction, while disk B recoils at 45°. The masses of the two disks are the same and the collision is inelastic. Calculate,

(a) the speed of each disk after the collision,
(b) the percentage of kinetic energy loss in the collision.

Solution

(a) The collision of the two disks is shown in Fig. 8.6. Here, u_A is the speed of disk A before the collision, while v_A and v_B are the speeds of disk A and B after the collision and m is the mass of each disk.

Applying the conservation of linear momentum in the x and y directions, we write,

$$mu_A = mv_A \cos 30^\circ + mv_B \cos 45^\circ,$$
$$0 = mv_A \sin 30^\circ - mv_B \sin 45^\circ.$$

Solving the two equations one gets the speeds of disk A and B after the collision as,

$$v_A = 22.0\,\text{m s}^{-1},$$
$$v_B = 15.5\,\text{m s}^{-1}.$$

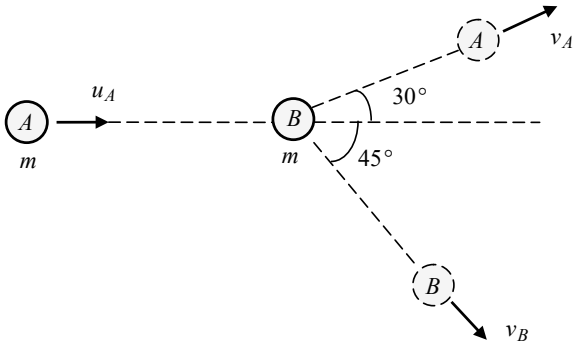

Fig. 8.6 Collision of two identical disks, Problem 8.6

(b) The kinetic energy before the collision is,

$$K_{init} = \frac{1}{2}mu_A^2,$$

because it is disk A that is moving at the instance. The kinetic energy after the collision is,

$$K_{final} = \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2.$$

Therefore, the fraction of kinetic energy loss is,

$$\begin{aligned}\frac{K_{final} - K_{init}}{K_{init}} &= \frac{\frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 - \frac{1}{2}mu_A^2}{\frac{1}{2}mu_A^2}, \\ &= \frac{v_A^2 + v_B^2 - u_A^2}{u_A^2}, \\ &= \frac{(22.0\,\mathrm{m/s})^2 + (15.5\,\mathrm{m/s})^2 - (30.0\,\mathrm{m/s})^2}{(30.0\,\mathrm{m/s})^2}, \\ &= -0.196.\end{aligned}$$

This means that 20% of the kinetic energy of disk A is lost in the collision.

- wxMaxima codes:

```
(%i3)    fpprintprec:5; ratprint:false; uA:30;
(fpprintprec)       5
(ratprint)   false
(uA)    30
(%i5)       solve([m*uA=m*vA*cos(30/180*%pi)+m*vB*cos (45/
180 * %pi), 0=m*vA*sin(30/180*%pi)-m*vB*sin(45/180*%pi)],
[vA,vB])$ float(%);
(%o5)   [[vA=21.962,vB=15.529]]
(%i7)   vA:rhs(%o5[1][1]); vB:rhs(%o5[1][2]);
(vA)    21.962
(vB)    15.529
(%i8)   Kinit: 1/2*m*uA^2;
(Kinit)  450*m
(%i9)   Kfinal: 1/2*m*vA^2 + 1/2*m*vB^2;
(Kfinal)  361.73*m
(%i10)   (Kfinal-Kinit)/Kinit;
(%o10)   -0.19615
```

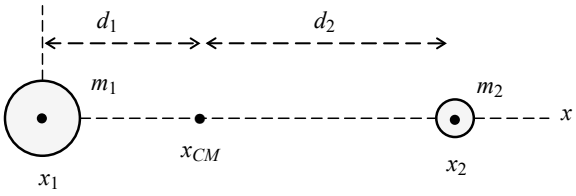

Fig. 8.7 Center of mass of a two-particle system, x_{CM}, Problem 8.7

Comments on the codes:

(%i3) Set floating point print precision to 5, internal rational number print to false, assign value of u_A.
(%i5) Solve $mu_A = mv_A \cos 30° + mv_B \cos 45°$ and $0 = mv_A \sin 30° - mv_B \sin 45°$ for v_A and v_B. Part (a).
(%i7), (%i8), (%i9) Assign values of v_A and v_B, calculate K_{init} and K_{final}.
(%i10) Calculate fraction of kinetic energy loss. Part (b).

Problem 8.7 What is the ratio of distances from the center of mass of a two-particle system with masses $m_1 = 4.0$ kg and $m_2 = 1.0$ kg?

Solution

Figure 8.7 shows the two-particle system and the relevant coordinates and distances. Here, $x_1 = 0$, $x_2 = d_1 + d_2$, and $x_{CM} = d_1$.

Using the definition of center of mass (Eq. 8.5), we have,

$$x_{CM} = d_1 = \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$
$$= \frac{(4.0\,\text{kg})(0) + (1.0\,\text{kg})(d_1 + d_2)}{4.0\,\text{kg} + 1.0\,\text{kg}} = \frac{d_1 + d_2}{5}.$$

Simplifying this equation gives the ratio of distances from the center of mass,

$$\frac{d_1}{d_2} = \frac{1}{4}.$$

- wxMaxima codes:

```
(%i1)  solve(d1=(d1+d2)/5, d1);
(%o1)  [d1=d2/4]
```

Comments on the codes:

(%i1) Solve $d_1 = \frac{d_1+d_2}{5}$ for d_1.

(%o1) The solution.

Furthermore,

$$\frac{d_1}{d_1+d_2}=\frac{m_2}{m_1+m_2}, \tag{8.1}$$

because,

$$d_1=\frac{m_1x_1+m_2x_2}{m_1+m_2}=\frac{m_2x_2}{m_1+m_2}=\frac{m_2(d_1+d_2)}{m_1+m_2}.$$

In addition,

$$\frac{d_1}{d_2}=\frac{m_2}{m_1}, \tag{8.2}$$

because,

$$\begin{aligned}\frac{d_1/d_1}{d_1/d_1+d_2/d_1}&=\frac{m_2/m_2}{m_1/m_2+m_2/m_2},\\ \frac{1}{1+d_2/d_1}&=\frac{1}{m_1/m_2+1},\\ \frac{d_1}{d_2}&=\frac{m_2}{m_1}.\end{aligned}$$

- wxMaxima codes:

```
(%i1)  solve(d1/(d1+d2)=m2/(m1+m2), d1);
(%o1)  [d1=(d2*m2)/m1]
```

Comments on the codes:

(%i1) Solve $\frac{d_1}{d_1+d_2}=\frac{m_2}{m_1+m_2}$ for d_1.
(%o1) The solution.

Equation (8.2) enables the center of mass of a two-particle system to be quickly determined. For example, if the masses ratio of a two-particle system is 3:1, then the ratio of distances from the center of mass is 1:3.

Problem 8.8 Calculate the center of mass of a semicircular plate of radius 1.0 m.

Solution

Figure 8.8 shows the semicircular plate and the relevant coordinates. Due to the symmetry, the center of mass in the x direction is along the y axis.

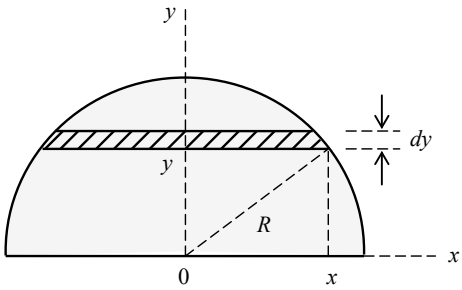

Fig. 8.8 Determining center of mass of a semicircular plate, Problem 8.8

Let the mass of the plate be M and the radius be R. The mass per unit area is,

$$\frac{M}{\frac{1}{2}\pi R^2} = \frac{2M}{\pi R^2}.$$

In the y direction the coordinate of the center of mass is given by (Eq. 8.6),

$$y_{CM} = \frac{\int y\, dm}{M},$$

where dm is the elementary mass of the plate. By the Pythagoras theorem, $R^2 = x^2 + y^2$ and $x = (R^2 - y^2)^{1/2}$. Considering a shaded strip of width dy, the elementary mass is,

$$dm = 2x\, dy \frac{2M}{\pi R^2} = \frac{4M}{\pi R^2}\sqrt{R^2 - y^2}\, dy.$$

In the y direction the coordinate of the center of mass is,

$$\begin{aligned} y_{CM} &= \frac{\int_0^R y \frac{4M}{\pi R^2}\sqrt{R^2 - y^2} dy}{M} = \int_0^R \frac{4}{\pi R^2} y\sqrt{R^2 - y^2} dy \\ &= \frac{4}{\pi R^2}\left[-\frac{1}{3}\left(R^2 - y^2\right)^{3/2}\right]_0^R = \frac{4R}{3\pi} = \frac{4(1.0\,\text{m})}{3\pi} \\ &= 0.42\,\text{m}. \end{aligned}$$

The coordinate of the center of mass of the semicircular plate is (0 m, 0.42 m).

- wxMaxima codes:

```
(%i2)    fpprintprec:5; R:1;
(fpprintprec)        5
(R)       1
(%i4)     yCM: integrate( 4/(%pi*R^2)*y*sqrt(R^2-y^2), y, 0,
R); float(%);
(yCM)    4/(3*%pi)
(%o4)    0.42441
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and assign value of R.
(%i4) Calculate y_{CM} and convert result to decimal value.

Problem 8.9 The structure of an ammonia molecule (NH_3) is pyramidal with nitrogen (N) atom at one vertex and three hydrogen (H) atoms at vertices of the pyramid base. The base is an equilateral triangle with the H atoms at the vertices. The distance between H atoms is 1.628×10^{-10} m. The distance between the center of the triangle and each of the H atom is 9.400×10^{-11} m. The distance between the N atom and each of the H atom is 1.014×10^{-10} m. Calculate the center of mass of the molecule with respect to the N atom. Relative molecular masses of H and N are 1 and 14, respectively.

Solution

Figure 8.9a is the structure of the NH_3 molecule. The center of mass of the triangular base is X. The distance NX is the height of the pyramid given by,

$$\begin{aligned} NX &= \sqrt{NH^2 - HX^2} \\ &= \sqrt{\left(1.014 \times 10^{-10}\ \text{m}\right)^2 - \left(9.400 \times 10^{-11}\ \text{m}\right)^2} \\ &= 3.803 \times 10^{-11}\ \text{m}. \end{aligned}$$

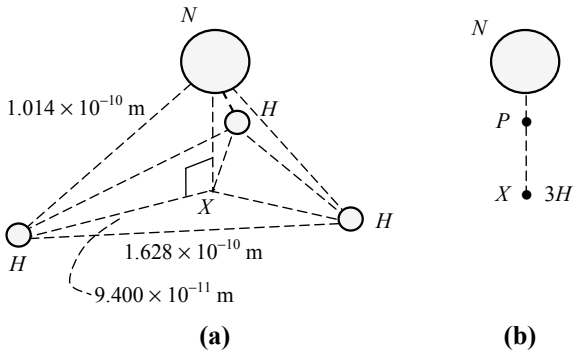

Fig. 8.9 **a** Structure of an ammonia molecule, **b** determining center of mass of an ammonia molecule, Problem 8.9

The mass of the three H atoms is centered at X. Figure 8.9b shows the two-particle system representing the NH_3 molecule. Let P be the center of mass of NH_3.

Using results of Problem 8.7, the distance of center of mass of NH_3 from the N atom, NP, is calculated as follows,

$$\frac{NP}{NX} = \frac{m_{3H}}{m_N + m_{3H}}$$
$$\frac{NP}{3.803 \times 10^{-11}} = \frac{3}{14+3}$$
$$NP = 6.710 \times 10^{-12} \text{ m}.$$

- wxMaxima codes:

```
(%i1)     fpprintprec:5;
(fpprintprec)          5
(%i2)     NX: sqrt(1.014e-10^2 - 9.4e-11^2);
(NX)      3.8026*10^-11
(%i3)     NP: NX*3/(14+3);
(NP)      6.7104*10^-12
```

Comments on the codes:

(%i2) Calculate NX.
(%i3) Calculate NP.

Problem 8.10 Figure 8.10 is an iron plate in the form of a quadrant of a circle. Calculate its center of mass.

Solution

Figure 8.11 shows the quadrant plate and a strip of width dx and height y of the quadrant. The strip is needed to calculate the center of mass by means of calculus.

The center of mass is calculated using Eq. (8.6),

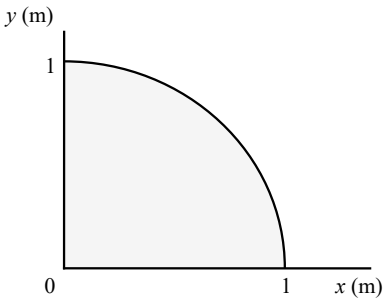

Fig. 8.10 An iron plate quadrant, Problem 8.10

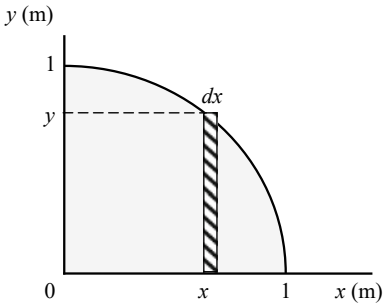

Fig. 8.11 Determining center of mass of an iron plate quadrant, Problem 8.10

$$x_{CM} = \frac{\int x\, dm}{\int dm} \quad \text{and} \quad y_{CM} = \frac{\int y\, dm}{\int dm},$$

where $\int dm$ is the mass of the quadrant and dm is the mass of the shaded strip. Assume that the mass per unit area of the plate is uniform. This means that the mass of any portion of the plate is proportional to the area of the portion. We have,

$$x_{CM} = \frac{\int x\, y dx}{\frac{1}{4}\pi(1^2)} = \frac{4}{\pi}\int_0^1 x\left(1-x^2\right)^{1/2} dx = \left[-\frac{4}{3\pi}\left(1-x^2\right)^{3/2}\right]_0^1 = \frac{4}{3\pi}$$
$$= 0.42 \text{ m}.$$

Similarly,

$$y_{CM} = 0.42 \text{ m},$$

which is deduced from symmetry of the problem. Therefore, the center of mass of the quadrant plate is (0.42 m, 0.42 m).

- wxMaxima codes:

```
(%i1)    fpprintprec:5;
(fpprintprec)        5
(%i3)    xCM: integrate(x*sqrt(1-x^2)/(%pi/4),x,0,1); float
(%);
(xCM)    4/(3*%pi)
(%o3)    0.42441
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i3) Calculate x_{CM} and convert to result to decimal value.

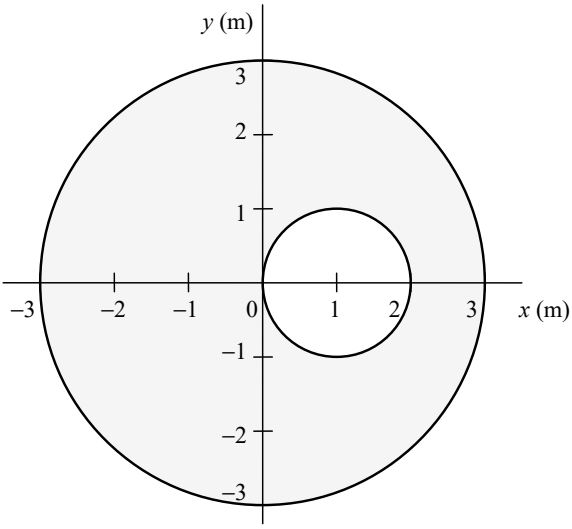

Fig. 8.12 A circular plate with a circular hole, Problem 8.11

Problem 8.11 Figure 8.12 is a circular plate with a circular hole in it. Locate the center of mass of the plate.

Solution

Assume that the mass per unit area of the plate is uniform, such that the mass of any portion of the plate is proportional to the area of the portion. By the law of symmetry, the center of mass must be along the x axis. Let the center of mass of the plate (Fig. 8.12) be $(x_A, 0)$, a point somewhere along the x axis from -3 to 0. Call this plate A (Fig. 8.13). Let the hole be filled with a circular plate of the same material and thickness, call this plate B, with center of mass $(x_B, 0) = (1, 0)$ (Fig. 8.13).

We know that the center of mass of the plate without the hole in Fig. 8.13 is (0, 0) and it is the center of mass of both plates A and B. In the x direction, we have,

$$\begin{aligned} x_{CM} &= \frac{x_A m_A + x_B m_B}{m_A + m_B}, \\ 0 &= \frac{x_A\left[\pi(3\text{ m})^2 - \pi(1\text{ m})^2\right] + (1\text{ m})\pi(1\text{ m})^2}{\pi(3\text{ m})^2}, \\ x_A &= -0.125\text{ m}. \end{aligned}$$

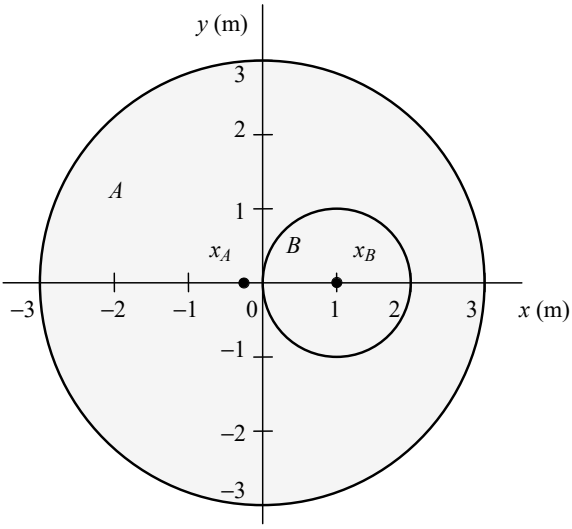

Fig. 8.13 Determining center of mass of a circular plate with a circular hole, Problem 8.11

The center mass of plate A, that is the center of mass in question is $(-0.125, 0)$ m.

- wxMaxima codes:

```
(%i2)    solve(0=(xA*(%pi*3^2-%pi) + %pi) / (%pi*3^2), xA)$
float(%);
(%o2)    [xA=-0.125]
```

Comments on the codes:

(%i2) Solve $0 = \frac{x_A[\pi(3)^2-\pi]+\pi}{\pi(3)^2}$ for x_A and get the decimal value.
(%o2) The solution.

Problem 8.12 Using Problem 8.4, with the figure redrawn as Fig. 8.14, sketch,

(a) change in momentum for each particle in the collision,

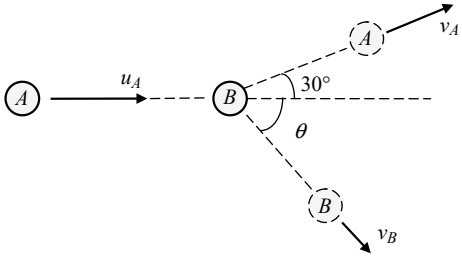

Fig. 8.14 A two-dimensional elastic collision between two identical particles, Problem 8.12

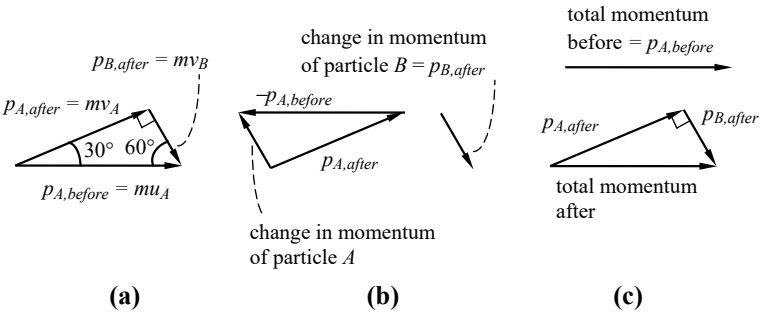

Fig. 8.15 **a** Momentum before and momentum after the collision, **b** change in momentum of particle A and change in momentum of particle B in the collision, **c** total momentum before and after the collision, Problem 8.12

(b) total momentum before and after the collision.

Particles A and B are of the same mass. Particle A hits particle B at rest.

Solution

(a) From Problem 8.4, the momentum vector diagram for the collision is as shown in Fig. 8.15a.

The changes in momentum for particle A and B are as shown in Fig. 8.15b. Change in momentum of a particle is the particle's momentum after the collision plus the negative of particle's momentum before the collision. Vector addition must be used. It can be seen that the change in momentum of particle B is the same in magnitude but opposite in direction to that of particle A.

(b) Figure 8.15c shows the total momentum before and after the collision. The total momentum before the collision is the momentum of particle A, because particle B is at rest. The total momentum after the collision is the vector addition of $p_{A,after}$ and $p_{B,after}$. The result shows that the total momentum before and after the collision are the same. This is in fact the conservation of linear momentum.

8.3 Summary

- A particle of mass, m, moving with the velocity, $\boldsymbol{v}$, has a linear momentum of $\boldsymbol{p} = m\boldsymbol{v}$.
- The conservation of linear momentum states that for an isolated system in which the bodies are interacting, the total momentum is always conserved.
- In an elastic collision, the linear momentum and the kinetic energy are conserved. In an inelastic collision, the linear momentum is conserved but not the kinetic energy.

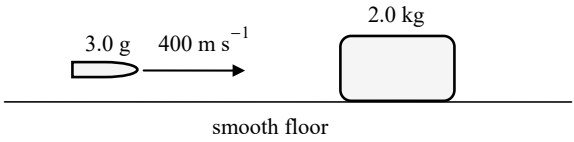

Fig. 8.16 A bullet hits a wooden block at rest, Exercise 8.3

8.4 Exercises

Exercise 8.1 A rifle of mass M at rest fires a bullet of mass m and the bullet flies off with velocity v. What is the recoil velocity of the rifle?

(Answer: $-mv/M$)

Exercise 8.2 An object of mass 3.0 kg is released from rest from a tall building. What is the magnitude of its momentum after 2.0 s?

(Answer: 59 kg m s^{-1})

Exercise 8.3 A bullet of mass 3.0 g at 400 m s^{-1} hits a wooden block of mass 2.0 kg at rest on a floor with negligible friction (Fig. 8.16). The bullet gets buried in the block. Calculate the speed of the block after the bullet hit.

(Answer: 0.60 m s^{-1})

Exercise 8.4 Sphere A of mass 0.10 kg with the velocity 3.0 m s^{-1} makes a head on elastic collision with a stationary sphere B mass of 0.20 kg. What are the speeds of the spheres A and B after the collision?

(Answer: $-$ 1.0 m s^{-1}, 2.0 m s^{-1})

Exercise 8.5 A sphere of mass 1.2 kg moving at a velocity of 20 cm s^{-1} makes a head on collision with an identical sphere at rest. The coefficient of restitution of the collision is 0.60. Calculate the loss of kinetic energy in the collision.

(Answer: 7.7×10^{-3} J)

Chapter 9
Rotational Motion

9.1 Basic Concepts and Formulae

(1) The angular speed of a particle rotating in a circle or a rigid body rotating about an axis is the time rate of change of angular displacement,

$$\omega = \frac{d\theta}{dt}. \tag{9.1}$$

The unit of ω is rad s^{-1} or s^{-1}. Every point of the body has the same angular speed.

The angular acceleration is the time rate of change of angular speed,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \tag{9.2}$$

The unit of α is rad s^{-2} or s^{-2}. Every point of the body has the same angular acceleration.

(2) For a rigid body rotating about an axis with a constant angular acceleration of α, the kinematic equations are,

$$\omega = \omega_0 + \alpha t, \tag{9.3}$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2, \tag{9.4}$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0), \tag{9.5}$$

where ω and θ are the angular speed and angular displacement at time t, while ω_0 and θ_0 are the angular speed and angular displacement at $t = 0$.

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W. M. S. Wan Hassan, *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-42678-0_9

(3) Angular speed, ω, and linear speed, v, angular acceleration, α, and tangential acceleration, a_T, of a rigid body rotating about an axis are related via,

$$v = r\omega, \tag{9.6}$$

$$a_T = r\alpha. \tag{9.7}$$

The direction of the tangential acceleration is parallel to the line touching the circumference of the circle. There is also centripetal acceleration, a_R, of magnitude,

$$a_R = \omega^2 r = \frac{v^2}{r}, \tag{9.8}$$

pointing to the center of the circle and at right angle to the tangential acceleration. The magnitude of the acceleration is,

$$\begin{aligned} a &= \sqrt{a_T^2 + a_R^2} \\ &= \sqrt{r^2\alpha^2 + \omega^4 r^2}. \end{aligned}$$

(4) The moment of inertia of a system of particles is,

$$I = \sum_i m_i r_i^2, \tag{9.9}$$

where m_i is the mass and r_i is the distance of ith particle from the axis of rotation. The moment of inertia of a rigid body is,

$$I = \int r^2 dm, \tag{9.10}$$

where r is the distance of elementary mass, dm, of the body from the axis of rotation. Moment of inertia is also known as the angular mass or rotational inertia.

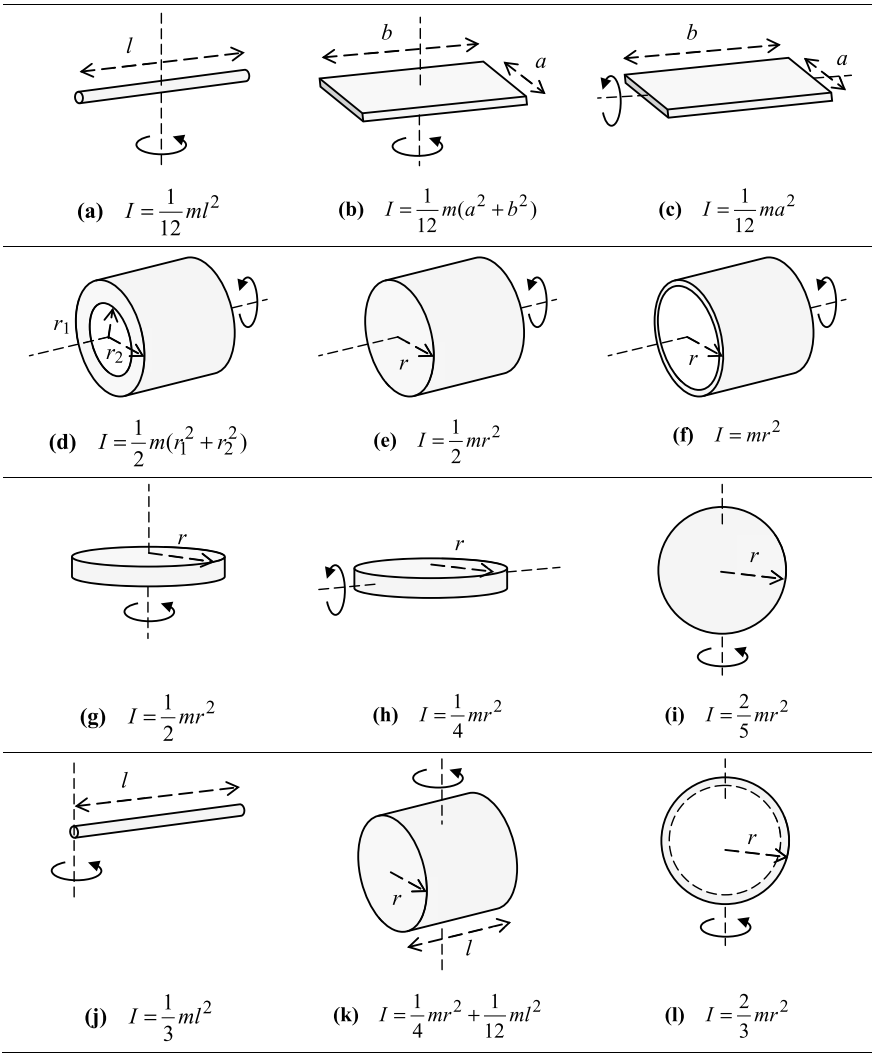
(5) Moments of inertia of a number of rigid bodies are shown in Fig. 9.1.

(6) The radius of gyration, k, the moment of inertia, I, and the mass, m, of the rigid body are related by,

$$I = mk^2. \tag{9.11}$$

(7) Parallel axes theorem: The moment of inertia about an axis parallel to the axis through the center of mass of a rigid body is,

$$I = I_{CM} + mh^2, \tag{9.12}$$



(a) $I = \frac{1}{12}ml^2$ **(b)** $I = \frac{1}{12}m(a^2 + b^2)$ **(c)** $I = \frac{1}{12}ma^2$

(d) $I = \frac{1}{2}m(r_1^2 + r_2^2)$ **(e)** $I = \frac{1}{2}mr^2$ **(f)** $I = mr^2$

(g) $I = \frac{1}{2}mr^2$ **(h)** $I = \frac{1}{4}mr^2$ **(i)** $I = \frac{2}{5}mr^2$

(j) $I = \frac{1}{3}ml^2$ **(k)** $I = \frac{1}{4}mr^2 + \frac{1}{12}ml^2$ **(l)** $I = \frac{2}{3}mr^2$

Fig. 9.1 Moments of inertia of rigid bodies

where I_{CM} is the moment of inertia of the body about the axis that coincides with its center of mass, m is the mass of the body, and h is the distance between the parallel axes.

(8) Perpendicular axes theorem for plate: For a plate on the xy plane, the moment of inertia about the z axis is,

$$I_z = I_x + I_y, \tag{9.13}$$

where I_x and I_y are the moments of inertia of the plate about the x and y axes, respectively.

(9) The kinetic energy of a rotating rigid body at angular speed, ω, about an axis is,

$$K = \frac{1}{2} I \omega^2, \tag{9.14}$$

where I is the moment of inertia of the rigid body about the axis.

(10) The magnitude of torque, τ, due to force, F, acting on a body is,

$$\tau = Fd, \tag{9.15}$$

where d is the distance from the force action line to the axis or origin.

(11) A net external torque on a rigid body will cause an angular acceleration of α,

$$\Sigma \tau = I \alpha, \tag{9.16}$$

where $\Sigma \tau$ is the net external torque on the body about the axis of rotation and I is the moment of inertia about the axis. This is the Newton's second law of rotational motion.

(12) The time rate of work done by external forces in rotating a rigid body about an axis, i.e. the power, is,

$$P = \tau \omega. \tag{9.17}$$

(13) The work-energy theorem for rotational motion: The net work done by external forces in rotating a rigid body about an axis is the change in rotational kinetic energy of the body,

$$W = \frac{1}{2} I \omega_{final}^2 - \frac{1}{2} I \omega_{init}^2. \tag{9.18}$$

(14) The total kinetic energy of a rigid body in translational and rotational motions is,

$$K = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} m v_{CM}^2, \tag{9.19}$$

where the first term on right of the equation is the rotational kinetic energy about an axis through the center of mass of the body and the second term is the translational kinetic energy of the center of mass.

(15) The torque, $\boldsymbol{\tau}$, due to a force, $\boldsymbol{F}$, about an origin or an axis is,

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}, \tag{9.20}$$

where $\boldsymbol{r}$ is the position vector of the force relative to the origin or the axis.

(16) The angular momentum, $\boldsymbol{L}$, of a particle with linear momentum, $\boldsymbol{p}$ $(= m\boldsymbol{v})$, is,

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times m\boldsymbol{v}, \tag{9.21}$$

where $\boldsymbol{r}$ is the position vector of the particle relative to an origin.

(17) The resultant external torque acting on a rigid body or a particle is the time rate of change of angular momentum of the rigid body or particle,

$$\Sigma\tau = \frac{d\boldsymbol{L}}{dt}. \tag{9.22}$$

(18) The z component of angular momentum of a rigid body rotating about the z axis is,

$$L_z = I\omega, \tag{9.23}$$

where I is the moment of inertia of the body about the axis of rotation and ω is the angular speed of the body.

(19) The net external torque acting on a rigid body is the product of its moment of inertia about an axis and the angular acceleration about the axis,

$$\Sigma\tau = I\alpha. \tag{9.24}$$

(20) If the net external torque acting on a rigid body is zero, the total angular momentum of the body is constant. The conservation of angular momentum yields,

$$I_{init}\omega_{init} = I_{final}\omega_{final}. \tag{9.25}$$

9.2 Problems and Solutions

Problem 9.1 A wheel rotates by 30 rad in 15 s about its axis.

(a) Express the angle in degree and number of turns.
(b) What is the average angular speed, ω?
(c) If the radius of the wheel is 10 cm, what is the distance, s, travelled by the rim?
(d) If initially the wheel is at rest, and it rotates with a constant acceleration, what is the angular acceleration, α?

Solution

(a) Angle 2π rad is 360° or a revolution. This means that 30 rad is,

$$30 \text{ rad} \times \frac{360^\circ}{2\pi \text{ rad}} = 1720^\circ,$$

and this is,

$$\frac{30 \text{ rad}}{2\pi \text{ rad/revolution}} = 4.8 \text{ revolutions}.$$

(b) The average angular speed, ω, is (Eq. 9.1),

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{30 \text{ rad}}{15 \text{ s}} = 2.0 \text{ rad s}^{-1}.$$

(c) The distance, s, travelled by the rim is,

$$s = r\theta = (10 \text{ cm})(30 \text{ rad}) = 300 \text{ cm}.$$

(d) For rotational motion with a constant angular acceleration, the angular displacement is (Eq. 9.4),

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2.$$

For this problem,

$$\theta_0 = 0, \quad \omega_0 = 0.$$

The angular acceleration, α, is,

$$\alpha = \frac{2\theta}{t^2} = \frac{2(30 \text{ rad})}{(15 \text{ s})^2} = 0.27 \text{ rad s}^{-2}.$$

- wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec)      5
(%i2) degree: 30*360/float(2*%pi);
(degree) 1718.9
(%i3) revolution: 30/float(2*%pi);
(revolution) 4.7746
(%i4) angular_speed: 30/15;
(angular_speed) 2
(%i5) distance: 10*30;
(distance) 300
(%i6) alpha: float(2*30/15^2);
```

```
(alpha) 0.26667
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i2), (%i3), (%i4), (%i5), (%i6) Calculate angle, number of revolutions, ω, s, and α.

Problem 9.2 A disk of radius 10 cm is rotating with angular deceleration -20 rad s^{-2} and angular speed 45 rad s^{-1} about its axis. Calculate,

(a) the angular speed, ω, at 2.5 s,
(b) the speed, v, of a point on the rim at 2.5 s,
(c) the angular displacement, θ, at 2.5 s,
(d) the acceleration, a, of a point on the rim at 2.5 s.

Solution

(a) The angular speed, ω, at 2.5 s is (Eq. 9.3),

$$\omega = \omega_0 + \alpha t = 45 \text{ rad/s} + (-20 \text{ rad/s}^2)(2.5 \text{ s}) = -5.0 \text{ rad s}^{-1}.$$

(b) The speed, v, of a point on the rim at 2.5 s is (Eq. 9.6),

$$v = |\omega r| = (5 \text{ rad/s})(0.10 \text{ m}) = 0.50 \text{ m s}^{-1}.$$

(c) The angular displacement, θ, at 2.5 s is (Eq. 9.4),

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = (45 \text{ rad/s})(2.5 \text{ s}) + \frac{1}{2}\left(-20 \text{ rad/s}^2\right)(2.5 \text{ s})^2 = 50 \text{ rad}.$$

(d) At 2.5 s, the centripetal acceleration, a_R, is (Eq. 9.8),

$$a_R = \frac{v^2}{r} = \frac{(0.50 \text{ m/s})^2}{0.10 \text{ m}} = 2.5 \text{ m s}^{-2}.$$

The tangential acceleration, a_T, is,

$$a_T = \alpha r = (-20 \text{ rad/s}^2)(0.10 \text{ m}) = -2.0 \text{ m s}^{-2}.$$

The total acceleration, a, is,

$$a = \sqrt{a_R^2 + a_T^2} = \sqrt{(2.5 \text{ m s}^{-2})^2 + (-2.0 \text{ m s}^{-2})^2} = 3.2 \text{ m s}^{-2}.$$

- wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec)      5
(%i2) omega: -20*2.5;
(omega) -50.0
(%i3) speed: 5*0.1;
(speed) 0.5
(%i4) angular_displacement: 45*2.5 + 0.5*(-20)*2.5^2;
(angular_displacement)  50.0
(%i5) aR: 0.5^2/0.1;
(aR)   2.5
(%i6) aT: -20*0.1;
(aT)   -2.0
(%i7) a: float( sqrt(aR^2 + aT^2) );
(a)   3.2016
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i2), (%i3), (%i4), (%i5), (%i6), (%i7) Calculate ω, v, θ, a_R, a_T, and a.

Problem 9.3 Figure 9.2 shows a disk of mass 0.20 kg and radius 0.10 m that is free to rotate about an axis perpendicular to the page at point O. Two forces, equal in magnitude, parallel to each other, but in opposite directions, act at different points of the disk. The magnitudes of the forces are, $F = 0.50$ N.

(a) Calculate the net torque, τ, on the disk.
(b) What is the angular acceleration, α, of the disk?

Solution

(a) The torque of the top force is (Eq. 9.20),

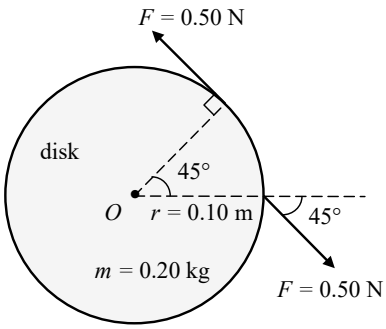


Fig. 9.2 Two forces acting on the rim of a disk, Problem 9.3

$$\tau_1 = rF\sin\theta = rF\sin 90^\circ = rF.$$

The torque of the bottom force is,

$$\tau_2 = rF\sin\theta = -rF\sin 45^\circ = -rF\sqrt{2}/2.$$

The magnitude of net torque about O is,

$$\begin{aligned}\Sigma\tau = \tau_1 + \tau_2 &= rF\left(1-\sqrt{2}/2\right) = 0.29\,rF \\ &= 0.29(0.10\text{ m})(0.50\text{ N}) = 0.015\text{ N m}.\end{aligned}$$

Using the right hand rule, the direction of net torque is out of the plane of the page. The net torque tends to rotate the disk counter clockwise about point O.

(b) The angular acceleration, α, of the disk is,

$$\begin{aligned}\alpha = \frac{\Sigma\tau}{I} &= \frac{rF\left(1-\sqrt{2}/2\right)}{\frac{1}{2}mr^2} \\ &= \frac{(0.10\text{ m})(0.50\text{ N})\left(1-\sqrt{2}/2\right)}{\frac{1}{2}(0.20\text{ kg})(0.10\text{ m})^2} = 15\text{ rad s}^{-2}.\end{aligned}$$

Here, $I = \frac{1}{2}mr^2$ is the moment of inertia of a disk about its center of mass (Fig. 9.1g).

- wxMaxima codes:

```
(%i4) fpprintprec:5; m:0.2; r:0.1; F:0.5;
(fpprintprec)      5
(m)    0.2
(r)    0.1
(F)    0.5
(%i6)  tau1: r*F*sin(float(%pi)/2); tau2: -r*F * sin(float
(%pi)/4);
(tau1) 0.05
(tau2) -0.035355
(%i7) nettorque: tau1+tau2;
(nettorque) 0.014645
(%i8) alpha: nettorque/(0.5*m*r^2);
(alpha) 14.645
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of m, r, and F.

(%i6) Calculate τ_1 and τ_2.

(%i7) Calculate net torque. Part (a).

(%i8) Calculate angular acceleration. Part (b).

Alternative solution:

(a) Calculation of net torque about O,

$$\begin{aligned}\boldsymbol{\tau}_1 = \boldsymbol{r} \times \boldsymbol{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.1\cos(\pi/4)\ \text{m} & 0.1\sin(\pi/4)\ \text{m} & 0 \\ -0.5\cos(\pi/4)\ \text{N} & 0.5\sin(\pi/4)\ \text{N} & 0 \end{vmatrix} \\ &= [(0.1\cos(\pi/4)\ \text{m})(0.5\sin(\pi/4)\ \text{N}) \\ &\quad - (0.1\sin(\pi/4)\ \text{m})(-0.5\cos(\pi/4)\ \text{N})]\mathbf{k} \\ &= 0.05\ \mathbf{k}\ \text{N m}.\end{aligned}$$

$$\begin{aligned}\boldsymbol{\tau}_2 = \boldsymbol{r} \times \boldsymbol{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.1\ \text{m} & 0 & 0 \\ 0.5\cos(\pi/4)\ \text{N} & -0.5\sin(\pi/4)\ \text{N} & 0 \end{vmatrix} \\ &= [(0.1\ \text{m})(-0.5\sin(\pi/4)\ \text{N})]\mathbf{k} \\ &= -0.035\ \mathbf{k}\ \text{N m}.\end{aligned}$$

Net torque $= \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 = 0.015\ \mathbf{k}$ N m. Its direction is out of the plane of the page. This tends to rotate the disk counter clockwise about point O.

(b) The angular acceleration, α, of the disk is,

$$\alpha = \frac{\Sigma\tau}{I} = \frac{\Sigma\tau}{\frac{1}{2}mr^2} = \frac{0.015\ \text{N m}}{\frac{1}{2}(0.20\ \text{kg})(0.10\ \text{m})^2} = 15\ \text{rad s}^{-2}.$$

- wxMaxima codes:

```
(%i4) fpprintprec:5; m:0.2; r:0.1; F:0.5;
(fpprintprec)      5
(m)    0.2
(r)    0.1
(F)    0.5
(%i5) load("vect");
(%o5) "C:/maxima-5.43.0/share/maxima/5.43.0/share/vector/
vect.mac"
```

```
(%i6) rvec: [r*cos(%pi/4), r*sin(%pi/4), 0];
(rvec) [0.1/sqrt(2),0.1/sqrt(2),0]
(%i7) Fvec: [-F*cos(%pi/4), F*sin(%pi/4), 0];
(Fvec) [-0.5/sqrt(2),0.5/sqrt(2),0]
(%i10) tau1: rvec~Fvec$ express(%)$ float(%);
(%o10) [0.0,0.0,0.05]
(%i11) rvec: [r, 0, 0];
(rvec) [0.1,0,0]
(%i12) Fvec: [F*cos(%pi/4), -F*sin(%pi/4), 0];
(Fvec) [0.5/sqrt(2),-0.5/sqrt(2),0]
(%i15) tau2: rvec~Fvec$ express(%)$ float(%);
(%o15) [0.0,0.0,-0.035355]
(%i18) net_torque: tau1+tau2$ express(%)$ float(%);
(%o18) [0.0,0.0,0.014645]
(%i19) nettorque: %[3];
(nettorque) 0.014645
(%i20) alpha : nettorque/(0.5*m*r^2);
(alpha) 14.645
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of m, r, and F.

(%i5) Load "vect" package.

(%i6), (%i7), (%i10) Define $\boldsymbol{r}$ and $\boldsymbol{F}$, and calculate $\boldsymbol{\tau}_1$.

(%i11), (%i12), (%i15) Define $\boldsymbol{r}$ and $\boldsymbol{F}$, and calculate $\boldsymbol{\tau}_2$.

(%i18) Calculate net torque. Part (a).

(%i19), (%i20) Calculate angular acceleration. Part (b).

Problem 9.4 Two forces of magnitude 5.0 N are applied to a solid sphere of radius 1.0 m (Fig. 9.3).

(a) What is the net torque on the sphere about an axis perpendicular to the page at point O?

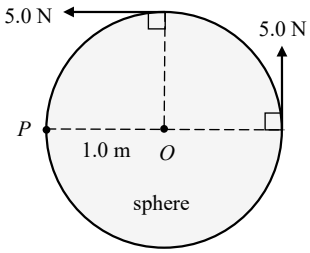


Fig. 9.3 Two forces acting on a sphere, Problem 9.4

(b) Calculate the magnitude of the net torque on the sphere about an axis perpendicular to the page at point *P*.

Solution

(a) The torque due to the 5.0 N force to the left is (Eq. 9.20),

$$\tau_1 = rF\sin\theta = (1.0\text{ m})(5.0\text{ N})\sin 90^\circ = 5.0\text{ N m}.$$

The torque due to the 5.0 N force to the top is,

$$\tau_2 = rF\sin\theta = (1.0\text{ m})(5.0\text{ N})\sin 90^\circ = 5.0\text{ N m}$$

The net torque on the sphere about an axis perpendicular to the page at point *O* is,

$$\Sigma\tau = \tau_1 + \tau_2 = 10\,\text{N m}.$$

Using the right hand rule, the direction of the net torque is out of the plane of the page. The net torque tends to rotate the sphere counter clockwise about point *O*.

(b) The torque due to the force to the left of 5.0 N is (Eq. 9.20),

$$\tau_1 = rF\sin\theta = (1.0\text{ m})(5.0\text{ N})\sin 90^\circ = 5.0\text{ N m}.$$

The torque due to the force to the top of 5.0 N is,

$$\tau_2 = rF\sin\theta = (2.0\text{ m})(5.0\text{ N})\sin 90^\circ = 10\text{ N m}.$$

The net torque on the sphere about an axis perpendicular to the page at point *P* is,

$$\Sigma\tau = \tau_1 + \tau_2 = 15\,\text{N m}.$$

Using the right hand rule, the direction of the net torque is out of the plane of the page. The net torque tends to rotate the sphere counter clockwise about point *P*.

- wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec)      5
(%i4) tau1: 1*5*sin(float(%pi/2)); tau2: 1*5*sin(float(%pi)/
2); nettorque: tau1+tau2;
```

```
(tau1) 5.0
(tau2) 5.0
(nettorque) 10.0
(%i7) tau1: 1*5*sin(float(%pi/2)); tau2: 2*5*sin(float(%pi)/
2); nettorque: tau1+tau2;
(tau1) 5.0
(tau2) 10.0
(nettorque) 15.0
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i4) Calculate τ_1, τ_2, and net torque. Part (a).

(%i7) Calculate τ_1, τ_2, and net torque. Part (b).

Alternative solution:

(a) Calculation of net torque about O,

$$\boldsymbol{\tau}_1 = \boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1\,\text{m} & 0 \\ -5\,\text{N} & 0 & 0 \end{vmatrix}$$
$$= 5\,\mathbf{k}\,\text{N m}.$$

$$\boldsymbol{\tau}_2 = \boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1\,\text{m} & 0 & 0 \\ 0 & 5\,\text{N} & 0 \end{vmatrix}$$
$$= 5\,\mathbf{k}\,\text{N m}.$$

Net torque $= \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 = 10\,\mathbf{k}\,\text{N m}$. The direction of the net torque is out of the plane of the page. The net torque tends to rotate the sphere counter clockwise about point O.

(b) Calculation of net torque about P,

$$\boldsymbol{\tau}_1 = \boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1\,\text{m} & 1\,\text{m} & 0 \\ -5\,\text{N} & 0 & 0 \end{vmatrix}$$
$$= 5\,\mathbf{k}\,\text{N m}.$$

$$\boldsymbol{\tau}_2 = \boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\,\text{m} & 0 & 0 \\ 0 & 5\,\text{N} & 0 \end{vmatrix}$$

$$= 10\,\mathbf{k}\ \text{N m}.$$

Net torque $= \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 = 15\,\mathbf{k}$ N m. The direction of the net torque is out of the plane of the page. The net torque tends to rotate the sphere counter clockwise about point P.

- wsMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec)      5
(%i2) load("vect");
(%o2) "C:/maxima-5.43.0/share/maxima/5.43.0/share/vector/
vect.mac"
(%i7)  rvec: [0, 1, 0]; Fvec: [-5, 0, 0]; tau1: rvec~Fvec$
express(%)
$ float(%);
(rvec) [0,1,0]
(Fvec) [-5,0,0]
(%o7) [0.0,0.0,5.0]
(%i12)  rvec: [1, 0, 0]; Fvec: [0, 5, 0]; tau2: rvec~Fvec$
express(%)
$ float(%);
(rvec) [1,0,0]
(Fvec) [0,5,0]
(%o12) [0.0,0.0,5.0]
(%i15) net_torque: tau1+tau2$ express(%)$ float(%);
(%o15) [0.0,0.0,10.0]
(%i20) rvec: [1, 1, 0]; Fvec: [-5, 0, 0]; tau1: rvec~Fvec
$ express(%)$ float(%);
(rvec) [1,1,0]
(Fvec) [-5,0,0]
(%o20) [0.0,0.0,5.0]
(%i25) rvec: [2, 0, 0]; Fvec: [0, 5, 0]; tau2: rvec~Fvec
$ express(%)$ float(%);
(rvec) [2,0,0]
(Fvec) [0,5,0]
(%o25) [0.0,0.0,10.0]
(%i28) net_torque: tau1+tau2$ express(%)$ float(%);
(%o28) [0.0,0.0,15.0]
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i2) Load "vect" package.

(%i7) Define $\boldsymbol{r}$ and $\boldsymbol{F}$, and calculate $\boldsymbol{\tau}_1$.

(%i12) Define $\boldsymbol{r}$ and $\boldsymbol{F}$, and calculate $\boldsymbol{\tau}_2$.

(%i15) Calculate net torque. Part (a).

(%i20) Define $\boldsymbol{r}$ and $\boldsymbol{F}$, and calculate $\boldsymbol{\tau}_1$.

(%i25) Define $\boldsymbol{r}$ and $\boldsymbol{F}$, and calculate $\boldsymbol{\tau}_2$.

(%i28) Calculate net torque. Part (b).

Problem 9.5 A force $\boldsymbol{F} = 5\mathbf{j} - 6\mathbf{k}$ N is applied on a particle located at $\boldsymbol{r} = 3\mathbf{j} + 5\mathbf{k}$ m. What is the torque, $\boldsymbol{\tau}$, on the particle with respect to the origin?

Solution

The torque, $\boldsymbol{\tau}$, on the particle with respect to the origin is (Eq. 9.20),

$$\begin{aligned}\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 \text{ m} & 3 \text{ m} & 5 \text{ m} \\ 0 \text{ N} & 5 \text{ N} & -6 \text{ N} \end{vmatrix} \\ &= [(3 \text{ m})(-6 \text{ N}) - (5 \text{ m})(5 \text{ N})]\mathbf{i} - 0\mathbf{j} + 0\mathbf{k}, \\ &= -43 \, \mathbf{i} \text{ N m}.\end{aligned}$$

- wxMaxima codes:

```
(%i1) load("vect");
(%o1) "C:/maxima-5.43.0/share/maxima/5.43.0/share/vector/
vect.mac"
(%i2) F: [0,5,-6] ;
(F)   [0,5,-6]
(%i3) r: [0,3,5] ;
(r)   [0,3,5]
(%i5) tau: r~F; express(%);
(tau) [0,3,5]~[0,5,-6]
(%o5) [-43,0,0]
```

Comments on the codes:

(%i1) Load "vect" package.

(%i2), (%i3) Assign vectors $\boldsymbol{F}$ and $\boldsymbol{r}$.

(%i5) Calculate vector product $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$.

Problem 9.6 A particle of mass 3.0 kg is moving at velocity 5.0 m s^{-1} along line PQ (Fig. 9.4).

(a) Calculate the angular momentum, $\boldsymbol{L}$, of the particle relative to the origin when the particle is at point P.

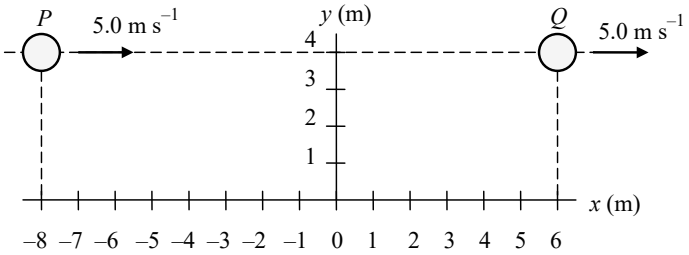


Fig. 9.4 Determining angular momentum of a particle at points *P* and *Q*, Problem 9.6

(b) What is the angular momentum, $\boldsymbol{L}$, of the particle when the particle is at point *Q*?

Solution

(a) Angular momentum is defined as $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times m\boldsymbol{v}$ (Eq. 9.21). The angular momentum of the particle relative to the origin when it is at point *P* is,

$$\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8\ \text{m} & 4\ \text{m} & 0 \\ 15\ \text{kg m/s} & 0 & 0 \end{vmatrix} = -60\ \text{kg m}^{-2}\ \text{s}^{-1}\ \mathbf{k}.$$

The direction of the angular momentum is into the plane of the paper, that is, in the negative *z* direction.

- wxMaxima codes:

```
(%i1) load("vect");
(%o1) "C:/maxima-5.43.0/share/maxima/5.43.0/share/vector/
vect.mac"
(%i2) r: [-8,4,0];
(r)   [-8,4,0]
(%i3) mv: [15,0,0];
(mv)  [15,0,0]
(%i5) L: r~mv; express(%);
(L)   [-8,4,0]~[15,0,0]
(%o5) [0,0,-60]
```

Comments on the codes:

(%i1) Load "vect" package.

(%i2), (%i3) Assign vectors $\boldsymbol{r}$ and $m\boldsymbol{v}$.

(%i5) Calculate vector product $\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v}$.

(b) Repeating the calculation for different position vector. The angular momentum of the particle relative to the origin when it is at point Q is,

$$\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6\text{ m} & 4\text{ m} & 0 \\ 15\text{ kg m/s} & 0 & 0 \end{vmatrix} = -60\text{ kg m}^{-2}\text{ s}^{-1}\,\mathbf{k}.$$

The direction of the angular momentum is into the plane of the paper, that is, in the negative z direction.

- wxMaxima codes:

```
(%i1) load("vect");
(%o1) "C:/maxima-5.43.0/share/maxima/5.43.0/share/vector/
vect.mac"
(%i2) r: [6,4,0];
(r)   [6,4,0]
(%i3) mv: [15,0,0];
(mv)  [15,0,0]
(%i5) L: r~mv; express(%);
(L)   [6,4,0]~[15,0,0]
(%o5) [0,0,-60]
```

Comments on the codes:

(%i1) Load "vect" package.

(%i2), (%i3) Assign vectors $\boldsymbol{r}$ and $m\boldsymbol{v}$.

(%i5) Calculate vector product $\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v}$.

Problem 9.7 An object of mass 2.0 kg is at coordinates $x = 2.0$ m, $y = 4.0$ m moving with a velocity of $\boldsymbol{v} = 3\,\mathbf{i} + 7\,\mathbf{j}$ m s^{-1}. A force of $-9.0\,\mathbf{i}$ N acts on the object. Calculate,

(a) the angular momentum, $\boldsymbol{L}$, of the object,
(b) the torque, $\boldsymbol{\tau}$, on the object,
(c) the rate of angular momentum change.

Solution

(a) The angular momentum, $\boldsymbol{L}$, of the object relative to the origin is (Eq. 9.21),

$$\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\text{ m} & 4\text{ m} & 0 \\ 6\text{ kg m/s} & 14\text{ kg m/s} & 0 \end{vmatrix} = 4.0\text{ kg m}^{-2}\text{ s}^{-1}\,\mathbf{k}.$$

- wxMaxima codes:

```
(%i1) load("vect");
(%o1) "C:/maxima-5.43.0/share/maxima/5.43.0/share/vector/
vect.mac"
(%i2) r: [2,4,0];
(r)   [2,4,0]
(%i3) mv: [6,14,0];
(mv)  [6,14,0]
(%i5) L: r~mv; express(%);
(L)   [2,4,0]~[6,14,0]
(%o5) [0,0,4]
```

Comments on the codes:

(%i1) Load "vect" package.

(%i2), (%i3) Assign vectors $\boldsymbol{r}$ and $m\boldsymbol{v}$.

(%i5) Calculate vector product $\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v}$.

(b) The torque, $\boldsymbol{\tau}$, on the object relative to the origin is,

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\text{ m} & 4\text{ m} & 0 \\ -9\text{ N} & 0 & 0 \end{vmatrix} = 36\text{ N m }\mathbf{k}.$$

- wxMaxima codes:

```
(%i1) load("vect");
(%o1) "C:/maxima-5.43.0/share/maxima/5.43.0/share/vector/
vect.mac"
(%i2) r: [2,4,0];
(r)   [2,4,0]
(%i3) F: [-9,0,0];
(F)   [-9,0,0]
(%i5) tau: r~F; express(%);
(tau) -[-9,0,0]~[2,4,0]
(%o5) [0,0,36]
```

Comments on the codes:

(%i1) Load "vect" package.

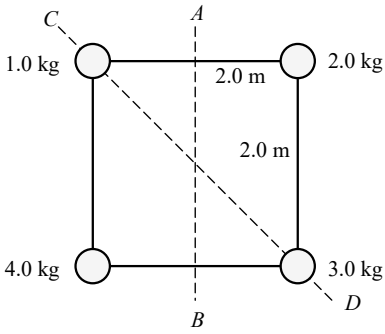


Fig. 9.5 Determining moments of inertia about different axes of rotation, Problem 9.8

(%i2), (%i3) Assign vectors $\boldsymbol{r}$ and $\boldsymbol{F}$.

(%i5) Calculate vector product $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$.

(c) The rate of angular momentum change = torque = 36 N m **k**.

Problem 9.8 Four particles are fixed at the vertices of a light rigid square wire of dimension 2.0 m by 2.0 m (Fig. 9.5). Calculate the moment of inertia, I, about the following axis of rotation:

(a) AB,
(b) CD.

Solution

(a) The moment of inertia of a system of discrete particles about an axis is $I = \sum_i m_i r_i^2$ as stated in Eq. (9.9). The moment of inertia about the axis AB is,

$$I_{AB} = (1.0\ \text{kg})(1.0\ \text{m})^2 + (2.0\ \text{kg})(1.0\ \text{m})^2 + (3.0\ \text{kg})(1.0\ \text{m})^2 + (4.0\ \text{kg})(1.0\ \text{m})^2 = 10\ \text{kg m}^2.$$

(b) The moment of inertia about the axis CD is,

$$I_{CD} = (1.0\ \text{kg})(0)^2 + (2.0\ \text{kg})(\sqrt{2}\ \text{m})^2 + (3.0\ \text{kg})(0)^2 + (4.0\ \text{kg})(\sqrt{2}\ \text{m})^2 = 12\ \text{kg m}^2.$$

- wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec)      5
(%i2) IAB: float( 1*1^2 + 2*1^2 + 3*1^2 + 4*1^2 );
(IAB) 10.0
(%i3) ICD: float( 1*0^2 + 2*sqrt(2)^2 + 3*0^2 + 4*sqrt(2)^2 );
```

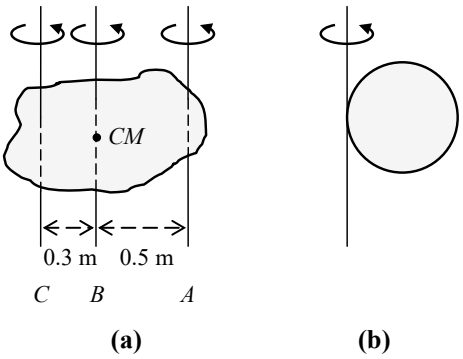


Fig. 9.6 **a** Moments of inertia about different parallel axes, **b** moment of inertia of a sphere about an axis that touches its surface, Problem 9.9

```
(ICD)  12.0
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i2), (%i3) Calculate I_{AB} and I_{CD}.

Problem 9.9

(a) A rigid body of mass 8.0 kg has a moment of inertia of 6.0 kg m^2 about the axis A (Fig. 9.6a). What is the moment of inertia about the axis B that passes through the center of mass of the body? What is the moment of inertia about the axis C?

(b) Calculate the moment of inertia of a sphere about an axis that touches the sphere as in Fig. 9.6b. The mass and radius of the sphere are m and r, respectively.

Solution

(a) Applying the parallel axes theorem to axes A and B (Eq. 9.12), we have,

$$I = I_{CM} + mh^2,$$
$$I_A = I_B + mh^2,$$
$$6.0 \text{ kg m}^2 = I_B + (8.0 \text{ kg})(0.50 \text{ m})^2,$$

where $I_{CM} = I_B$ is the moment of inertia about the axis through the center of mass, I_A is the moment of inertia about axis A, and h is the distance between axes A and B. Therefore, the moment of inertia about the axis B is,

$$I_B = I_{CM} = 4.0 \text{ kg m}^2.$$

The moment of inertia of the rigid body about the axis C is,

$$\begin{aligned} I_C &= I_{CM} + mh^2 = I_B + mh^2 \\ &= 4.0 \text{ kg m}^2 + (8.0 \text{ kg})(0.30 \text{ m})^2 \\ &= 4.7 \text{ kg m}^2, \end{aligned}$$

where h is the distance between axes C and B.

- wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec)      5
(%i2) IB: float(6 - 8*0.5^2);
(IB)  4.0
(%i3) IC: float(IB + 8*0.3^2);
(IC)  4.72
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i2), (%i3) Calculate I_B and I_C.

(b) We know that the moment of inertia of a sphere about an axis through its center of mass is (Fig. 9.1i),

$$I_{CM} = \frac{2}{5}mr^2.$$

We apply the parallel axes theorem (Eq. 9.12), to the axis that touches the sphere and the axis through the center of mass. The moment of inertia of a sphere about an axis that touches the sphere is,

$$\begin{aligned} I &= I_{CM} + mh^2 = \frac{2}{5}mr^2 + mr^2 \\ &= \frac{7}{5}mr^2, \end{aligned}$$

where h is the distance between the two axes, that is, the radius of the sphere.

Problem 9.10 Determine the moment of inertia, I, of a rectangular plate of mass, m, and dimension, $a \times b$, about an axis perpendicular to the plate through the center of the plate (Fig. 9.7a).

Solution

For a plate, the perpendicular axes theorem (Eq. 9.13) can be applied as in Fig. 9.7b. First, we calculate the moments of inertia of the plate about the x and y axes. Then,

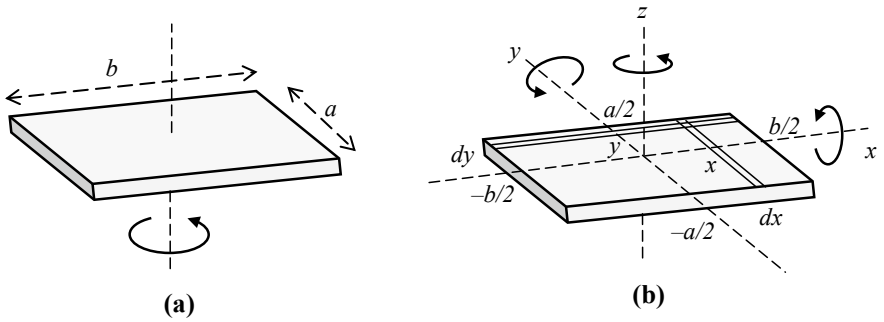


Fig. 9.7 **a** What is moment of inertia of the plate about an axis perpendicular to the plate? **b** Determining the moment of inertia by the perpendicular axis theorem, Problem 9.10

we apply the perpendicular axes theorem to calculate the moment of inertia about the z axis.

The moment of inertia of the plate about the x axis is (Eq. 9.10),

$$I_x = \int y^2 dm = \int_{-a/2}^{a/2} y^2 \left(\frac{m}{a} dy\right) = \frac{m}{3a}\left[y^3\right]_{-a/2}^{a/2} = \frac{1}{12} m a^2,$$

where the elementary mass is $dm = (m/a)\, dy$ and its distance to the x axis is y.

Similarly, the moment of inertia of the plate about the y axis is (Eq. 9.10),

$$I_y = \int x^2 dm = \int_{-b/2}^{b/2} x^2 \left(\frac{m}{b} dx\right) = \frac{m}{3b}\left[x^3\right]_{-b/2}^{b/2} = \frac{1}{12} m b^2,$$

where the elementary mass is $dm = (m/b)\, dx$ and its distance to the y axis is x.

Applying the perpendicular axes theorem (Eq. 9.13), the moment of inertia, I, of the rectangular plate about the z axis is,

$$I_z = I_x + I_y = \frac{1}{12} m a^2 + \frac{1}{12} m b^2 = \frac{1}{12} m\left(a^2 + b^2\right),$$
$$I = I_z = \frac{1}{12} m\left(a^2 + b^2\right),$$

because the z axis is perpendicular to the x and y axes. This result is shown in Fig. 9.1b.

- wxMaxima codes:

```
(%i1) Ix: integrate( (m/a)*y^2, y, -a/2, a/2 );
(Ix)  (a^2*m)/12
(%i2) Iy: integrate( (m/b)*x^2, x, -b/2, b/2 );
(Iy)  (b^2*m)/12
(%i3) Iz: Ix + Iy;
(Iz)  (b^2*m)/12+(a^2*m)/12
(%i4) ratsimp(%);
(%o4) ((b^2+a^2)*m)/12
```

Comments on the codes:

(%i1), (%i2) Calculate define integrals $I_x = \int_{-a/2}^{a/2} y^2\left(\frac{m}{a}dy\right)$ and $I_y = \int_{-b/2}^{b/2} x^2\left(\frac{m}{b}dx\right)$.

(%i3), (%i4) Calculate $I_z = I_x + I_y$ and simplify.

Problem 9.11

(a) A uniform solid cylinder rotates about its symmetry axis A at an angular speed of 4.0 rad s^{-1} (Fig. 9.8a). The mass, radius, and length of the cylinder are 5.0 kg, 3.0 cm, and 8.0 cm, respectively. Calculate its rotational kinetic energy.

(b) The cylinder rotates at the same angular speed about axis B which is parallel to axis A (Fig. 9.8a). What is the rotational kinetic energy?

(c) The cylinder rotates at the same angular speed about axis C. Axis C passes through the center of mass and is perpendicular to axis A (Fig. 9.8a). Calculate the rotational kinetic energy.

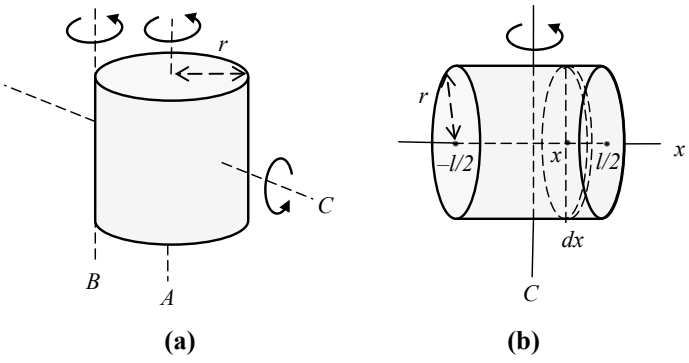


Fig. 9.8 Problem 9.11

Solution

(a) The rotational kinetic energy of the cylinder rotating about axis A is (Eq. 9.14),

$$\begin{aligned} K_A &= \frac{1}{2} I_A \omega^2 = \frac{1}{2}\left(\frac{1}{2} m r^2\right)\omega^2 \\ &= \frac{1}{2} \times \frac{1}{2} \times (5.0\ \text{kg})(0.030\ \text{m})^2 (4.0\ \text{rad/s})^2 \\ &= 0.018\ \text{J}. \end{aligned}$$

The moment of inertia of a cylinder about its axis is shown in Fig. 9.1e.

(b) Applying the parallel axes theorem (Eq. 9.12), the moment of inertia of the culinder about axis B is,

$$I_B = I_A + mh^2 = \frac{1}{2} m r^2 + m r^2 = \frac{3}{2} m r^2.$$

The rotational kinetic energy about axis B is (Eq. 9.14),

$$\begin{aligned} K_B &= \frac{1}{2} I_B \omega^2 = \frac{1}{2}\left(\frac{3}{2} m r^2\right)\omega^2 = \frac{3}{4} m r^2 \omega^2 \\ &= \frac{3}{4} \times (5.0\ \text{kg})(0.030\ \text{m})^2 (4.0\ \text{rad/s})^2 \\ &= 0.054\ \text{J}. \end{aligned}$$

(c) First, we calculate the moment of inertia of the cylinder about axis C (Fig. 9.8b). Consider a disk of thickness dx of mass $dm = dx\, m/l$. Applying the parallel axes theorem (Eq. 9.12), the moment of inertia of the disk about axis C is,

$$\begin{aligned} I &= I_{CM} + mh^2, \\ dI &= \frac{1}{4} dm\, r^2 + dm\, x^2 \\ &= \frac{1}{4} \frac{r^2 m}{l} dx + \frac{m}{l} x^2\, dx, \end{aligned}$$

where $\frac{1}{4} dm\, r^2$ is the moment of inertia of the disk about a diameter of the disk through the center of mass (Fig. 9.1h). The moment of inertia of the whole cylinder about axis C is obtained by the integration,

$$\begin{aligned} I_C &= \int dI = \int_{-l/2}^{l/2} \frac{1}{4} \frac{r^2 m}{l} dx + \int_{-l/2}^{l/2} \frac{m}{l} x^2\, dx \\ &= \frac{1}{4} \frac{r^2 m}{l} [x]_{-l/2}^{l/2} + \frac{m}{l} \left[x^3/3\right]_{-l/2}^{l/2} \end{aligned}$$

$$= \frac{1}{4}r^2 m + \frac{1}{12}ml^2.$$

Thus, the rotational kinetic energy of the cylinder about axis C is (Eq. 9.14),

$$\begin{aligned} K_C &= \frac{1}{2}I_C\omega^2 = \frac{1}{2}\left(\frac{1}{4}mr^2 + \frac{1}{12}ml^2\right)\omega^2 \\ &= \frac{1}{2}\left(\frac{(5.0\text{ kg})(0.030\text{ m})^2}{4} + \frac{(5.0\text{ kg})(0.080\text{ m})^2}{12}\right)(4.0\text{ rad/s})^2 \\ &= 0.030\text{ J}. \end{aligned}$$

- wxMaxima codes:

```
(%i6)  fpprintprec:5; ratprint:false; omega:4; m:5; r:3e-2;
l:8e-2;
(fpprintprec)      5
(ratprint)  false
(omega)      4
(m)    5
(r)    0.03
(l)    0.08
(%i7)  IA: 1/2*m*r^2;
(IA)  0.00225
(%i8)  KA: 1/2*IA*omega^2;
(KA)  0.018
(%i9)  IB: IA + m*r^2;
(IB)  0.00675
(%i10) KB: 1/2*IB*omega^2;
(KB)  0.054
(%i11) IC: integrate(r^2*m/(4*l),x,-l/2,l/2) + integrate
(m/l*x^2,x,-l/2,l/2);
(IC)  0.0037917
(%i12) KC: 1/2*IC*omega^2;
(KC)  0.030333
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of ω, m, r, and l.

(%i7), (%i8) Calculate I_A and K_A. Part (a).

(%i9), (%i10) Calculate I_B and K_B. Part (b).

(%i11), (%i12) Calculate I_C and K_C. Part (c).

- Alternative calculation:

```
(%i1) fpprintprec:5;
(fpprintprec)      5
(%i3) IA: (1/2)*m*r^2; KA: (1/2)*IA*omega^2;
(IA)  (m*r^2)/2
(KA)  (m*omega^2*r^2)/4
(%i4) subst([omega=4, m=5, r=0.03, l=0.08], [IA, KA]);
(%o4) [0.00225,0.018]
(%i6) IB: IA + m*r^2; KB: (1/2)*IB*omega^2;
(IB)  (3*m*r^2)/2
(KB)  (3*m*omega^2*r^2)/4
(%i7) subst([omega=4, m=5, r=0.03, l=0.08], [IB, KB]);
(%o7) [0.00675,0.054]
(%i9) IC: integrate((1/4)*r^2*m/l,x,-l/2,l/2) + integrate
(m/l*x^2,x,-l/2,l/2); KC: (1/2)*IC*omega^2;
(IC)  (m*r^2)/4+(l^2*m)/12
(KC)  (omega^2*((m*r^2)/4+(l^2*m)/12))/2
(%i10) subst([omega=4, m=5, r=0.03, l=0.08], [IC, KC]);
(%o10) [0.0037917,0.030333]
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i3) Assign I_A and K_A in symbols.

(%i4) Substitute values of ω, m, r, and l to get numerical values of I_A and K_A.

(%i6) Assign I_B and K_B in symbols.

(%i7) Substitute values of ω, m, r, and l to get numerical values of I_B and K_B.

(%i9) Assign I_C and K_C in symbols.

(%i10) Substitute values of ω, m, r, and l to get numerical values of I_C and K_C.

Problem 9.12 The radius of gyration, k, of a rigid body of mass, m, about an axis is the distance of a point particle of mass, m, to the axis such that the moment of inertia of the rigid body is $I = mk^2$. This means that $k = (I/m)^{1/2}$. Determine the radius of gyration of,

(a) a disk about its symmetry axis normal to its circular surface,
(b) a ring about its symmetry axis normal to the plane of the ring,
(c) a uniform rod about an axis perpendicular to the rod at one of its ends,
(d) a uniform rod about an axis perpendicular to the rod at its center,
(e) a spherical shell about an axis through its center,
(f) a solid sphere about an axis through its center.

Solution

The radius of gyration is calculated from the formula $k = (I/m)^{1/2}$, by substituting the appropriate moment of inertia into the formula. Formulae of moments of inertia can be referred to Fig. 9.1.

(a) A disk of mass, m, radius, r, about its symmetry axis normal to its circular surface,

$$k = \sqrt{\frac{I}{m}} = \sqrt{\frac{\frac{1}{2}mr^2}{m}} = \sqrt{\frac{1}{2}}r = 0.71r.$$

(b) A ring of mass, m, radius, r, about its symmetry axis normal to the plane of the ring,

$$k = \sqrt{\frac{I}{m}} = \sqrt{\frac{mr^2}{m}} = r.$$

(c) A uniform rod of mass, m, length, l, about an axis perpendicular to the rod at one of its ends,

$$k = \sqrt{\frac{I}{m}} = \sqrt{\frac{\frac{1}{3}ml^2}{m}} = \sqrt{\frac{1}{3}}l = 0.58l.$$

(d) A uniform rod of mass, m, length, l, about an axis perpendicular to the rod at its center,

$$k = \sqrt{\frac{I}{m}} = \sqrt{\frac{\frac{1}{12}ml^2}{m}} = \sqrt{\frac{1}{12}}l = 0.29l.$$

(e) A spherical shell of mass, m, radius, r, about an axis through its center,

$$k = \sqrt{\frac{I}{m}} = \sqrt{\frac{\frac{2}{3}mr^2}{m}} = \sqrt{\frac{2}{3}}r = 0.82r.$$

(f) A solid sphere of mass, m, radius, r, about an axis through its center,

$$k = \sqrt{\frac{I}{m}} = \sqrt{\frac{\frac{2}{5}mr^2}{m}} = \sqrt{\frac{2}{5}}r = 0.63r.$$

- wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec)      5
(%i3) k: sqrt((1/2)*m*r^2/m)$ float(%);
(%o3) 0.70711*abs(r)
(%i5) k: sqrt(m*r^2/m)$ float(%);
(%o5) abs(r)
(%i7) k: sqrt((1/3)*m*l^2/m)$ float(%);
(%o7) 0.57735*abs(l)
(%i9) k: sqrt((1/12)*m*l^2/m)$ float(%);
(%o9) 0.28868*abs(l)
(%i11) k: sqrt((2/3)*m*r^2/m)$ float(%);
(%o11) 0.8165*abs(r)
(%i13) k: sqrt((2/5)*m*r^2/m)$ float(%);
(%o13) 0.63246*abs(r)
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i3) Calculate radius of gyration. Part (a).

(%i5) Calculate radius of gyration. Part (b).

(%i7) Calculate radius of gyration. Part (c).

(%i9) Calculate radius of gyration. Part (d).

(%i11) Calculate radius of gyration. Part (e).

(%i13) Calculate radius of gyration. Part (f).

Figure 9.9 shows the radii of gyration, k, that have been calculated.

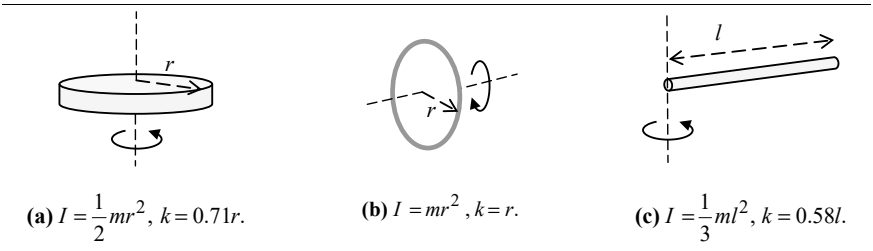
Problem 9.13 A wheel of radius 20 cm is wrapped by a light string where the wheel can rotate about an axle with negligible friction. The string is pulled by a constant force of 6.0 N and it moves a distance of 50 cm in 1.0 s.

(a) What is the angular acceleration of the wheel?
(b) Calculate the moment of inertia of the wheel.
(c) If the wheel is a disk, what is the mass of the disk?

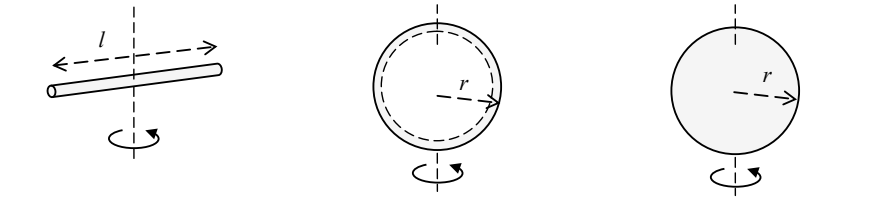
Solution

(a) Figure 9.10 shows the wheel of radius, R, and moment of inertia, I, and the string being pulled by the force, F.

The string move a distance of 50 cm is 1.0 s. The linear acceleration, a, of the string can be calculated using the kinematic equation,



(a) $I = \frac{1}{2}mr^2,\ k = 0.71r.$ **(b)** $I = mr^2,\ k = r.$ **(c)** $I = \frac{1}{3}ml^2,\ k = 0.58l.$



(d) $I = \frac{1}{12}ml^2,\ k = 0.29l.$ **(e)** $I = \frac{2}{3}mr^2,\ k = 0.82r.$ **(f)** $I = \frac{2}{5}mr^2,\ k = 0.63r.$

Fig. 9.9 Moment of inertia and radius of gyration of rigid bodies

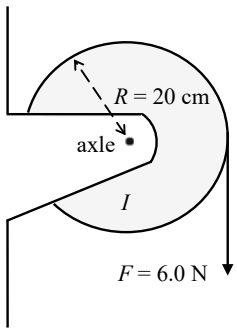


Fig. 9.10 A light string wrapped around a wheel is pulled by a force, Problem 9.13

$$s = ut + \frac{1}{2}at^2,$$
$$0.50\ \text{m} = 0 + \frac{1}{2}a(1.0\ \text{s})^2,$$
$$a = 1.0\ \text{m s}^{-2}.$$

The angular acceleration, α, is calculated from the relation $a = \alpha R$,

$$\alpha = \frac{a}{R} = \frac{1.0\ \text{m/s}^2}{0.20\ \text{m}} = 5.0\ \text{rad s}^{-2}.$$

(b) The torque is $\tau = RF$ and it causes the angular acceleration. The moment of inertia, I, of the wheel is calculated as follows,

$$\begin{aligned}\tau &= I\alpha,\\ RF &= I\alpha,\\ I &= \frac{RF}{\alpha} = \frac{(0.20\text{ m})(6.0\text{ N})}{5.0\text{ rad/s}^2} = 0.24\text{ kg m}^2.\end{aligned}$$

(c) The moment of inertia of a disk of mass, M, and radius, R, about its symmetry axis is $I = MR^2/2$ (Fig. 9.1g). The mass, M, of the disk is obtained as follows,

$$\begin{aligned}I &= \frac{1}{2}MR^2,\\ M &= \frac{2I}{R^2} = \frac{2(0.24\text{ kg m}^2)}{(0.20\text{ m})^2} = 12\text{ kg}.\end{aligned}$$

- wxMaxima codes:

```
(%i5) fpprintprec:5; R:0.2; F:6; s:0.5; t:1;
(fpprintprec)      5
(R)    0.2
(F)    6
(s)    0.5
(t)    1
(%i6) a: 2*s/t^2;
(a)    1.0
(%i7) alpha: a/R;
(alpha) 5.0
(%i8) tau: R*F;
(tau) 1.2
(%i9) I: tau/alpha;
(I)    0.24
(%i10) M: 2*I/R^2;
(M)    12.0
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of R, F, s, and t.

(%i6), (%i7) Calculate a and α. Part (a).

(%i8), (%i9) Calculate τ and I. Part (b).

(%i10) Calculate M. Part (c).

- Alternative calculation:

```
(%i7)  fpprintprec:5; ratprint:false; R:0.2; F:6; tau:R*F;
s:0.5; t:1;
(fpprintprec)       5
(ratprint)  false
(R)   0.2
(F)   6
(tau) 1.2
(s)   0.5
(t)   1
(%i8) solve(s=(1/2)*a*t^2, a);
(%o8) [a=1]
(%i9) a: rhs(%[1]);
(a)   1
(%i10) alpha: a/R;
(alpha) 5.0
(%i12) solve(tau=I*alpha, I)$ float(%);
(%o12) [I=0.24]
(%i13) I: rhs(%[1]);
(I)   0.24
(%i14) solve(I=(1/2)*M*R^2, M);
(%o14) [M=12]
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of R, F, τ, s, and t.

(%i8), (%i9) Solve $s = \frac{1}{2}at^2$ for a and assign value of a.

(%i10) Assign value of α.

(%i12), (%i13) Solve $\tau = I\alpha$ for I and assign value of I.

(%i14) Solve $I = \frac{1}{2}MR^2$ for M.

Problem 9.14 A disk of mass 12 kg and radius of 20 cm is free to rotate about an axle with negligible friction. A light string is warped around the disk and a block of mass 5.0 kg is tied to the string (Fig. 9.11). The system is released from rest, the disk rotates, and the block moves down.

(a) What is the angular acceleration of the disk?
(b) Calculate the time taken for the block to move a distance of 50 cm down.
(c) What is the speed of the block at a distance of 50 cm down from the initial position?

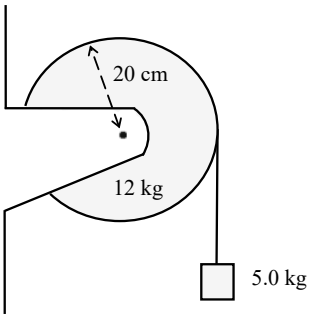


Fig. 9.11 A disk-mass system of Problem 9.14

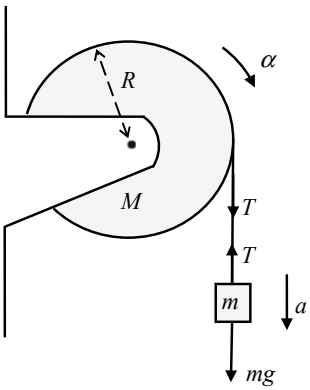


Fig. 9.12 Forces on the mass and torque on the disk of Problem 9.14

Solution

(a) Figure 9.12 shows the forces acting on the disk and the block. The net force acting on the block is $mg - T$, where mg is the weight of the block and T is the tension in the string. This net force causes the block to move down with linear acceleration, a.

Applying the Newton's second law to the block, we write,

$$mg - T = ma. \tag{9.1}$$

The linear acceleration is,

$$a = \alpha R, \tag{9.2}$$

where α is the angular acceleration of the disk and R is its radius. So,

$$mg - T = mR\alpha.$$

The torque on the disk is $\tau = TR$ and this causes the disk to rotate with angular acceleration α,

$$TR = I\alpha, \tag{9.3}$$

where I is the moment of inertia of the disk. From the two equations, we write,

$$mgR - I\alpha = mR^2\alpha,$$

which gives,

$$\alpha = \frac{mgR}{I + mR^2}.$$

For a disk of mass, M, and radius, R, its moment of inertia is,

$$I = MR^2/2. \tag{9.4}$$

The angular acceleration, α, of the disk is,

$$\begin{aligned}\alpha &= \frac{mgR}{\frac{1}{2}MR^2 + mR^2} = \frac{mg}{\frac{1}{2}MR + mR} \\ &= \frac{(5.0\text{ kg})(9.8\text{ N/kg})}{\frac{1}{2}(12\text{ kg})(0.20\text{ m}) + (5.0\text{ kg})(0.20\text{ m})} \\ &= 22\text{ rad s}^{-2}.\end{aligned}$$

(b) The time, t, taken for the block to fall by a distance of 50 cm, is obtained as follows,

$$s = ut + \frac{1}{2}at^2,\ u = 0 \Rightarrow s = \frac{1}{2}at^2,$$

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2s}{\alpha R}} = \sqrt{\frac{2(0.50\text{ m})}{(22\text{ rad/s}^2)(0.20\text{ m})}} = 0.47\text{ s}.$$

(c) The speed, v, of the block at a distance of 50 cm down from the initial position is,

$$v^2 - u^2 = 2as,\quad u = 0 \Rightarrow v^2 = 2as,$$

$$v = \sqrt{2as} = \sqrt{2(22\text{ rad/s}^2)(0.50\text{ m})} = 2.1\text{ m s}^{-1}.$$

- wxMaxima codes:

```
(%i7) fpprintprec:5; ratprint:false; M:12; R:0.2; m:5; g:9.8;
s:0.5;
(fpprintprec)      5
(ratprint)  false
(M)   12
(R)   0.2
(m)   5
(g)   9.8
(s)   0.5
(%i8) I: 0.5*M*R^2;
(I)   0.24
(%i10) solve([m*g-T=m*R*alpha, T*R=I*alpha],[alpha, T])$
float(%);
(%o10) [[alpha=22.273,T=26.727]]
(%i11) alpha: rhs(%[1][1]);
(alpha) 22.273
(%i12) a: alpha*R;
(a) 4.4545
(%i14) solve(s=0.5*a*t^2, t)$ float(%);
(%o14) [t=-0.4738,t=0.4738]
(%i15) v: sqrt(2*a*s);
(v) 2.1106
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of M, R, m, g, and s.

(%i8) Calculate I.

(%i10) Solve $mg - T = mR\alpha$ and $TR = I\alpha$ for α and T.

(%i11) Assign value of α.

(%i12) Calculate a. Part (a).

(%i14) Solve $s = \frac{1}{2}at^2$ for t. Part (b).

(%i15) Calculate v. Part (c).

- Alternative calculation:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)      5
(ratprint)  false
```

```
(%i6)  eq1: m*g-T=m*a; eq2: a=alpha*R; eq3: T*R=I*alpha; eq4:
I=(1/2)*M*R^2;
(eq1)  g*m-T=a*m
(eq2)  a=R*alpha
(eq3)  R*T=I*alpha
(eq4)  I=(M*R^2)/2
(%i7)  solve([eq1,eq2,eq3,eq4], [alpha,a,T,I]);
(%o7)  [[alpha=(2*g*m)/(2*R*m+M*R),a=(2*g*m)/(2*m+M),
       T=(M*g*m)/(2*m+M),I=(M*R^2)/2]]
(%i8)  subst([R=0.2, M=12, m=5, g=9.8], %);
(%o8)  [[alpha=22.273,a=4.4545,T=26.727,I=0.24]]
(%i9)  a: rhs(%[1][2]);
(a)    4.4545
(%i10) s:0.5;
(s)    0.5
(%i12) solve(s=(1/2)*a*t^2, t)$ float(%);
(%o12) [t=-0.4738,t=0.4738]
(%i14) solve(v^2=2*a*s, v)$ float(%);
(%o14) [v=-2.1106,v=2.1106]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i6) Assign Eqs. (9.1), (9.2), (9.3), and (9.4) as eq1, eq2, eq3, and eq4, respectively.

(%i7) Solve Eqs. (9.1), (9.2), (9.3), and (9.4) for α, a, T, and I. Part (a).

(%i8) Substitute values of R, M, m, and g to get numerical values of α, a, T, and I.

(%i9), (%i10) Assign values of a and s.

(%i12) Solve $s = \frac{1}{2}at^2$ for t. Part (b).

(%i14) Solve $v^2 = 2as$ for v. Part (c).

Alternative solution of part (c): We apply the conservation of energy to the initial and final instances (Eq. 7.10). In the initial instance, we have the potential energy of the block. This energy is transferred to the rotational kinetic energy of the disk and the translational kinetic energy of the block in the final instance. We have,

$$\begin{aligned} U_{init} + K_{init} &= U_{final} + K_{final}, \\ mgs &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2, \\ mgs &= \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2, \end{aligned}$$

$$v = \sqrt{\frac{mgs}{\frac{1}{4}M + \frac{1}{2}m}} = \sqrt{\frac{(5.0\text{ kg})(9.8\text{ N/kg})(0.50\text{ m})}{\frac{1}{4}(12\text{ kg}) + \frac{1}{2}(5.0\text{ kg})}}$$
$$= 2.1\text{ m s}^{-1}.$$

In the initial instance: The potential energy, U_{init} of the block is mgs and of the disk is zero. The kinetic energy, K_{init} of both block and disk are zero.

In the final instance: The potential energy, U_{final} of both block and disk are zero. The kinetic energy, K_{final} of the block is $mv^2/2$ and of the disk is $I\omega^2/2$.

- wxMaxima codes:

```
(%i7) fpprintprec:5; ratprint:false; M:12; R:0.2; m:5; g:9.8;
s:0.5;
(fpprintprec)     5
(ratprint)  false
(M)   12
(R)   0.2
(m)   5
(g)   9.8
(s)   0.5
(%i8) I: 0.5*M*R^2;
(I)   0.24
(%i10) solve(m*g*s = 0.5*I*(v/R)^2 + 0.5*m*v^2, v)$ float(%);
(%o10) [v=-2.1106,v=2.1106]
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of M, R, m, g, and s.

(%i8) Calculate I.

(%i10) Solve $mgs = 0.5 \times I(v/R)^2 + 0.5 \times mv^2$ for v.

Problem 9.15 Block 1 of mass, $m_1 = 5.0$ kg, is connected to the block 2 of mass, $m_2 = 10$ kg, by a light string through a pulley (Fig. 9.13). The table top has negligible friction, the moment of inertia of the pulley is $I = 1.0$ kg m^2, and its radius is $r = 20$ cm. The system is released from rest. Determine,

(a) the acceleration, a, of the blocks,
(b) the tensions in the string, T_1 and T_2,
(c) the speed, v, of block 1 when it falls by a distance of 40 cm.

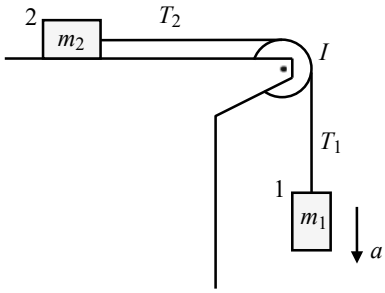


Fig. 9.13 Blocks connected by a string and pulley of Problem 9.15

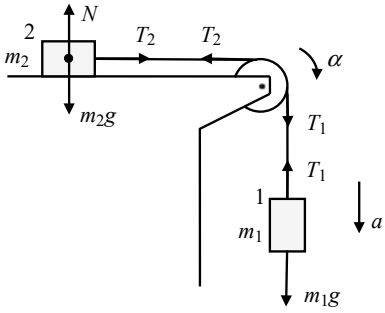


Fig. 9.14 Forces acting on the blocks and pulley, Problem 9.15

Solution

(a) Figure 9.14 shows the forces acting on both blocks and the pulley. For the block 1, the forces acting on it are its weight, m_1g, and the tension, T_1, in the string. The block moves with acceleration of a.

For the block 2, the forces acting on it are the tension in the string, T_2, which causes it to move with acceleration of a. The weight of the block, m_2g, and the normal reaction of the table on the block is N, that are same in magnitudes and opposite in directions, i.e. $N = m_2g$. For the pulley, two forces, namely, T_1 and T_2 act on it.

Applying the Newton's second law to the blocks 1 and 2, we obtain,

$$m_1g - T_1 = m_1a, \tag{9.1}$$

$$T_2 = m_2a. \tag{9.2}$$

The pulley rotates with angular acceleration of α due to net torque $\Sigma\tau = (T_1 - T_2)r$ on it. The angular acceleration is related to linear acceleration by $\alpha = a/R$. We write,

$$\Sigma\tau = I\alpha,$$

$$(T_1 - T_2)r = I\frac{a}{r}. \tag{9.3}$$

Solving Eqs. (9.1), (9.2), and (9.3) one gets the values of a, T_1, and T_2. From Eqs. (9.1) and (9.2), one gets,

$$-T_1 + T_2 = (m_1 + m_2)a - m_1 g.$$

From Eq. (9.3),

$$T_1 - T_2 = I\frac{a}{r^2}.$$

From these two equations, we have,

$$0 = \left(m_1 + m_2 + \frac{I}{r^2}\right)a - m_1 g.$$

The acceleration, a, of the blocks is,

$$a = \frac{m_1 g}{m_1 + m_2 + \frac{I}{r^2}} = \frac{(5.0\text{ kg})(9.8\text{ m/s}^2)}{5.0\text{ kg} + 10\text{ kg} + \frac{1.0\text{ kg m}^2}{(0.20\text{ m})^2}} = 1.2\text{ m s}^{-2}.$$

(b) From Eq. (9.2), the tension in the upper string is obtained as,

$$T_2 = m_2 a = (10\text{ kg})(1.23\text{ m/s}^2) = 12\text{ N}.$$

From Eq. (9.1), the tension in the lower string achieves,

$$T_1 = m_1(g - a) = (5.0\text{ kg})(9.8\text{ m/s}^2 - 1.2\text{ m/s}^2) = 43\text{ N}.$$

(c) The speed, v, of the block 1 when it falls by a distance of 40 cm is calculated as follows,

$$v^2 - u^2 = 2as,\ u = 0 \Rightarrow v^2 = 2as,$$
$$v = \sqrt{2as} = \sqrt{2(1.2\ \text{m/s}^2)(0.40\ \text{m})} = 0.99\ \text{m s}^{-1}.$$

- wxMaxima codes:

```
(%i8) fpprintprec:5; ratprint:false; m1:5; m2:10; I:1; r:20e-
2; g:9.8; s:0.4;
(fpprintprec)     5
(ratprint)  false
(m1)   5
(m2)   10
(I)    1
(r)    0.2
(g)    9.8
(s)    0.4
(%i10)  solve([m1*g-T1=m1*a, T2=m2*a, (T1-T2)*r=I*a/r], [a,
T1, T2])$ float(%);
(%o10)  [[a=1.225,T1=42.875,T2=12.25]]
(%i11)  a: rhs(%[1][1]);
(a)  1.225
(%i13)  solve( v^2=2*a*s, v)$ float(%);
(%o13)  [v=-0.98995,v=0.98995]
```

Comments on the codes:

(%i8) Set floating point print precision to 5, internal rational number print to false, assign values of m_1, m_2, I, r, g, and s.

(%i10) Solve $m_1 g - T_1 = m_1 a$, $T_2 = m_2 a$ and $(T_1 - T_2)r = Ia/r$ for a, T_1, and T_2. Part (a) and (b).

(%i11) Assign value of a.

(%i13) Solve $v^2 = 2as$ for v. Part (c).

- Alternative calculation:

```
(%i2)  fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i5)  eq1: m1*g-T1=m1*a; eq2: T2=m2*a; eq3: (T1-T2)*r=I*a/r;
(eq1)  g*m1-T1=a*m1
(eq2)  T2=a*m2
(eq3)  (T1-T2)*r=(I*a)/r
```

```
(%i6) solve([eq1,eq2,eq3], [a,T1,T2]);
(%o6) [[a=(g*m1*r^2)/(m2*r^2+m1*r^2+I),
       T1=(g*m1*(m2*r^2+I))/(m2*r^2+m1*r^2+I),
       T2=(g*m1*m2*r^2)/(m2*r^2+m1*r^2+I)]]
(%i7) subst([m1=5, m2=10, I=1, r=0.2, g=9.8], %);
(%o7) [[a=1.225,T1=42.875,T2=12.25]]
(%i8) a: rhs(%[1][1]);
(a)   1.225
(%i9) s:0.4;
(s)   0.4
(%i11) solve(v^2=2*a*s, v)$ float(%);
(%o11) [v=-0.98995,v=0.98995]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i5) Assign Eqs. (9.1), (9.2), and (9.3) as eq1, eq2, and eq3, respectively.

(%i6) Solve Eqs. (9.1), (9.2), and (9.3) in symbols for a, T_1, and T_2.

(%i7) Substitute values of m_1, m_2, I, r, and g to get numerical values of a, T_1, and T_2. Part (a) and (b).

(%i8), (%i9) Assign values of a and s.

(%i11) Solve $v^2 = 2as$ for v. Part (c).

Alternative solution of part (c): We apply the conservation of energy to the initial and final instances (Eq. 7.10). The potential energy of the block 1 is transferred to rotational kinetic energy of the pulley and kinetic energies of the blocks 1 and 2.

$$\begin{aligned}
U_{init} + K_{init} &= U_{final} + K_{final}, \\
m_1 g s &= \frac{1}{2} I \omega^2 + \frac{1}{2}(m_1 + m_2) v^2, \\
m_1 g s &= \frac{1}{2} I \left(\frac{v}{r}\right)^2 + \frac{1}{2}(m_1 + m_2) v^2, \\
(5.0\ \text{kg})(9.8\ \text{N/kg})(0.40\ \text{m}) &= \frac{1}{2}(1.0\ \text{kg m}^2)\left(\frac{v}{0.20\ \text{m}}\right)^2 + \frac{1}{2}(5.0\ \text{kg} + 10\ \text{kg}) v^2 \\
v &= 0.99\ \text{m s}^{-1}.
\end{aligned}$$

In the initial instance: The potential energy, U_{init} of the pulley is zero, that of block 1 is m_1gs, and that of block 2 is zero. The kinetic energy, K_{init} of the pulley and both blocks are zero.

In the final instance: The potential energy, U_{final} of the pulley and both blocks are zero. The kinetic energy, K_{final} of the pulley is $I\omega^2/2$, that of the blocks is $(m_1 + m_2)v^2/2$.

- wxMaxima codes:

```
(%i8) fpprintprec:5; ratprint:false; m1:5; m2:10; I:1; r:20e-
2; g:9.8; s:0.4;
(fpprintprec)     5
(ratprint)  false
(m1)   5
(m2)   10
(I)    1
(r)    0.2
(g)    9.8
(s)    0.4
(%i10) solve(m1*g*s = 0.5*I*(v/r)^2 + 0.5*(m1+m2)*v^2, v)$
float(%);
(%o10) [v=-0.98995,v=0.98995]
```

Comments on the codes:

(%i8) Set floating point print precision to 5, internal rational number print to false, assign values of m_1, m_2, I, r, g, and s.

(%i10) Solve $m_1gs = \frac{1}{2}I(\frac{v}{r})^2 + \frac{1}{2}(m_1 + m_2)v^2$ for v.

Problem 9.16 A solid cylinder of mass, $M = 2.0$ kg, and radius, $R = 10$ cm, is axled horizontally about its symmetry axis so that it is free to rotate with negligible friction (Fig. 9.15). Two blocks each of mass, $m = 3.0$ kg, are tied to strings wound around the cylinder. The system is released from rest. Determine the tension in the strings and the acceleration of the blocks.

Solution

The forces acting on the blocks and the cylinder are shown in Fig. 9.16. The forces acting on each block are its weight, mg, and the tension, T, in the string. The net downward force is $mg - T$ and this force causes the block to move down with the

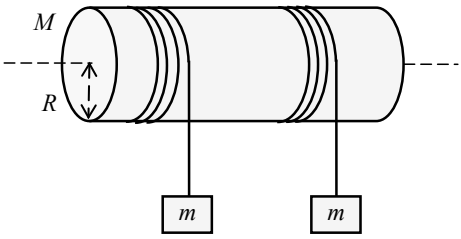


Fig. 9.15 Cylinder and two blocks of Problem 9.16

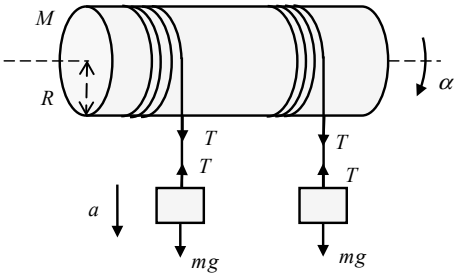


Fig. 9.16 Forces acting on the blocks and cylinder, Problem 9.16

acceleration of a. The net torque on the cylinder is $2RT$ that causes the cylinder to rotate with angular acceleration of $\alpha = a/R$.

Applying the Newton's second law to a block, we have,

$$\Sigma F = ma,$$

$$mg - T = ma. \tag{9.1}$$

The cylinder rotates with the angular acceleration of α due to the net torque,

$$\Sigma \tau = I\alpha,$$
$$2RT = \left(\frac{1}{2}MR^2\right)\frac{a}{R},$$
$$T = \frac{1}{4}Ma. \tag{9.2}$$

Equations (9.1) and (9.2) are solved for a and T. Adding Eqs. (9.1) and (9.2) one obtains,

$$mg = a\left(m + \frac{M}{4}\right).$$

This means that the acceleration, a, of the blocks is,

$$a = \frac{mg}{m + \frac{M}{4}} = \frac{4mg}{4m + M} = \frac{4(3.0\text{ kg})(9.8\text{ m/s}^2)}{4(3.0\text{ kg}) + 2.0\text{ kg}} = 8.4\text{ m s}^{-2}.$$

From Eq. (9.2), the tension, T, in the string is,

$$T = \frac{1}{4}Ma = \frac{Mmg}{4m + M} = \frac{(2.0\text{ kg})(3.0\text{ kg})(9.8\text{ m/s}^2)}{4(3.0\text{ kg}) + 2.0\text{ kg}} = 4.2\text{ N}.$$

- wxMaxima codes:

```
(%i6)   fpprintprec:5; ratprint:false; M:2; m:3; R:10e-2;
g:9.8;
(fpprintprec)       5
(ratprint)  false
(M)   2
(m)   3
(R)   0.1
(g)   9.8
(%i8) solve([m*g-T=m*a, T=1/4*M*a], [a,T])$ float(%);
(%o8) [[a=8.4,T=4.2]]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of M, m, R, and g.

(%i8) Solve $mg - T = ma$ and $T = \frac{1}{4}Ma$ for a and T.

- Alternative calculation:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)      5
(ratprint)  false
(%i3) solve([m*g-T=m*a, T=1/4*M*a], [a,T]);
(%o3) [[a=(4*g*m)/(4*m+M),T=(M*g*m)/(4*m+M)]]
(%i4) subst([M=2, m=3, R=10e-2, g=9.8], %);
(%o4) [[a=8.4,T=4.2]]
```

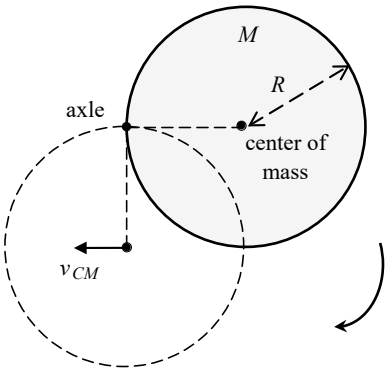


Fig. 9.17 A disk free to rotate about an axle, Problem 9.17

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i3) Solve $mg - T = ma$ and $T = \frac{1}{4}Ma$ in symbols for a and T.

(%o3) Solutions in symbols.

(%i4) Substitute values of M, m, R, and g into the solutions to get numerical values of a and T.

Problem 9.17

(a) A disk of radius, $R = 10$ cm, and mass, $M = 0.50$ kg, is free to rotate about an axle on its rim (Fig. 9.17). The disk is released from rest as shown. What is the speed, v_{CM}, of the center of mass of the disk at its lowest position?
(b) Repeat the calculation for a ring of radius, $R = 10$ cm, mass, $M = 0.50$ kg.

Solution

(a) Figure 9.18 shows the disk in initial and final instances. In the figure P is the axle, CM is the center of mass, v_{CM} and ω are the linear speed and angular speed of the center of mass at its lowest point, respectively. The reference line is the horizontal line through the lowest point of the center of mass.

Applying the conservation of energy to the initial and final instances (Eq. 7.10), we write,

$$U_{init} + K_{init} = U_{final} + K_{final},$$
$$MgR + 0 = 0 + \frac{1}{2} I_P \omega^2,$$
$$MgR = \frac{1}{2} I_P \omega^2, \tag{9.1}$$

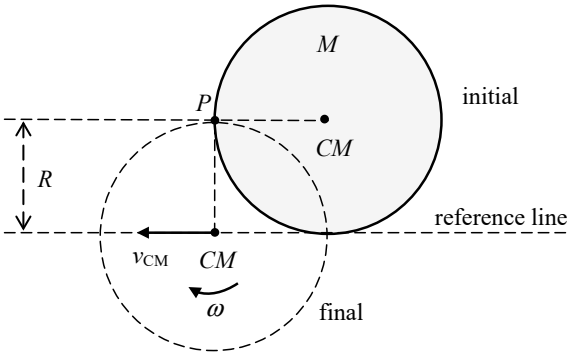


Fig. 9.18 The disk at its initial and final instances, Problem 9.17

where MgR is the gravitational potential energy of the disk in initial instance and $I_P\omega^2/2$ is the rotational kinetic energy of the disk in the final instance. The quantity, I_P, is the moment of inertia of the disk about axle P and this can be calculated by the parallel axes theorem (Eq. 9.12) as,

$$I_P = I_{CM,disk} + MR^2 = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2. \tag{9.2}$$

Substituting Eq. (9.2) in Eq. (9.1) one obtains,

$$MgR = \frac{1}{2}\left(\frac{3}{2}MR^2\right)\left(\frac{v_{CM}}{R}\right)^2,$$

because,

$$v_{CM} = \omega R. \tag{9.3}$$

Solving the equation for v_{CM}, the speed of the center of mass of the disk at its lowest position is given by,

$$v_{CM} = 2\sqrt{\frac{gR}{3}} = 2\sqrt{\frac{(9.8\ \text{m/s}^2)(0.10\ \text{m})}{3}} = 1.1\ \text{m s}^{-1}.$$

- wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; R:10e-2; M:0.5; g:9.8;
(fpprintprec)      5
(ratprint)  false
(R)    0.1
(M)    0.5
```

```
(g)   9.8
(%i6) IP: (3/2)*M*R^2;
(IP)  0.0075
(%i8) solve(M*g*R=1/2*IP*(vCM/R)^2, vCM)$ float(%);
(%o8) [vCM=-1.1431,vCM=1.1431]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of R, M, and g.

(%i6) Calculate I_P.

(%i8) Solve $MgR = \frac{1}{2} I_P \left(\frac{v_{CM}}{R}\right)^2$ for v_{CM}.

(b) For a ring of radius, $R = 10$ cm, mass, $M = 0.50$ kg, the moment of inertia about the axle P is,

$$I_P = I_{CM,ring} + MR^2 = MR^2 + MR^2 = 2MR^2, \tag{9.4}$$

and this is different from the one for a disk in part (a). Substitution of Eq. (9.4) into Eq. (9.1) one obtains,

$$MgR = \frac{1}{2}\left(2MR^2\right)\left(\frac{v_{CM}}{R}\right)^2.$$

Solving the equation for v_{CM}, the speed of the center of mass of the ring at its lowest position is,

$$v_{CM} = \sqrt{gR} = \sqrt{(9.8\ \text{m/s}^2)(0.10\ \text{m})} = 0.99\ \text{m s}^{-1}.$$

- wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; R: 10e-2; M: 0.5; g: 9.8;
(fpprintprec)     5
(ratprint)  false
(R)   0.1
(M)   0.5
(g)   9.8
(%i6) IP: 2*M*R^2;
(IP)  0.01
(%i8) solve(M*g*R=1/2*IP*(vCM/R)^2, vCM)$ float(%);
(%o8) [vCM=-0.98995,vCM=0.98995]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of R, M, and g.

(%i6) Calculate I_P.

(%i8) Solve $MgR = \frac{1}{2} I_P \left(\frac{v_{CM}}{R}\right)^2$ for v_{CM}.

- Alternative calculation:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)      5
(ratprint)  false
(%i5) eq1: M*g*R=(1/2)*IP*omega^2; eq2: IP=(3/2)*M*R^2; eq3:
vCM=omega*R;
(eq1) M*R*g=(IP*omega^2)/2
(eq2) IP=(3*M*R^2)/2
(eq3) vCM=R*omega
(%i6) solve([eq1,eq2,eq3], [vCM,omega,IP]);
(%o6) [[vCM=(2*sqrt(R*g))/sqrt(3),omega=(2*sqrt(R*g))/
(sqrt(3)*R),
        IP=(3*M*R^2)/2],
       [vCM=-(2*sqrt(R*g))/sqrt(3),omega=-(2*sqrt(R*g))/
(sqrt(3)*R),
        IP=(3*M*R^2)/2]]
(%i8) subst([R=0.1, M=0.5, g=9.8], %)$ float(%);
(%o8) [[vCM=1.1431,omega=11.431,IP=0.0075],
       [vCM=-1.1431,omega=-11.431,IP=0.0075]]
(%i9) eq4: IP=2*M*R^2;
(eq4) IP=2*M*R^2
(%i10) solve([eq1,eq4,eq3], [vCM,omega,IP]);
(%o10) [[vCM=sqrt(R*g),omega=sqrt(R*g)/R,IP=2*M*R^2],
       [vCM=-sqrt(R*g),omega=-sqrt(R*g)/R,IP=2*M*R^2]]
(%i12) subst([R=0.1, M=0.5, g=9.8], %)$ float(%);
(%o12) [[vCM=0.98995,omega=9.8995,IP=0.01],
       [vCM=-0.98995,omega=-9.8995,IP=0.01]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i5) Assign Eqs. (9.1), (9.2), and (9.3) as eq1, eq2, and eq3, respectively.

(%i6) Solve Eqs. (9.1), (9.2), and (9.3) in symbols for v_{CM}, ω, and I_P.

(%i8) Substitute values of R, M, and G to get numerical values of v_{CM}, ω, and I_P. Part (a).

(%i9) Assign Eq. (9.4) as eq4.

(%i10) Solve Eqs. (9.1), (9.4), and (9.3) in symbols for v_{CM}, ω, and I_P.

(%i12) Substitute values of R, M, and G to get numerical values of v_{CM}, ω, and I_P. Part (b).

Problem 9.18 A marble of radius, $r = 1.0$ cm, is placed in a hemispherical bowl of radius, $R = 10$ cm. The marble is released from rest at angle, $\theta = 45°$, from the vertical and it rolls down without slipping (Fig. 9.19). What are the angular speed, ω, and linear speed, v_{CM}, of the marble when it reaches the bottom of the bowl?

Solution

Figure 9.20 shows the marble in the initial and final instances. The mass of the marble is m and the reference is the horizontal line through the center of mass of the marble at its lowest point.

In the initial instance, the distance of center of mass to the reference line is,

$$(R - r) - (R - r)\cos q = (R - r)(1 - \cos\theta).$$

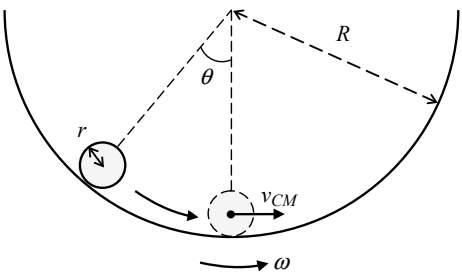


Fig. 9.19 A marble in a hemispherical bowl, Problem 9.8

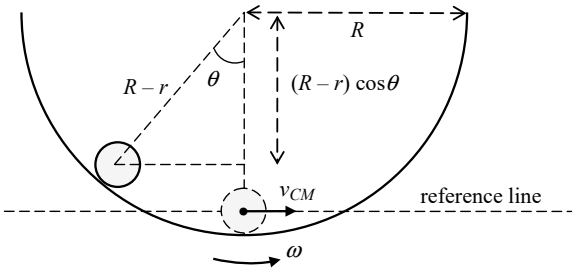


Fig. 9.20 The marble in the initial and final instances, Problem 9.8

The gravitational potential energy of the marble relative to the reference line is,

$$U_{init} = mg(R - r)(1 - \cos\theta).$$

In the final instance, the marble is rolling. The angular speed is ω, the linear speed is v_{CM}, and the kinetic energy of the marble is,

$$K_{final} = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2.$$

The first term in the right side of this equation is the translational kinetic energy and the second term is the rotational kinetic energy of the marble.

Applying the conservation of energy, that is, the total energy in the initial and final instances are the same (Eq. 7.10), we have,

$$\begin{aligned}
U_{init} + K_{init} &= U_{final} + K_{final},\\
mg(R - r)(1 - \cos\theta) + 0 &= 0 + \frac{1}{2}mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2,\\
mg(R - r)(1 - \cos\theta) &= \frac{1}{2}mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2 = \frac{1}{2}mr^2\omega^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\omega^2\\
&= \frac{7}{10}mr^2\omega^2\\
&= \frac{7}{10}mv_{CM}^2,
\end{aligned}$$

where the fact that the moment of inertia of the sphere $I_{CM} = 2mr^2/5$ has been used. The angular speed, ω, and linear speed, v_{CM}, of the marble when it reaches the bottom of the bowl are,

$$\begin{aligned}
\omega &= \frac{1}{r}\sqrt{\frac{10}{7}g(R - r)(1 - \cos\theta)}\\
&= \frac{1}{0.010\text{ m}}\left(\frac{10}{7}(9.8\text{ m/s}^2)(0.10\text{ m} - 0.010\text{ m})(1 - \cos 45^\circ)\right)^{1/2}\\
&= 61\text{ rad s}^{-1}.
\end{aligned}$$

$$\begin{aligned}
v_{CM} &= \sqrt{\frac{10}{7}g(R - r)(1 - \cos\theta)}\\
&= \left(\frac{10}{7}(9.8\text{ m/s}^2)(0.10\text{ m} - 0.010\text{ m})(1 - \cos 45^\circ)\right)^{1/2}
\end{aligned}$$

$= 0.61 \text{ m s}^{-1}$.

- wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)      5
(ratprint)  false
(%i3)  solve([m*g*(R-r)*(1-cos(theta))=7/10*m*r^2*omega^2,
vCM=omega*r], [omega, vCM]);
(%o3)          [[omega=-(sqrt(10)*sqrt(R*g-g*r)*sqrt(1-cos
(theta)))/(sqrt(7)*r),
         vCM=-(sqrt(10)*sqrt(R*g-g*r)*sqrt(1-cos(theta)))/
sqrt(7)],
       [omega=(sqrt(10)*sqrt(R*g-g*r)*sqrt(1-cos(theta)))/
(sqrt(7)*r),
          vCM=(sqrt(10)*sqrt(R*g-g*r)*sqrt(1-cos(theta)))/
sqrt(7)]]
(%i5) subst([R=0.1, r=0.01, theta=45/180*float(%pi), g=9.8],
%)$ float(%);
(%o5)     [[omega=-60.749,  vCM=-0.60749],  [omega=60.749,
vCM=0.60749]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i3) Solve $mg(R-r)(1-\cos\theta) = \frac{7}{10}mr^2\omega^2$ and $v_{CM} = \omega r$ in symbols for ω and v_{CM}.

(%o3) The solutions in symbols.

(%i5) Substitute values of R, r, θ, and g into the solutions to get numerical values of ω and v_{CM}.

Problem 9.19 A uniform rod of length, $l = 2.0$ m, and mass, $m = 2.0$ kg, is pivoted about a horizontal axis with the negligible friction at its bottom end P (Fig. 9.21). The rod is released from rest from vertical position as shown in the figure. The rod then falls down and swings about P to positions 1 and 2.

(a) At position 1, determine the angular speed, angular acceleration, and x and y components of acceleration of the center of mass. Determine also the horizontal and vertical components of the force of the pivot on the rod.
(b) Calculate all the quantities in part (a) when the rod is at position 2.

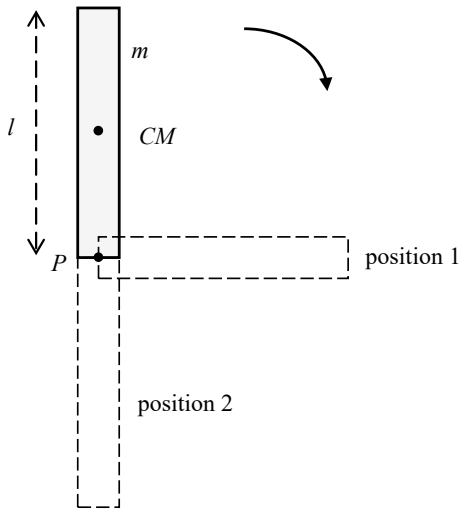


Fig. 9.21 A rod pivoted at P released from rest to positions 1 and 2, Problem 9.19

Solution

(a) Apply the conservation of energy (Eq. 7.10). The reference is a horizontal line through the pivot P (Fig. 9.22a). Initially, the gravitational potential energy of the rod is $mgl/2$, as the distance of the center of mass to the reference line is $l/2$.

The kinetic energy of the rod at the final instance (at position 1) is $I_P\omega^2/2$, where $I_P = ml^2/3$ is the moment of inertia of the rod about pivot P and ω is the angular speed of the rod. The moment of inertia of the rod about a perpendicular axis at one of its end is $I_P = ml^2/3$ because,

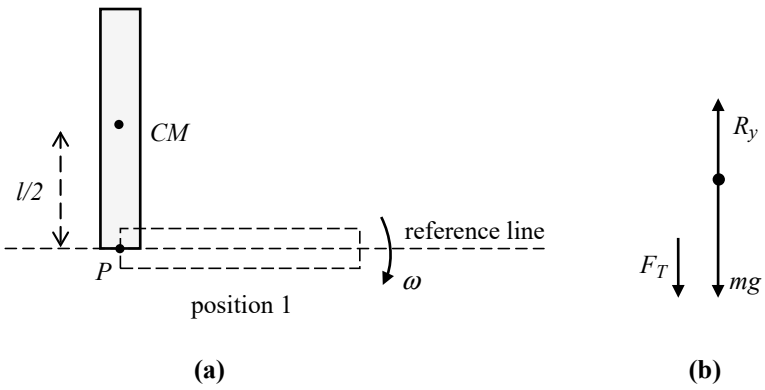


Fig. 9.22 **a** The rod at initial position and position 1, **b** at position 1, tangential force is weight of the rod minus vertical reaction of the pivot on the rod

$$I_P = I_{CM} + mh^2 = \frac{1}{12}ml^2 + m\left(\frac{l}{2}\right)^2 = \frac{1}{3}ml^2,$$

where the parallel axes theorem has been applied and the moment of inertia of a rod about a perpendicular axis at the rod center $I_{CM} = ml^2/12$ has been used (Fig. 9.1a, j).

The conservation of energy at the initial and final instances gives (Eq. 7.10),

$$\begin{aligned} U_{init} + K_{init} &= U_{final} + K_{final}, \\ mg\frac{l}{2} + 0 &= 0 + \frac{1}{2}I_P\omega^2, \\ mg\frac{l}{2} &= \frac{1}{2}I_P\omega^2 = \frac{1}{2}\left(\frac{1}{3}ml^2\right)\omega^2. \end{aligned}$$

The angular speed of the center of mass of the rod at position 1 is given by,

$$\omega = \sqrt{\frac{3g}{l}} = \sqrt{\frac{3(9.8\ \text{m/s}^2)}{2.0\ \text{m}}} = 3.8\ \text{rad s}^{-1}.$$

At position 1, the torque about point P is $mgl/2$, thus,

$$\begin{aligned} \Sigma\tau &= I\alpha, \\ mg\frac{l}{2} &= I_P\alpha. \end{aligned}$$

From this equation, the angular acceleration of the center of mass is,

$$\alpha = \frac{mg\frac{l}{2}}{I_P} = \frac{mg\frac{l}{2}}{\frac{1}{3}ml^2} = \frac{3g}{2l} = \frac{3(9.8\ \text{m/s}^2)}{2(2.0\ \text{m})} = 7.4\ \text{rad s}^{-2}.$$

The x component of the acceleration of the CM is its centripetal acceleration (Eq. 9.8) given by,

$$a_r = \omega^2 r = \frac{3g}{l}\frac{l}{2} = \frac{3g}{2} = \frac{3(9.8\,\text{m/s}^2)}{2} = 15\,\text{m s}^{-2},$$

directed to the negative x direction. The y component of acceleration of the CM is its tangential acceleration (Eq. 9.7),

$$a_T = \alpha r = \frac{3g}{2l}\frac{l}{2} = \frac{3g}{4} = \frac{3(9.8\ \text{m/s}^2)}{4} = 7.3\ \text{m s}^{-2},$$

directed to the negative y direction.

The horizontal component of the force on the rod is the centripetal force on the rod,

$$F_r = ma_r = \frac{3gm}{2} = \frac{3(9.8\ \text{m/s}^2)(2.0\ \text{kg})}{2} = 29\ \text{N},$$

directed to the negative x direction. This means that the horizontal component of the force on the rod by the pivot, R_x, is opposite in direction and the same magnitude as F_r i.e. $R_x = 29$ N in the positive x direction and R_x is also called the horizontal component of the reaction of the pivot on the rod,

$$R_x = 29\,\text{N}.$$

The vertical component of the force on the rod is the tangential force given by,

$$F_T = ma_T = \frac{3gm}{4},$$

directed to the negative y direction. From Fig. 9.22b, the vertical component of force on the rod by the pivot, R_y, is calculated as follows,

$$\begin{aligned} F_T &= mg - R_y, \\ R_y &= mg - F_T = mg - \frac{3gm}{4} = \frac{1}{4}mg = \frac{1}{4}(2.0\ \text{kg})(9.8\ \text{m/s}^2) \\ &= 4.9\ \text{N}. \end{aligned}$$

The direction of R_y is to the positive y direction and R_y is also called the vertical component of the reaction of the pivot on the rod.

- wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; l:2; m:2; g:9.8;
(fpprintprec)      5
(ratprint)  false
(l)   2
(m)   2
(g)   9.8
(%i7) IP: 1/3*m*l^2$ float(%);
(%o7) 2.6667
(%i9) solve(m*g*l/2=1/2*IP*omega^2, omega)$ float(%);
(%o9) [omega=-3.8341,omega=3.8341]
(%i10) omega: rhs(%[2]);
(omega) 3.8341
(%i12) solve(m*g*l/2=IP*alpha, alpha)$ float(%);
```

```
(%o12) [alpha=7.35]
(%i13) alpha: rhs(%[1]);
(alpha) 7.35
(%i14) ar: omega^2*l/2;
(ar) 14.7
(%i15) aT: alpha*l/2;
(aT) 7.35
(%i16) Fr: m*ar;
(Fr) 29.4
(%i17) Rx: Fr;
(Rx) 29.4
(%i18) FT: m*aT;
(Ft) 14.7
(%i19) Ry: m*g-FT;
(Ry) 4.9
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of l, m, and g.

(%i7) Calculate I_P.

(%i9) Solve $mgl/2 = 1/2 \times I_P\omega^2$ for ω.

(%i10) Assign value of ω.

(%i12) Solve $mgl/2 = I_P\alpha$ for α.

(%i13) Assign value of α.

(%i14), (%i15) Calculate a_r and a_T.

(%i16) Calculate F_r or R_x.

(%i18), (%i19) Calculate F_T and R_y.

(b) Figure 9.23a shows the rod in the initial and final (at position 2) instances and the reference line. Initially, the gravitational potential energy of the rod is mgl, as the distance of the center of mass of the rod to the reference line is l. In the final instance, the rotational kinetic energy of the rod is $I_P\omega^2/2$, where $I_P = ml^2/3$ is the moment of inertia of the rod about the pivot P and ω is the angular speed of the rod at position 2.

The conservation of energy implies (Eq. 7.10),

$$U_{init} + K_{init} = U_{final} + K_{final},$$
$$mgl + 0 = 0 + \frac{1}{2} I_P\omega^2,$$

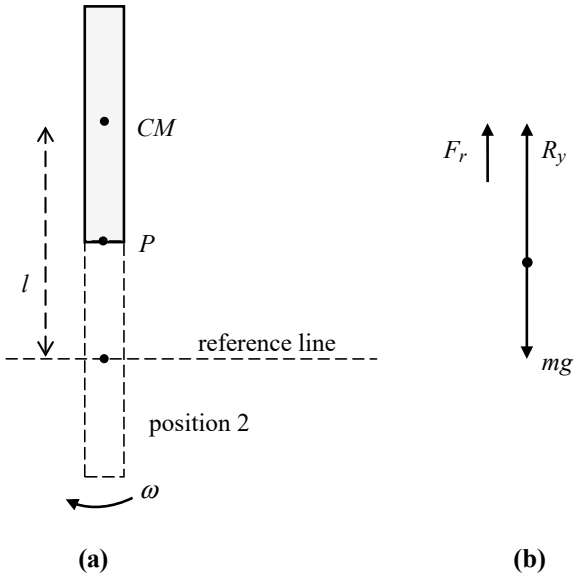


Fig. 9.23 **a** The rod at initial position and position 2, **b** at position 2, centripetal force is vertical reaction of pivot on the rod minus weight of the rod

$$mgl = \frac{1}{2} I_P \omega^2 = \frac{1}{2}\left(\frac{1}{3} ml^2\right)\omega^2.$$

The angular speed of the rod at position 2 is,

$$\omega = \sqrt{\frac{6g}{l}} = \sqrt{\frac{6(9.8\ \text{m/s}^2)}{2.0\ \text{m}}} = 5.4\ \text{rad s}^{-1}.$$

At position 2, the torque about P is zero and the angular acceleration is,

$$\alpha = \frac{\Sigma\tau}{I_P} = 0.$$

The x component of acceleration of the CM, that is, the tangential acceleration is zero,

$$a_T = \alpha r = \alpha(l/2) = 0.$$

The y component of acceleration of the CM, that is, the centripetal acceleration is,

$$a_r = \omega^2 r = \frac{6g}{l}\frac{l}{2} = 3g = 3(9.8\ \text{m/s}^2) = 29\ \text{m s}^{-2},$$

directed to the positive y direction.

The horizontal component of force on the rod by the pivot is zero,

$$R_x = 0.$$

The vertical component of force on the rod is the centripetal force,

$$F_r = ma_r = 3mg,$$

directed to the positive y direction. From Fig. 9.23b, one gets,

$$\begin{aligned} F_r &= R_y - mg, \\ R_y &= F_r + mg = 3mg + mg = 4mg = 4(2.0\text{ kg})(9.8\text{ m/s}^2) \\ &= 78\text{ N}, \end{aligned}$$

where R_y is the vertical component of the force on the rod by the pivot, that is, the vertical component of the reaction of the pivot on the rod. The direction of R_y is to the positive y direction (Fig. 9.23b).

- wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; l:2; m:2; g:9.8;
(fpprintprec)      5
(ratprint)  false
(l)   2
(m)   2
(g)   9.8
(%i7) IP: 1/3*m*l^2$ float(%);
(%o7) 2.6667
(%i9) solve(m*g*l=1/2*IP*omega^2, omega)$ float(%);
(%o9) [omega=-5.4222,omega=5.4222]
(%i10) omega: rhs(%[2]);
(omega) 5.4222
(%i11) ar: omega^2*l/2;
(ar)  29.4
(%i12) Fr: m*ar;
(Fr)  58.8
(%i13) Ry: Fr+m*g;
(Ry)  78.4
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of l, m, and g.

(%i7) Calculate I_P.

(%i9) Solve $mgl/2 = 1/2 \times I_P\omega^2$ for ω.

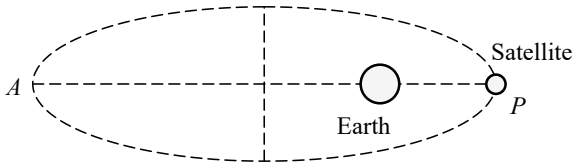


Fig. 9.24 A satellite orbits the Earth in an elliptical path, Problem 9.20

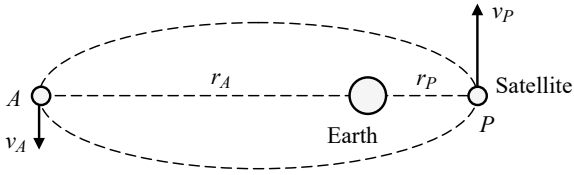


Fig. 9.25 The satellite at *perigee P* and *apogee A*, Problem 9.20

(%i10) Assign value of ω.

(%i11), (%i12), (%i13) Calculate a_r, F_r, and R_y.

Problem 9.20 A satellite orbits the earth in an elliptical path (Fig. 9.24). The speed of the satellite at the *perigee*, *P*, is 5.0 km s^{-1}. If the *apogee* distance is nine times the *perigee* distance, what is the speed of the satellite at the *apogee*, *A*? *Perigee* is the closest point of the satellite to the earth in the elliptical orbit while *apogee* is the furthest one.

Solution

Figure 9.25 shows the positions of the satellite in its elliptical orbit around the earth.

We apply the conservation of angular momentum in this problem (Eq. 9.25). There is no external torque. Let the mass of the satellite be *m*. The angular momentum of the satellite at *P* is,

$$L_P = r_P m v_P,$$

and at *A* is,

$$L_A = r_A m v_A.$$

Using the conservation of angular momentum (Eq. 9.25), the speed, v_A, of the satellite is obtained as follows,

$$\begin{aligned} L_P &= L_A, \\ r_P m v_P &= r_A m v_A, \\ r_P v_P &= r_A v_A, \\ r_P(5.0 \text{ km/s}) &= (9r_P)v_A, \\ v_A &= 0.56 \text{ km s}^{-1}. \end{aligned}$$

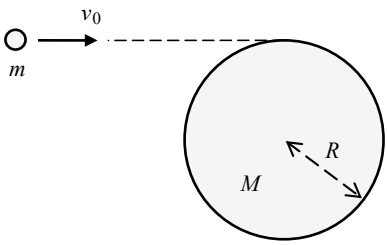


Fig. 9.26 A particle hits and sticks to a sphere, Problem 9.21

- wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec)      5
(%i3) solve(rP*5=9*rP*vA, vA)$ float(%);
(%o3) [vA=0.55556]
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i3) Solve $r_P \times 5 = 9 \times r_P v_A$ for v_A.

Problem 9.21 A particle of mass, $m = 10$ g, moving at the speed of $v_0 = 5.0$ m s^{-1} hits and sticks to an edge of a sphere of mass, $M = 1.0$ kg, and radius, $R = 20$ cm, as in Fig. 9.26.

(a) If the sphere is initially at rest, then calculate the angular speed of the system after the hit.
(b) Calculate the loss of energy in the hit.

Solution

(a) We apply the conservation of angular momentum to the problem (Eq. 9.25). The magnitude of the angular momentum before the hit is,

$$| \boldsymbol{r} \times \boldsymbol{p} | = Rmv_0.$$

Let the angular speed after the hit be ω. The angular momentum after the hit is,

$$\begin{aligned} I_{particle}\omega + I_{sphere}\omega &= mR^2\omega + \frac{2}{5}MR^2\omega, \\ &= \left(m + \frac{2}{5}M\right)R^2\omega. \end{aligned}$$

Applying the conservation of angular momentum (Eq. 9.25), before and after the hit gives,

$$Rmv_0 = \left(m + \frac{2}{5}M\right)R^2\omega,$$

and the angular speed of the system after the hit is,

$$\omega = \frac{mv_0}{\left(m + \frac{2}{5}M\right)R} = \frac{(0.010\text{ kg})(5.0\text{ m/s})}{\left(0.010\text{ kg} + \frac{2}{5}(1.0\text{ kg})\right)(0.20\text{ m})} = 0.61\text{ rad s}^{-1}.$$

(b) The initial kinetic energy of the particle is,

$$K_{init} = \frac{1}{2}mv_0^2 = \frac{1}{2}(0.010\text{ kg})(5.0\text{ m/s})^2 = 0.125\text{ J}.$$

The final kinetic energy is the rotational kinetic energies of the particle and the sphere, that is,

$$\begin{aligned} K_{final} &= \frac{1}{2}(I_{particle} + I_{sphere})\omega^2 = \frac{1}{2}\left(mR^2 + \frac{2}{5}MR^2\right)\omega^2 \\ &= \frac{1}{2}\left((0.010\text{ kg})(0.20\text{ m})^2 + \frac{2}{5}(1.0\text{ kg})(0.20\text{ m})^2\right)(0.61\text{ rad/s})^2 \\ &= 0.0030\text{ J}. \end{aligned}$$

Thus, the loss in kinetic energy is given by,

$$\begin{aligned} K_{loss} &= K_{final} - K_{init} \\ &= -0.122\text{ J}. \end{aligned}$$

- wxMaxima codes:

```
(%i6)  fpprintprec:5; ratprint:false; m:10e-3; v0:5; M:1;
R:20e-2;
(fpprintprec)       5
(ratprint)  false
(m)   0.01
(v0)  5
(M)   1
(R)   0.2
(%i8) solve(R*m*v0=(m+2/5*M)*R^2*omega, omega)$ float(%);
(%o8) [omega=0.60976]
(%i9) omega: rhs(%[1]);
```

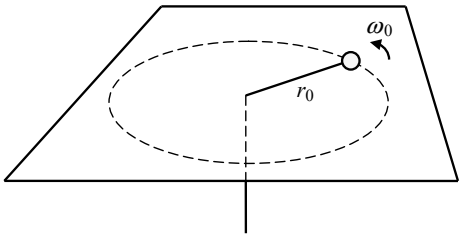


Fig. 9.27 An object rotating on a horizontal surface. Friction between the object and the surface is negligible, Problem 9.22

```
(omega) 0.60976
(%i10) Kinit: 1/2*m*v0^2;
(Kinit) 0.125
(%i11) Kfinal: 1/2*omega^2*(m*R^2+2/5*M*R^2);
(Kfinal) 0.0030488
(%i12) Kloss: Kfinal-Kinit;
(Kloss) -0.12195
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m, v_0, M, and R.

(%i8) Solve $Rmv_0 = \left(m + \frac{2}{5}M\right)R^2\omega$ for ω.

(%i9) Assign value of ω.

(%i10), (%i11), (%i12) Calculate K_{init}, K_{final}, and K_{loss}.

Problem 9.22 An object of mass $m = 50$ g is tied to a string through a hole and rotates on a horizontal surface with negligible friction (Fig. 9.27). Initially, the object rotates at angular speed of $\omega_0 = 5.0$ rad s^{-1} in a circle of radius $r_0 = 30$ cm. The string is then pulled so that the radius of the circle becomes $r_1 = 10$ cm.

(a) What is the final angular speed of the object?
(b) Calculate the work done to change the radius of rotation from r_0 to r_1.

Solution

(a) Applying the conservation of angular momentum to the initial and final instances (Eq. 9.25), we have,

$$\begin{aligned} L_0 &= L_1, \\ I_0\omega_0 &= I_1\omega_1, \\ mr_0^2\omega_0 &= mr_1^2\omega_1. \end{aligned}$$

The final angular speed, ω_1, of the object is,

$$\omega_1 = \frac{r_0^2}{r_1^2}\omega_0 = \frac{(30\text{ cm})^2}{(10\text{ cm})^2}(5.0\text{ rad/s}) = 45\text{ rad s}^{-1}.$$

(b) The work done to change the radius of rotation from r_0 to r is,

$$\begin{aligned}\text{work} &= \text{final kinetic energy} - \text{initial kinetic energy}\\ &= \frac{1}{2}I_1\omega_1^2 - \frac{1}{2}I_0\omega_0^2 = \frac{1}{2}mr_1^2\omega_1^2 - \frac{1}{2}mr_0^2\omega_0^2\\ &= \frac{1}{2}(0.050\text{ kg})(0.10\text{ m})^2(45\text{ rad/s})^2\\ &\quad - \frac{1}{2}(0.050\text{ kg})(0.30\text{ m})^2(5.0\text{ rad/s})^2\\ &= 0.45\text{ J}.\end{aligned}$$

- wxMaxima codes:

```
(%i6)   fpprintprec:5; ratprint:false; m:50e-3; omega0:5;
r0:0.3; r1:0.1;
(fpprintprec)       5
(ratprint)  false
(m)   0.05
(omega0)    5
(r0)  0.3
(r1)  0.1
(%i8)  I0: m*r0^2; I1: m*r1^2;
(I0)  0.0045
(I1)  5.0*10^-4
(%i10)  solve(r0^2*omega0 = r1^2*omega1, omega1)$ float(%);
(%o10)  [omega1=45.0]
(%i11)  omega1: rhs(%[1]);
(omega1)  45.0
(%i12)  Work: 1/2*I1*omega1^2 - 1/2*I0*omega0^2;
(Work)  0.45
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m, ω_0, r_0, and r_1.

(%i8) Calculate I_0 and I_1.

(%i10) Solve $r_0^2\omega_0 = r_1^2\omega_1$ for ω_1.

(%i11) Assign value of ω_1.

(%i12) Calculate the work.

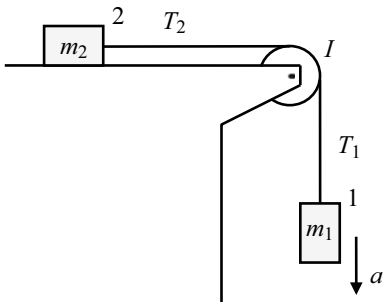


Fig. 9.28 A pulley and two blocks connected by a string of Problem 9.23

Problem 9.23 Block 1 of mass $m_1 = 5.0$ kg is connected to the block 2 of mass $m_2 = 10$ kg by a light string through a pulley (Fig. 9.28). The coefficient of kinetic friction between the table surface and block 2 is $\mu_k = 0.15$. The moment of inertia of the pulley is $I = 2.0$ kg m^2 and the radius of the pulley is $r = 20$ cm. The system is released form rest. Determine,

(a) the acceleration, a, of the blocks,
(b) the tensions in the string, T_1 and T_2,
(c) the speed of block 1 when it falls a distance of 40 cm.

Solution

(a) Figure 9.29 shows the forces acting on both blocks. For the block 1, the forces are its weight, $m_1 g$, and the tension in the string, T_1. The block moves downward with an acceleration of a.

For the block 2, in the x direction, the forces are the tension in the string, T_2, to the right and the friction, $\mu_k m_2 g$, to the left. The net force on the block 2 is $T_2 - \mu_k m_2 g$ to the right. This force causes the block 2 to move with acceleration, a. The weight of the block, $m_2 g$, and the normal reaction of the table on the block, N, are same in magnitude and opposite in directions, that is, $N = m_2 g$. For the pulley, two forces namely, T_1 and T_2, act on it.

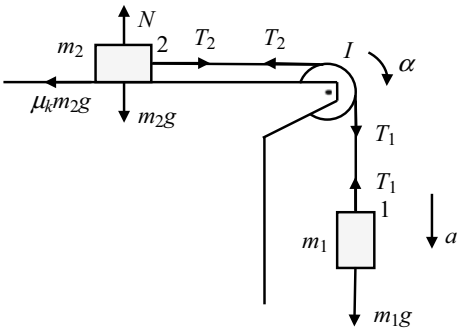


Fig. 9.29 Forces on the pulley and the blocks, Problem 9.23

Applying the Newton's second law to the blocks 1 and 2, we have,

$$m_1 g - T_1 = m_1 a, \tag{9.1}$$

$$T_2 - \mu_k m_2 g = m_2 a. \tag{9.2}$$

The pulley rotates with angular acceleration of α due to net torque, $\Sigma\tau = (T_1 - T_2)r$ on it. The angular acceleration is related to linear acceleration by, $\alpha = a/R$. We write,

$$\Sigma\tau = I\alpha,$$

$$(T_1 - T_2)r = I\frac{a}{r}. \tag{9.3}$$

Solving Eqs. (9.1), (9.2), and (9.3) one gets a, T_1, and T_2. From Eqs. (9.1) and (9.2), one gets,

$$-T_1 + T_2 = (m_1 + m_2)a - (m_1 - \mu_k m_2)g.$$

From Eq. (9.3),

$$T_1 - T_2 = I\frac{a}{r^2}.$$

From these two equations, one obtains,

$$0 = \left(m_1 + m_2 + \frac{I}{r^2}\right)a - (m_1 - \mu_k m_2)g.$$

This means that the acceleration, a, of the blocks is,

$$a = \frac{(m_1 - \mu_k m_2)g}{m_1 + m_2 + \frac{I}{r^2}} = \frac{(5.0\text{ kg} - 0.15 \times 10\text{ kg})(9.8\text{ m/s}^2)}{5.0\text{ kg} + 10\text{ kg} + \frac{2.0\text{ kg m}^2}{(0.20\text{ m})^2}} = 0.53\text{ m s}^{-2}.$$

(b) From Eq. (9.2), the tension, T_2, in the upper string is,

$$T_2 = m_2 a + \mu_k m_2 g = (10\text{ kg})(0.53\text{ m/s}^2) + 0.15(10\text{ kg})(9.8\text{ m/s}^2) = 20\text{ N}.$$

From Eq. (9.1), the tension, T_1, in the lower string is,

$$T_1 = m_1(g - a) = (5.0\text{ kg})(9.8\text{ m/s}^2 - 0.53\text{ m/s}^2) = 46\text{ N}.$$

(c) The speed, v, of block 1 when it falls a distance 40 cm is calculated as follows,

$$v^2 - u^2 = 2as,\ u = 0 \Rightarrow v^2 = 2as,$$

$$v = \sqrt{2as} = \sqrt{2(0.53\ \text{m/s}^2)(0.4\ \text{m})} = 0.65\ \text{m s}^{-1}.$$

- wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)      5
(ratprint)  false
(%i5)  eq1: m1*g-T1=m1*a; eq2: T2-mu_k*m2*g=m2*a; eq3: (T1-
T2)*r=I*a/r;
(eq1) g*m1-T1=a*m1
(eq2) T2-g*m2*mu_k=a*m2
(eq3) (T1-T2)*r=(I*a)/r
(%i6) solve([eq1,eq2,eq3], [a,T1,T2]);
(%o6) [[a=(g*m1*r^2-g*m2*mu_k*r^2)/(m2*r^2+m1*r^2+I),
       T1=(g*m1*(m2*mu_k*r^2+m2*r^2+I))/(m2*r^2+m1*r^2+I),
       T2=(g*m1*(m2*mu_k*r^2+m2*r^2)+I*g*m2*mu_k)/(m2*r^2+
m1*r^2+I)]]
(%i8) subst([m1=5, m2=10, mu_k=0.15, I=2, r=0.2, g=9.8], %)$
float(%);
(%o8) [[a=0.52769,T1=46.362,T2=19.977]]
(%i10) a: rhs(%[1][1]); s:0.4;
(a)   0.52769
(s)   0.4
(%i12) solve(v^2=2*a*s, v)$ float(%);
(%o12) [v=-0.64973,v=0.64973]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i5) Assign Eqs. (9.1), (9.2), and (9.3) as eq1, eq2, and eq3, respectively.

(%i6) Solve Eqs. (9.1), (9.2), and (9.3) in symbols for a, T_1, and T_2.

(%o6) The solutions in symbols.

(%i8) Substitute values of m_1, m_2, μ_k, I, r, and g into the solutions to get numerical values of a, T_1, and T_2. Part (a) and (b).

(%i10) Assign values of a and s.

(%i12) Solve $v^2 = 2as$ for v. Part (c).

Alternative solution for part (c): Applying the conservation of energy to the initial and final instances, the speed, v, of block 1 when it falls a distance of $s = 40$ cm is calculated as follows,

$$\begin{aligned} U_{init} + K_{init} &= U_{final} + K_{final} + W, \\ m_1 gs + 0 &= 0 + \frac{1}{2} I\omega^2 + \frac{1}{2}(m_1 + m_2)v^2 + \mu_k m_2 gs, \\ m_1 gs &= \frac{1}{2} I\omega^2 + \frac{1}{2}(m_1 + m_2)v^2 + \mu_k m_2 gs, \\ m_1 gs &= \frac{1}{2} I\left(\frac{v}{r}\right)^2 + \frac{1}{2}(m_1 + m_2)v^2 + \mu_k m_2 gs, \end{aligned}$$

$$\begin{aligned} (5.0\text{ kg})(9.8\text{ N/kg})(0.40\text{ m}) &= \frac{1}{2}(2.0\text{ kg m}^2)\left(\frac{v}{0.20\text{ m}}\right)^2 + \frac{1}{2}(5.0\text{ kg} + 10\text{ kg})v^2 \\ &\quad + 0.15(10\text{ kg})(9.8\text{ N/kg})(0.40\text{ m}), \\ v &= 0.65\text{ m s}^{-1}. \end{aligned}$$

- wxMaxima codes:

```
(%i9) fpprintprec:5; ratprint:false; m1:5; m2:10; I:2; r:20e-
2; g:9.8; s:0.4; mu_k:0.15;
(fpprintprec)      5
(ratprint)  false
(m1)  5
(m2)  10
(I)   2
(r)   0.2
(g)   9.8
(s)   0.4
(mu_k)  0.15
(%i11)  solve(m1*g*s=0.5*I*(v/r)^2+0.5*(m1+m2)*v^2+mu_k*
m2*g*s,v)$ float(%);
(%o11) [v=-0.64973,v=0.64973]
```

Comments on the codes:

(%i9) Set floating point print precision to 5, internal rational number print to false, assign values of m_1, m_2, I, r, g, s, and μ_k.

(%i11) Solve $m_1 gs = 0.5 \times I(v/r)^2 + 0.5 \times (m_1 + m_2)v^2 + \mu_k m_2 gs$ for v.

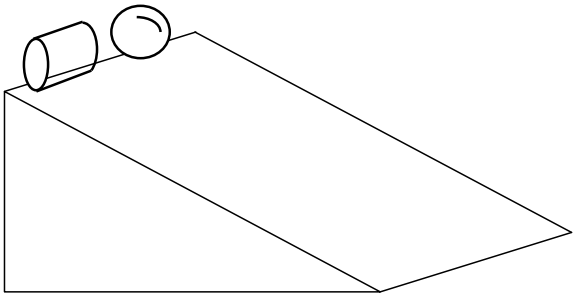


Fig. 9.30 Sphere and cylinder of the same mass and radius released to roll down an incline, Problem 9.24

Problem 9.24 A cylinder and a sphere of the same radius and mass are released at the same time from rest from an incline and they roll down the incline without slipping (Fig. 9.30). Which will have a higher speed when they reach the bottom? Which will reach the bottom first?

Solution

Rotational inertia of a sphere ($I_{sphere} = 2mr^2/5$) is smaller than that of a cylinder ($I_{cylinder} = mr^2/2$) (Fig. 9.1e, i). The torque on the sphere and the cylinder is the same, hence, the sphere will have higher angular acceleration (and linear acceleration) rolling down the incline, because $\tau = I\alpha$. The sphere will have higher speed than the cylinder at the bottom of the incline. Higher angular acceleration also means the sphere takes less time than the cylinder to reach the bottom of the incline.

Problem 9.25 Show that in Problem 9.24, the sphere will reach the bottom of incline at higher speed and in shorter time than the cylinder.

Solution

Figure 9.31 shows the sphere and the cylinder at height h on the incline of angle θ.

The potential energy of the object at the top of the incline is converted to the translational and rotational kinetic energies at the bottom.

For the sphere, one has,

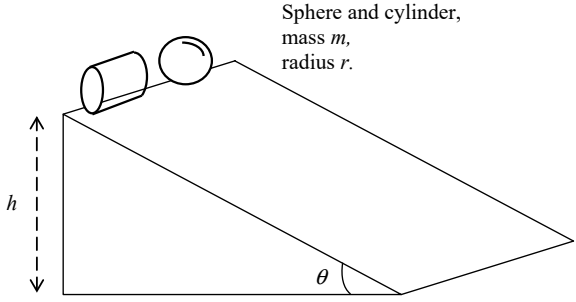
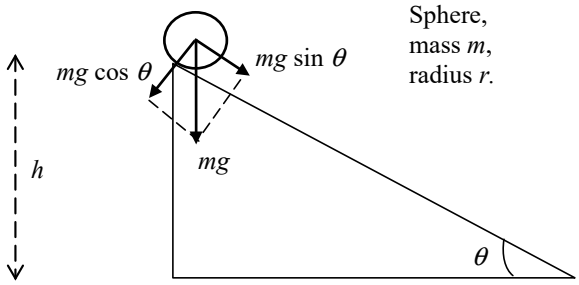


Fig. 9.31 Sphere and cylinder of the same mass and radius released to roll down an incline, from the same height, at the same time, Problem 9.25

Fig. 9.32 Problem 9.25



$$
\begin{aligned}
mgh &= \frac{1}{2}mv_{sphere}^2 + \frac{1}{2}I\omega_{sphere}^2 \\
&= \frac{1}{2}mv_{sphere}^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\left(\frac{v_{sphere}}{r}\right)^2 \\
&= \frac{7}{10}mv_{sphere}^2, \\
v_{sphere} &= \sqrt{\frac{10}{7}gh} = 1.20\sqrt{gh}.
\end{aligned}
$$

Figure 9.32 shows the forces acting on the sphere.
The torque on the sphere is,

$$\tau = rF = rmg\sin\theta.$$

Applying the Newton's second law for the rotational motion, we calculate the linear acceleration of the sphere,

$$
\begin{aligned}
\tau &= I\alpha, \\
rmg\sin\theta &= \frac{2}{5}mr^2 \cdot \frac{a_{sphere}}{r}, \\
a_{sphere} &= \frac{5}{2}g\sin\theta.
\end{aligned}
$$

The time taken by the sphere to reach the bottom is,

$$t_{sphere} = \frac{v-u}{a} = \frac{v_{sphere}-0}{a_{sphere}} = \frac{\sqrt{\frac{10}{7}gh}}{\frac{5}{2}g\sin\theta} = 0.478\frac{\sqrt{gh}}{g\sin\theta}.$$

For the cylinder, one has,

$$mgh = \frac{1}{2}mv_{cylinder}^2 + \frac{1}{2}I\omega_{cylinder}^2$$

$$
\begin{aligned}
&= \frac{1}{2}mv_{cylinder}^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{v_{cylinder}}{r}\right)^2 \\
&= \frac{3}{4}mv_{cylinder}^2, \\
v_{cylinder} &= \sqrt{\frac{4}{3}gh} = 1.15\sqrt{gh}.
\end{aligned}
$$

The torque on the cylinder is,

$$\tau = rF = rmg\sin\theta.$$

Applying the Newton's second law for the rotational motion, we calculate the linear acceleration of the cylinder,

$$
\begin{aligned}
\tau &= I\alpha, \\
rmg\sin\theta &= \frac{1}{2}mr^2 \cdot \frac{a_{cylinder}}{r}, \\
a_{cylinder} &= 2g\sin\theta.
\end{aligned}
$$

The time taken by the cylinder to reach the bottom is,

$$t_{cylinder} = \frac{v-u}{a} = \frac{v_{cylinder}-0}{a_{cylinder}} = \frac{\sqrt{\frac{4}{3}gh}}{2g\sin\theta} = 0.577\frac{\sqrt{gh}}{g\sin\theta}.$$

The calculations show that $v_{sphere} > v_{cylinder}$ and $t_{sphere} < t_{cylinder}$. This means that the sphere will reach the bottom of incline at higher speed and in shorter time than the cylinder.

- wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i4)       solve(m*g*h=0.5*m*v_sphere^2+0.5*0.4*m*r^2*(v_
sphere/ r)^2, v_sphere)$ float(%);
(%o4) [v_sphere=-1.1952*sqrt(g*h),v_sphere=1.1952*sqrt(g*h)]
(%i5) v_sphere: rhs(%[2]);
(v_sphere) 1.1952*sqrt(g*h)
(%i7)     solve(r*m*g*sin(theta)=0.4*m*r^2*a_sphere/r,   a_
sphere)$ float(%);
(%o7) [a_sphere=2.5*g*sin(theta)]
(%i8) a_sphere: rhs(%[1]);
(a_sphere) 2.5*g*sin(theta)
(%i9) t_sphere: v_sphere/a_sphere;
```

```
(t_sphere) (0.47809*sqrt(g*h))/(g*sin(theta))
(%i11) solve(m*g*h = 0.5*m*v_cylinder^2 + 0.5*0.5*m*r^2*(v_
cylinder/r)^2, v_cylinder)$ float(%);
(%o11)    [v_cylinder = -1.1547*sqrt(g*h), v_cylinder =
1.1547*sqrt(g*h)]
(%i12) v_cylinder: rhs(%[2]);
(v_cylinder) 1.1547*sqrt(g*h)
(%i14) solve(r*m*g*sin(theta) = 0.5*m*r^2*a_cylinder/r, a_
cylinder)$ float(%);
(%o14) [a_cylinder=2.0*g*sin(theta)]
(%i15) a_cylinder: rhs(%[1]);
(a_cylinder) 2.0*g*sin(theta)
(%i16) t_cylinder: v_cylinder/a_cylinder;
(t_cylinder) (0.57735*sqrt(g*h))/(g*sin(theta))
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i4), (%i5) Solve $mgh = \frac{1}{2}mv_{sphere}^2 + \frac{1}{2}(\frac{2}{5}mr^2)(\frac{v_{sphere}}{r})^2$ for v_{sphere} and assign v_{sphere}.

(%i7), (%i8), (%i9) Solve $rmg\sin\theta = \frac{2}{5}mr^2 \cdot \frac{a_{sphere}}{r}$ for a_{sphere}, assign a_{sphere}, and calculate t_{sphere}.

(%i11), (%i12) Solve $mgh = \frac{1}{2}mv_{cylinder}^2 + \frac{1}{2}(\frac{1}{2}mr^2)(\frac{v_{cylinder}}{r})^2$ for $v_{cylinder}$ and assign $v_{cylinder}$.

(%i14), (%i15), (%i16) Solve $rmg\sin\theta = \frac{1}{2}mr^2 \cdot \frac{a_{cylinder}}{r}$ for $a_{cylinder}$, assign $a_{cylinder}$, and calculate $t_{cylinder}$.

Problem 9.26 Figure 9.33 is an Atwood machine. Two blocks of masses $m_1 = 6.0$ kg and $m_2 = 1.0$ kg are connected by a string that passes over a pulley of radius $R = 0.20$ m and moment of inertia $I = 1.5$ kg m^2. The distance of block 1 to the floor is $h = 2.0$ m. The blocks are released from rest. Calculate,

(a) the acceleration, a, of the blocks and the tensions, T_1 and T_2, in the string,
(b) the speed, v, at which block 1 strikes the floor.

Solution

(a) Figure 9.34 shows the forces acting on the blocks and their acceleration a. The net force acting on the block 1 is $m_1g - T_1$ toward the ground, where m_1g is the weight of the block and T_1 is the tension in the string. This net force accelerates the block 1. The net force on the block 2 is $T_2 - m_2g$ pointing upward, where T_2 is the tension in the string and m_2g is the weight of the block. This net force accelerates the block 2. The torque acting on the pulley is $(T_1 - T_2)R$.

Applying the Newton's second law to the block 1 and block 2 gives,

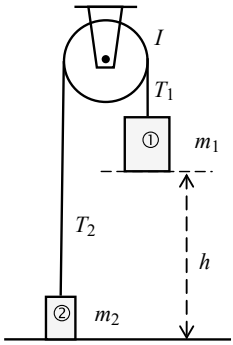


Fig. 9.33 Atwood machine, Problem 9.26

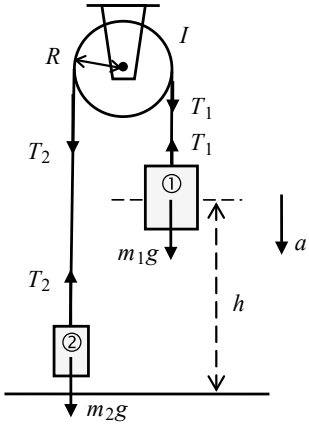


Fig. 9.34 Forces on the blocks and pulley, Problem 9.26

$$m_1 g - T_1 = m_1 a, \tag{9.1}$$

$$T_2 - m_2 g = m_2 a. \tag{9.2}$$

The torque rotates the pulley with angular acceleration α,

$$(T_1 - T_2)R = I\alpha,$$

$$(T_1 - T_2)R = \frac{Ia}{R},$$

$$T_1 - T_2 = \frac{Ia}{R^2}, \tag{9.3}$$

because $\alpha = a/R$. Adding Eqs. (9.1), (9.2), and (9.3) gives,

$$(m_1 - m_2)g = \left(m_1 + m_2 + \frac{I}{R^2}\right)a.$$

Thus, the acceleration of the block is,

$$a = \frac{(m_1 - m_2)g}{\left(m_1 + m_2 + \frac{I}{R^2}\right)} = \frac{(6.0\text{ kg} - 1.0\text{ kg})9.8\text{ m/s}^2}{\left(6.0\text{ kg} + 1.0\text{ kg} + \frac{1.5\text{ kg m}^2}{(0.20\text{ m})^2}\right)}$$
$$= 1.1\text{ m s}^{-2}.$$

From Eq. (9.1), the tension,

$$T_1 = m_1 g - m_1 a = m_1 g - \frac{m_1(m_1 - m_2)g}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}$$
$$= (6.0\text{ kg})(9.8\text{ m/s}^2) - \frac{(6.0\text{ kg})(6.0\text{ kg} - 1.0\text{ kg})9.8\text{ m/s}^2}{\left(6.0\text{ kg} + 1.0\text{ kg} + \frac{1.5\text{ kg m}^2}{(0.20\text{ m})^2}\right)}$$
$$= 52\text{ N}.$$

From Eq. (9.2), the tension,

$$T_2 = m_2 g + m_2 a = m_2 g + \frac{m_2(m_1 - m_2)g}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}$$
$$= (1.0\text{ kg})(9.8\text{ m/s}^2) + \frac{(1.0\text{ kg})(6.0\text{ kg} - 1.0\text{ kg})9.8\text{ m/s}^2}{\left(6.0\text{ kg} + 1.0\text{ kg} + \frac{1.5\text{ kg m}^2}{(0.20\text{ m})^2}\right)}$$
$$= 11\text{ N}.$$

(b) The speed at which block 1 strikes the floor is calculated as follows,

$$v^2 - u^2 = 2as \Rightarrow v^2 = 2ah, \tag{9.4}$$

$$v = \sqrt{2ah} = \sqrt{\frac{2(m_1 - m_2)gh}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}} = \sqrt{\frac{2(6.0\text{ kg} - 1.0\text{ kg})(9.8\text{ m/s}^2)(2.0\text{ m})}{\left(6.0\text{ kg} + 1.0\text{ kg} + \frac{5.0\text{ kg m}^2}{(0.20\text{ m})^2}\right)}},$$
$$= 2.1\text{ m s}^{-1}.$$

- wxMaxima codes:

```
(%i8)   fpprintprec:5; ratprint:false; m1:6; m2:1; R:0.2;
I:1.5; g:9.8; h:2;
```

```
(fpprintprec)      5
(ratprint)  false
(m1)  6
(m2)  1
(R)   0.2
(I)   1.5
(g)   9.8
(h)   2
(%i10)  solve([m1*g-T1=m1*a, T2-m2*g=m2*a, T1-T2=I*a/R^2],
[a,T1,T2])$ float(%);
(%o10)  [[a=1.1011,T1=52.193,T2=10.901]]
(%i11)  a: rhs(%[1][1]);
(a)   1.1011
(%i12)  v: sqrt(2*a*h);
(v)   2.0987
(%i13)  a: (m1-m2)*g/(m1+m2+I/R^2);
(a)   1.1011
(%i14)  T1: m1*g-m1*(m1-m2)*g/(m1+m2+I/R^2);
(T1)  52.193
(%i15)  T2: m2*g+m2*(m1-m2)*g/(m1+m2+I/R^2);
(T2)  10.901
(%i16)  v: sqrt(2*a*h);
(v)   2.0987
```

Comments on the codes:

(%i8) Set floating point print precision to 5, internal rational number print to false, assign values of m_1, m_2, R, I, g, and h.

(%i10) Solve Eqs. (9.1), (9.2), and (9.3) for a, T_1, and T_2. Part (a).

(%i11) Assign value of a.

(%i12) Calculate v. Part (b).

(%i13), (%i14), (%i15), (%i16) Direct calculation of a, T_1, T_2, and v.

- Alternative calculation:

```
(%i2)  fpprintprec:5; ratprint:false;
(fpprintprec)      5
(ratprint)  false
(%i5)  eq1: m1*g-T1=m1*a; eq2: T2-m2*g=m2*a; eq3: T1-T2=I*a/
R^2;
(eq1)  g*m1-T1=a*m1
(eq2)  T2-g*m2=a*m2
(eq3)  T1-T2=(I*a)/R^2
(%i6)  solve([eq1,eq2,eq3], [a,T1,T2]);
```

```
(%o6)  [[a=(R^2*g*m1-R^2*g*m2)/(R^2*m2+R^2*m1+I),
        T1=(g*m1*(2*R^2*m2+I))/(R^2*m2+R^2*m1+I),
        T2=(2*R^2*g*m1*m2+I*g*m2)/(R^2*m2+R^2*m1+I)]]
(%i8)  subst([m1=6, m2=1, R=0.2, I=1.5, g=9.8], %)$ float(%);
(%o8)  [[a=1.1011,T1=52.193,T2=10.901]]
(%i10)  a: rhs(%[1][1]); h:2;
(a)   1.1011
(h)   2
(%i11)  eq4: v^2=2*a*h;
(eq4)  v^2=4.4045
(%i13)  solve(eq4, v)$ float(%);
(%o13)  [v=-2.0987,v=2.0987]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i5) Assign Eqs. (9.1), (9.2), and (9.3) as eq1, eq2, and eq3, respectively.

(%i6) Solve Eqs. (9.1), (9.2), and (9.3) in symbols for a, T_1, and T_2.

(%i8) Substitute values of m_1, m_2, R, I, and g into the solutions to get numerical values of a, T_1, and T_2. Part (a).

(%i10) Assign values of a and h.

(%i11) Assign Eq. (9.4) as eq4.

(%i13) Solve Eq. (9.4) for v. Part (b).

Alternative solution for part (b): Figure 9.35 shows the two blocks in the initial and final instances. The reference level is as indicated and the speed at which the block 1 hits the floor is v.

Applying the conservation of energy to the initial and final instances (Eq. 7.10), one obtains,

$$\begin{aligned}\text{potential energy of block 1} &= \text{potential energy of block 2}\\ &\quad + \text{translational kinetic energy of block 1 and 2}\\ &\quad + \text{rotational kinetic energy of the pulley}\end{aligned}$$

$$\begin{aligned}m_1gh &= m_2gh + \frac{1}{2}(m_1+m_2)v^2 + \frac{1}{2}I\omega^2\\ &= m_2gh + \frac{1}{2}(m_1+m_2)v^2 + \frac{1}{2}I\frac{v^2}{R^2}\\ &= m_2gh + \frac{1}{2}\left(m_1+m_2+\frac{I}{R^2}\right)v^2.\end{aligned}$$

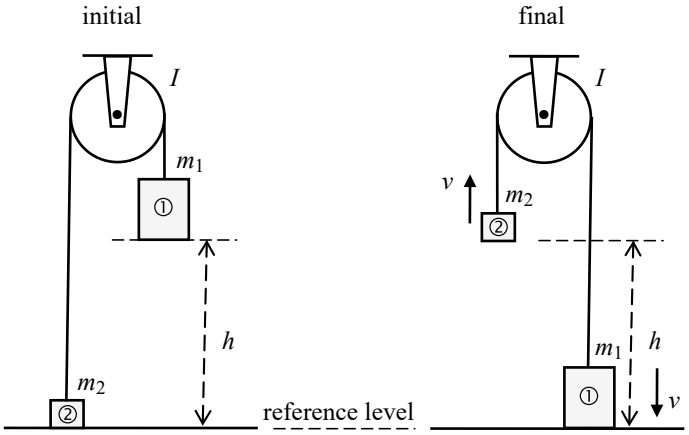


Fig. 9.35 Problem 9.26

The speed, v, at which the block 1 hits the floor is,

$$v = \sqrt{\frac{2(m_1 - m_2)gh}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}} = \sqrt{\frac{2(6.0\text{ kg} - 1.0\text{ kg})(9.8\text{ m/s}^2)(2.0\text{ m})}{\left(6.0\text{ kg} + 1.0\text{ kg} + \frac{5.0\text{ kg m}^2}{(0.20\text{ m})^2}\right)}}$$
$$= 2.1\text{ m s}^{-1}.$$

- wxMaxima codes:

```
(%i8)  fpprintprec:5; ratprint:false; m1:6; m2:1; R:0.2; I:
1.5; g:9.8; h:2;
(fpprintprec)      5
(ratprint)  false
(m1)  6
(m2)  1
(R)   0.2
(I)   1.5
(g)   9.8
(h)   2
(%i10)   solve(m1*g*h=m2*g*h+0.5  *  (m1+m2+I/R^2)*v^2,  v)$
float(%);
(%o10)  [v=-2.0987,v=2.0987]
```

Comments on the codes:

(%i8) Set floating point print precision to 5, internal rational number print to false, assign values of m_1, m_2, R, I, g, and h.

(%i10) Solve $m_1gh = m_2gh + \frac{1}{2}(m_1 + m_2 + \frac{I}{R^2})v^2$ for v.

9.3 Summary

- Table 9.1 compares the expressions of rotational motion with those of translational motion.
- Torque acting on a particle is $\tau = r \times F = \frac{dL}{dt}$.
- Angular momentum of a particle is $L = r \times p = r \times mv$.
- Moment of inertia of a system of particles is $I = \sum_i m_i r_i^2$.
- Moment of inertia of a rigid body is $I = \int r^2 dm$.
- Work-energy theorem for a rotating rigid body is $W = \int \tau \, d\theta = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$.
- Conservation of angular momentum is $I_{init}\omega_{init} = I_{final}\omega_{final}$.

Table 9.1 Rotational and translational motions

Rotational motion		Translational motion	
Angular displacement	$\theta - \theta_0$	Displacement	$x - x_0$
Angular velocity	$\omega = \frac{d\theta}{dt}$	Velocity	$v = \frac{dx}{dt}$
Angular acceleration	$\alpha = \frac{d\omega}{dt}$	Acceleration	$a = \frac{dv}{dt}$
Moment of inertia	I	Mass	m
Resultant torque, Newton's second law	$\Sigma\tau = I\alpha$	Resultant force, Newton's second law	$\Sigma F = ma$
$\alpha =$ constant	$\omega = \omega_0 + \alpha t$ $\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$ $\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0)$	$a =$ constant	$v = v_0 + at$ $x - x_0 = v_0 t + \frac{1}{2}at^2$ $v^2 - v_0^2 = 2a(x - x_0)$
Work	$W = \int \tau \, d\theta$	Work	$W = \int F \, dx$
Kinetic energy	$K = \frac{1}{2}I\omega^2$	Kinetic energy	$K = \frac{1}{2}mv^2$
Power	$P = \tau \cdot \omega$	Power	$P = F \cdot v$
Angular momentum	$L = I\omega$	Linear momentum	$p = mv$
Torque	$\tau = \frac{dL}{dt}$	Force	$F = \frac{dp}{dt}$

9.4 Exercises

Exercise 9.1 In 8.0 s, a car accelerates uniformly from rest to such a speed that its wheels are turning at 6.0 revolutions per second. How many radians does the wheel turn and what is the angular acceleration?

(Answer: 150 rad, 4.7 rad s^{-2})

Exercise 9.2 A constant torque is applied to a flywheel such that it attains angular speed of 1200 revolutions per minute from rest in 15 s. The rotational inertia of the flywheel is 2.1×10^{-3} kg m^2. Calculate,

(a) the angular acceleration,
(b) the torque,
(c) the angle turned,
(d) the work done.

(Answer: (a) 8.4 rad s^{-2}, (b) 0.018 N m, (c) 940 rad, (d) 17 J)

Exercise 9.3 A light rod is pivoted at one end and free to swing about it. Two masses, $2m$ and m are attached to the rod at distances b and $3b$, respectively, from the pivot (Fig. 9.36). The rod is held horizontal and released. Calculate,

(a) angular acceleration of the rod at the instant of release,
(b) angular speed of the rod at its lowest position.

(Answer: (a) $\alpha = \frac{5g}{11b}$, (b) $\omega = \sqrt{\frac{10g}{11b}}$)

Exercise 9.4 A solid cylinder and a thin walled pipe, both have the same mass and radius, are simultaneously released from rest at the upper end of an incline at inclination of θ. Both roll without slipping.

(a) Calculate the acceleration of center of mass of each object.
(b) Which object reach the bottom first?

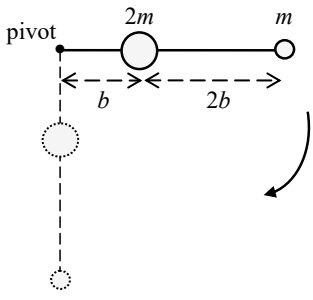


Fig. 9.36 Two masses attached to a rod. The rod is free to rotate about a pivot, Exercise 9.3

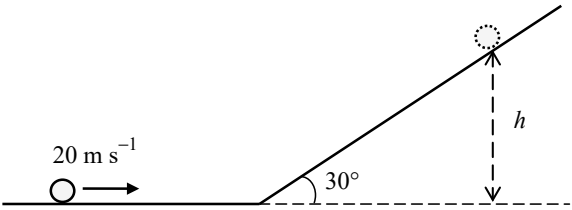


Fig. 9.37 A sphere rolls an incline up to height h, Exercise 9.5

(Answer: (a) $a_{\text{cylinder}} = 2g \sin\theta$, $a_{pipe} = g \sin\theta$, (b) the cylinder)

Exercise 9.5 A solid sphere rolls on a horizontal surface at 20 m s^{-1} (Fig. 9.37). It then rolls up an incline of 30° above the horizontal until it stops momentarily at height of h, before it rolls back down. Assuming there is no slipping and friction losses are negligible, determine h.

(Answer: $h = 29$ m)

Chapter 10
Statics of Rigid Body

10.1 Basic Concepts and Formulae

(1) A rigid body is in mechanical equilibrium if,

   i. the net external force acting on the body is zero, $\Sigma \boldsymbol{F} = 0$. This is the condition of translational equilibrium.
   ii. the net external torque on the body about any point or axis is zero, $\Sigma \boldsymbol{\tau} = 0$. This is the condition of rotational equilibrium.

(2) When a rigid body is in mechanical equilibrium under the actions of two forces, the forces are of the same magnitudes, opposite in directions, and have the same line of action.

(3) When a rigid body is in mechanical equilibrium under the actions of three forces, the resultant of the forces is zero and the lines of action of the forces intersect at a point.

(4) The center of gravity is,

$$\boldsymbol{r}_{CG} = \frac{\sum_i m_i g_i \boldsymbol{r}_i}{\sum_i m_i g_i}, \tag{10.1}$$

where m_i is the mass of the ith particle, r_i is its position vector, and g_i is acceleration of gravity at particle position. If the acceleration of gravity is uniform, the center of gravity coincides with the center of mass, that is,

$$\boldsymbol{r}_{CM} = \frac{\sum_i m_i \boldsymbol{r}_i}{\sum_i m_i}. \tag{10.2}$$

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10.2 Ploblems and Solutions

Problem 10.1 A uniform plank of mass 80 kg is suspended to a ceiling by two strings. A block of mass 30 kg is suspended at one end of the plank (Fig. 10.1). Determine the tensions, T_1, T_2, and T_3, in the strings.

Solution

Figure 10.2 shows the plank and the forces related to the problem. Here, T_1 and T_2 are tensions in the upper strings, T_3 is tension in the lower string, and m_pg is the weight of the plank. The weight m_pg acts at the middle of the plank because the plank is uniform, thus, the middle of the plank is the center of gravity.

The tension, T_3, in the third string is,

$$T_3 = mg = (30\,\text{kg})(9.8\,\text{N/kg}) = 294\,\text{N}.$$

The weight of the plank is,

$$m_pg = (80\,\text{kg})(9.8\,\text{N/kg}) = 784\,\text{N}.$$

The first equilibrium condition or the translational equilibrium condition, $\Sigma \boldsymbol{F} = 0$, gives,

$$T_1 + T_2 - m_pg - T_3 = 0, \tag{10.1}$$

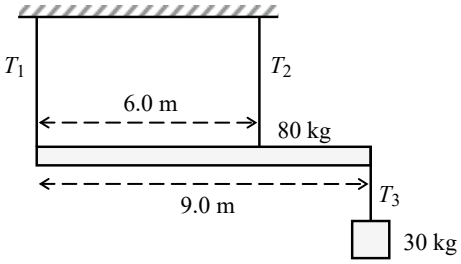

Fig. 10.1 A plank and a mass suspended by two strings, Problem 10.1

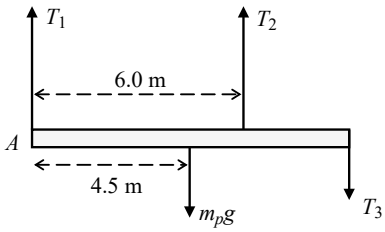

Fig. 10.2 Forces on the plank, Problem 10.1

$$T_1 + T_2 = m_p g + T_3 = (80\,\text{kg})(9.8\,\text{N/kg}) + 294\,\text{N} = 1078\,\text{N}. \tag{10.2}$$

The second equilibrium condition or the rotational equilibrium condition, $\Sigma \boldsymbol{\tau} = 0$, taken at point A, gives,

$$T_2(6.0\,\text{m}) - m_p g(4.5\,\text{m}) - T_3(9.0\,\text{m}) = 0, \tag{10.3}$$

$$T_2(6.0\,\text{m}) - (80 \times 9.8\,\text{N})(4.5\,\text{m}) - (30 \times 9.8\,\text{N})(9.0\,\text{m}) = 0. \tag{10.4}$$

From Eq. (10.4), the tension, T_2, in the second string is,

$$T_2 = 1029\,\text{N},$$

and from Eq. (10.2), the tension, T_1, in the first string is,

$$\begin{aligned} T_1 &= 1078\,\text{N} - T_2 = 1078\,\text{N} - 1029\,\text{N} \\ &= 49\,\text{N}. \end{aligned}$$

- wxMaxima codes:

```
(%i5)   fpprintprec:5; ratprint:false; mp:80; m:30; g:9.8;
(fpprintprec)       5
(ratprint)  false
(mp)    80
(m)     30
(g)     9.8
(%i6)   T3: m*g;
(T3)    294.0
(%i8)   solve([T1+T2-mp*g-T3=0, T2*6-mp*g*4.5-T3*9=0], [T1,
T2]) $ float(%);
(%o8)   [[T1=49.0,T2=1029.0]]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of m_p, m, and g.

(%i6) Calculate T_3.

(%i8) Solve $T_1 + T_2 - m_p g - T_3 = 0$ (Eq. 10.1) and $T_2(6) - m_k g(4.5) - T_3(9) = 0$ (Eq. 10.3) for T_1 and T_2.

Problem 10.2 A uniform plank of mass 10 kg is placed on two supports as in Fig. 10.3. Determine the reaction of each support on the plank, namely, R and S.

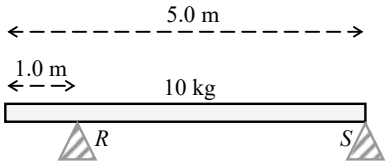

Fig. 10.3 A plank on two supports, Problem 10.2

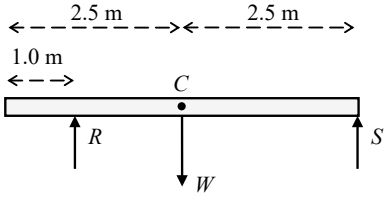

Fig. 10.4 Forces on the plank, Problem 10.2

Solution

The forces acting on the plank are shown in Fig. 10.4. Here, R and S are reactions of the supports on the plank, while W is the weight of the plank. The weight, W, acts at the middle because the plank is uniform.

Applying the first equilibrium condition, $\Sigma F = 0$, we write,

$$R + S - W = 0.$$

This gives,

$$R + S = W = (10\,\text{kg})(9.8\,\text{N/kg}) = 98\,\text{N}. \tag{10.1}$$

Applying the second equilibrium condition, $\Sigma\tau = 0$, at point C, we obtain,

$$S(2.5\,\text{m}) - R(1.5\,\text{m}) = 0. \tag{10.2}$$

From Eqs. (10.1) and (10.2),

$$S(2.5\,\text{m}) - (98\,\text{N} - S)(1.5\,\text{m}) = 0.$$

Therefore, the reaction, S, is,

$$S = \frac{(98\,\text{N})(1.5\,\text{m})}{2.5\,\text{m} + 1.5\,\text{m}} = 37\,\text{N},$$

and from Eq. (10.1), the reaction, R, is,

$$R = 98\,\text{N} - S = 98\,\text{N} - 37\,\text{N} = 61\,\text{N}.$$

- wxMaxima codes:

```
(%i2)   fpprintprec:5; ratprint:false;
(fpprintprec)       5
(ratprint)  false
(%i4)   solve([R+S=98, S*2.5-R*1.5=0], [R,S])$ float(%);
(%o4)   [[R=61.25,S=36.75]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i4) Solve $R + S = 98$ (Eq. 10.1) and $S(2.5) - R(1.5) = 0$ (Eq. 10.2) for R and S.

Problem 10.3 A uniform bar of length 4.0 m and mass 10 kg supports a load of 20 kg (Fig. 10.5). Determine the tension, T, in the string and the reaction, R, of the pivot P on the bar.

Solution

Figure 10.6 shows the forces that act on the bar. Here, T is the tension in the string, $m_1 g$ is the tension in the string that hangs the load, $m_2 g$ is the weight of the bar, while R_x and R_y are the x and y components of reaction, R, of the pivot P on the bar. The length of the bar is l.

Applying the first equilibrium condition, that is, the net force on the bar is zero, in the x and y directions,

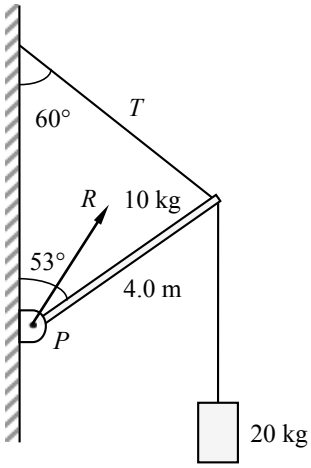

Fig. 10.5 A bar and a string supporting a mass, Problem 10.3

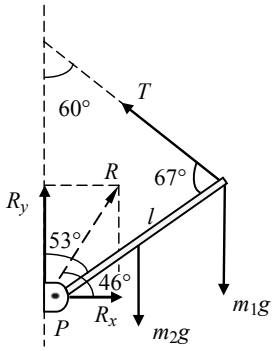

Fig. 10.6 Forces on the bar, Problem 10.3

$$\Sigma F_x = R_x - T\sin 60^\circ = 0, \tag{10.1}$$

$$\Sigma F_y = R_y + T\cos 60^\circ - m_2 g - m_1 g = 0. \tag{10.2}$$

Applying the second equilibrium condition, that is, the net torque on the bar is zero, at the pivot P,

$$\Sigma\tau = lT\sin 67^\circ - \frac{l}{2}m_2 g\sin 53^\circ - lm_1 g\sin 53^\circ = 0. \tag{10.3}$$

From Eq. (10.3), the tension, T, in the string is,

$$\begin{aligned} T &= \frac{\frac{l}{2}m_2 g\sin 53^\circ + lm_1 g\sin 53^\circ}{l\sin 67^\circ} = \left(\frac{1}{2}m_2 + m_1\right)\frac{g\sin 53^\circ}{\sin 67^\circ} \\ &= \left(\frac{1}{2}(10\,\text{kg}) + 20\,\text{kg}\right)\frac{(9.8\,\text{N/kg})\sin 53^\circ}{\sin 67^\circ} \\ &= 213\,\text{N}. \end{aligned}$$

From Eq. (10.1), the reaction, R_x, of the pivot on the bar in the x direction is,

$$R_x = T\sin 60^\circ = (213\,\text{N})\sin 60^\circ = 184\,\text{N}.$$

From Eq. (10.2) the reaction, R_y, of the pivot on the bar in the y direction is,

$$\begin{aligned} R_y &= (m_2 + m_1)g - T\cos 60^\circ = (10\,\text{kg} + 20\,\text{kg})(9.8\,\text{N/kg}) - (213\,\text{N})\cos 60^\circ \\ &= 188\,\text{N}. \end{aligned}$$

The magnitude of the reaction, R, of the pivot on the bar is,

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(184\text{ N})^2 + (188\text{ N})^2} = 263\text{ N},$$

and the direction is given by the angle,

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{188\ \text{N}}{184\ \text{N}}\right) = 46^\circ.$$

The reaction R and the angle θ are shown in Fig. 10.6.

- wxMaxima codes:

```
(%i9)    fpprintprec:5; ratprint:false; m1:20; m2:10; l:4;
g:9.8;  angle60:float(60/180  *  %pi);  angle67:float(67/
180*%pi);
angle53:float(53/180*%pi);
(fpprintprec)       5
(ratprint)   false
(m1)    20
(m2)    10
(l)     4
(g)     9.8
(angle60)  1.0472
(angle67)  1.1694
(angle53)  0.92502
(%i11)     solve([Rx-T*sin(angle60)=0,   Ry+T*cos(angle60)-
m2*g    -m1*g=0,    l*T*sin(angle67)-l/2*m2*g*sin(angle53)-
l*m1*g*sin(angle53)=0], [T,Rx,Ry])$ float(%);
(%o11) [[T=212.56,Rx=184.09,Ry=187.72]]
(%i12) Rx: rhs(%o11[1][2]);
(Rx)   184.09
(%i13) Ry: rhs(%o11[1][3]);
(Ry)   187.72
(%i14) R: sqrt(Rx^2 + Ry^2);
(R)    262.92
(%i15) theta: atan(Ry/Rx);
(theta)  0.79517
(%i16) theta_deg: float(theta*180/%pi);
(theta_deg)  45.56
```

Comments on the codes:

(%i9) Set floating point print precision to 5, internal rational number print to false, assign values of m_1, m_2, l, g, and angles in radian.

(%i11) Solve $R_x - T\sin 60^\circ = 0$, $R_y + T\cos 60^\circ - m_2 g - m_1 g = 0$, and $lT\sin 67^\circ - \frac{l}{2}m_2 g \sin 53^\circ - l m_1 g \sin 53^\circ = 0$ for T, R_x, and R_y.

(%i12), (%i13) Assign values of R_x and R_y.

(%i14) Calculate R.

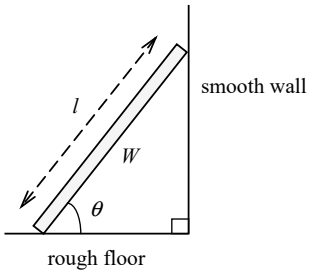

Fig. 10.7 A plank leaning against a smooth wall, Problem 10.4

(%i15), (%i16) Calculate θ and convert the angle to degree.

Problem 10.4 A uniform plank of weight, W, and length, l, leans on a vertical smooth wall with the lower end of the plank on a rough floor (Fig. 10.7). The coefficient of static friction between the floor and the plank is $\mu = 0.30$. Calculate the minimum angle, θ, so that the plank would not slide down.

Solution

The forces acting on the plank are shown in Fig. 10.8. Here, W is the weight of the plank, N is the reaction of the floor on the plank, f is the friction, and M is the reaction of the wall on the plank. M is normal to the wall because the wall is smooth. The length of the plank is l.

Applying the first equilibrium condition, the net force is zero, in the x and y directions, we have,

$$\Sigma F_x = f - M = 0,$$

$$\Sigma F_y = N - W = 0.$$

Applying the second equilibrium condition, the net torque is zero about point P, gives,

$$\Sigma \tau = Ml \sin\theta - W(l/2)\cos\theta = 0.$$

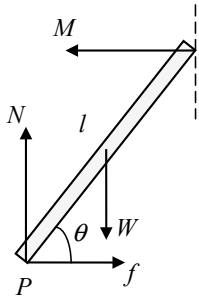

Fig. 10.8 Forces on the plank, Problem 10.4

The friction is,

$$f = \mu N = \mu W = M.$$

So we have $M = \mu W$ and,

$$\mu W l \sin\theta - W(l/2)\cos\theta = 0.$$

This means that,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{W(l/2)}{\mu W l} = \frac{1}{2\mu}.$$

The minimum angle, θ, so that the plank does not slide down is,

$$\theta = \tan^{-1}\left(\frac{1}{2\mu}\right) = \tan^{-1}\left(\frac{1}{2(0.30)}\right) = 59°.$$

- wxMaxima codes:

```
(%i1)   fpprintprec:5;
(fpprintprec)       5
(%i2)          solve(mu*W*l*sin(theta)-W*l/2*cos(theta)=0,
sin(theta));
(%o2)   [sin(theta)=cos(theta)/(2*mu)]
(%i3)   mu:0.3;
(mu)    0.3
(%i5)    theta: atan(1/(2*mu)); theta_deg: float(theta/%pi
*180);
(theta)   1.0304
(theta_deg)   59.036
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i2) Solve $\mu W l \sin\theta - W(l/2)\cos\theta = 0$ for $\sin\theta$.

(%i3) Assign value of μ.

(%i5) Calculate θ and convert the angle to degree.

Problem 10.5 Two planks of the same length, l, and weight, W, are connected by a hinge and are made to stand on a floor with negligible friction by a string (Fig. 10.9). The angle between the planks is θ and the string is h away from the floor. Determine the tension, T, in the string and the reaction, D, of the hinge on the plank.

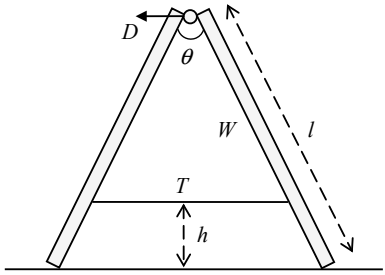

Fig. 10.9 Two planks erected on a smooth floor by means of a hinge and a string, Problem 10.5

Solution

Figure 10.10a shows the forces acting on the planks. R is the reaction of the floor on the plank. The reaction, R, is perpendicular to the floor because the friction between the floor and the plank is negligible. By symmetry, the reactions of the floor on the left and right planks are the same, R. W is the weight of the plank acting downward, T is the tension in the string, and D is the reaction of hinge on the left plank.

Consider forces acting on plank AB (the left plank). The net force on left plank is zero. This means that in the x and y directions,

$$T - D = 0, \tag{10.1}$$

$$R - W = 0. \tag{10.2}$$

The moment of force about A of the left plank is zero. This means that,

$$W(l/2)\sin(\theta/2) + T[l\cos(\theta/2) - h] - Rl\sin(\theta/2) = 0. \tag{10.3}$$

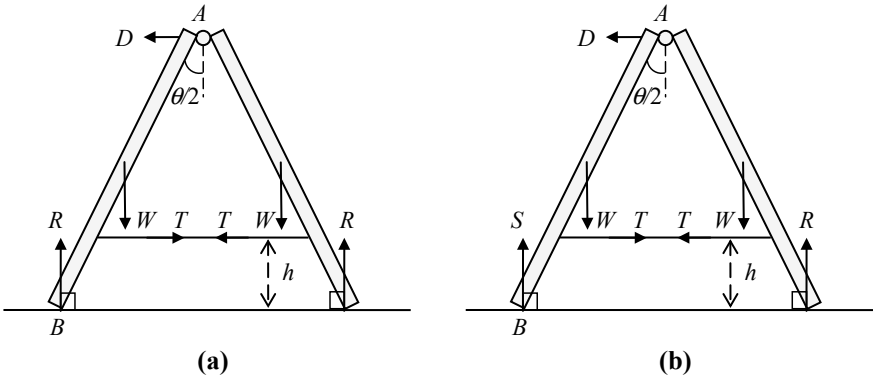

Fig. 10.10 Forces on the plank. **a** Assumption that normal reactions R of the floor on the planks are the same, **b** assumption that the normal reactions are different, one is S the other is R, Problem 10.5

From Eq. (10.2), $R = W$. Equation (10.3) gives the tension, T, in the string as,

$$T = \frac{(Wl/2) \cdot \sin(\theta/2)}{l\cos(\theta/2) - h}.$$

The reaction, D, of the hinge on the left plank is (Eq. 10.1),

$$D = T = \frac{(Wl/2) \cdot \sin(\theta/2)}{l\cos(\theta/2) - h}.$$

The magnitudes of D and T are the same, but they are opposite in directions.

- wxMaxima codes:

```
(%i2)   fpprintprec:5; ratprint:false;
(fpprintprec)       5
(ratprint)   false
(%i3)   eq1: T-D=0;
(eq1)   T-D=0
(%i4)   eq2: R-W=0;
(eq2)   R-W=0
(%i5)    eq3: W*l/2*sin(theta/2)+T*(l*cos(theta/2)-h)-R*l*
sin(theta/2)=0;
(eq3)                    (W*l*sin(theta/2))/2-R*l*sin(theta/
2)+T*(l*cos(theta/ 2)-h)=0
(%i6)   solve([eq1, eq2, eq3], [T, D, R]);
(%o6)   [[T=(W*l*sin(theta/2))/(2*l*cos(theta/2)-2*h),
           D=(W*l*sin(theta/2))/(2*l*cos(theta/2)-2*h), R=
W]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i3), (%i4), (%i5) Assign Eqs. (10.1), (10.2), and (10.3) as eq1, eq2, and eq3, respectively.

(%i6) Solve Eqs. (10.1), (10.2), and (10.3) for T, D, and R.

Alternative solution: If we labeled the normal reaction of the floor on the left plank at B as S instead of R, Fig. 10.10b, would the result be the same? Let us do the calculation.

In the y direction, the net force on the planks is zero. This means that,

$$S + R - 2W = 0. \tag{10.1}$$

In the x direction, the net force on the left plank is zero. This implies that,

$$T - D = 0. \tag{10.2}$$

The moment of force about A of the left plank is zero. This means that,

$$W(l/2)\sin(\theta/2) + T[l\cos(\theta/2) - h] - Sl\sin(\theta/2) = 0. \tag{10.3}$$

The moment of force about A of the right plank is zero. This means that,

$$Rl\sin(\theta/2) - W(l/2)\sin(\theta/2) - T[l\cos(\theta/2) - h] = 0. \tag{10.4}$$

Solving Eqs. (10.1), (10.2), (10.3), and (10.4) for T, D, R, and S gives,

$$T = D = \frac{(Wl/2)\cdot\sin(\theta/2)}{l\cos(\theta/2) - h},$$

$$R = S = W.$$

Thus, we get the same results as the former solution.

- wxMaxima codes:

```
(%i2)   fpprintprec:5; ratprint:false;
(fpprintprec)       5
(ratprint)  false
(%i3)   eq1: S+R-2*W=0;
(eq1)   -2*W+S+R=0
(%i4)   eq2: T-D=0;
(eq2)   T-D=0
(%i5)     eq3: W*l/2*sin(theta/2)+T*(l*cos(theta/2)-h)-S*l*
sin(theta/2)=0;
(eq3)        (W*l*sin(theta/2))/2-S*l*sin(theta/2)+T*(l*cos
(theta/2)-h)=0
(%i6)     eq4: R*l*sin(theta/2)-W*l/2*sin(theta/2)-T*(l*cos
(theta/2)-h)=0;
(eq4)       -(W*l*sin(theta/2))/2+R*l*sin(theta/2)-T*(l*cos
(theta/2)-h)=0
(%i7)   solve([eq1,eq2,eq3,eq4], [T,D,R,S]);
(%o7)   [[T=(W*l*sin(theta/2))/(2*l*cos(theta/2)-2*h),
D=(W*l*sin(theta/2))/(2*l*cos(theta/2)-2*h),R=W,S=W]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

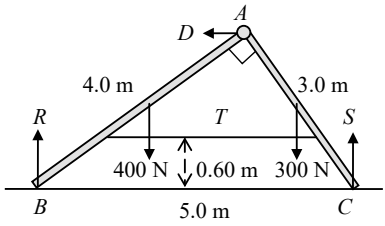

Fig. 10.11 Two planks of different lengths erected on a smooth floor by means of a hinge and a string, Problem 10.6

(%i3), (%i4), (%i5), (%i6) Assign Eqs. (10.1), (10.2), (10.3), and (10.4) as eq1, eq2, eq3, and eq4, respectively.

(%i7) Solve Eqs. (10.1), (10.2), (10.3), and (10.4) for T, D, R, and S.

Problem 10.6 A plank of length 4.0 m and weight of 400 N and a shorter plank of length 3.0 m and weight of 300 N are hinged at one of their ends and erected of a floor with negligible friction by a string (Fig. 10.11). The string is 0.60 m away from the floor.

(a) Calculate the reactions, R and S, of the floor on the plank, the tension, T, in the string, and the reaction, D, of the hinge on the long plank.
(b) A load of 1000 N is suspended from A. Calculate the reactions, R and S, of the floor on the plank, the tension, T, in the string, and the reaction, D, of the hinge on the long plank.

Solution

(a) Figure 10.12 shows the two planks and the forces acting on them. Here, R and S are the reactions of the floor on planks at B and C, respectively. The reactions are normal to the floor because the floor has negligible friction. The weight of long plank, 400 N, is directed downward from its center of gravity. Similarly, the weight of short plank, 300 N, is directed downward from its center of gravity. The tension in the string is T and the reaction of the hinge on the long plank is D. The distance BC is 5.0 m as calculated by the Pythagoras theorem and the angle is calculated as $\theta = 36.9°$.

The net force is zero in the y direction. This means that,

$$R + S - 400\ \text{N} - 300\ \text{N} = 0, \tag{10.1}$$

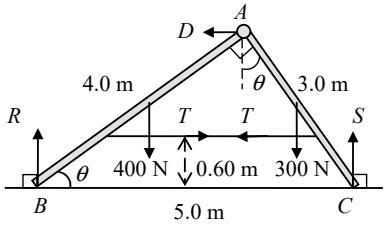

Fig. 10.12 Forces on the planks, Problem 10.6

where R and S are the reactions of the floor on the plank at B and C, respectively.

From the figure, $\cos\theta = 4/5$, $\sin\theta = 3/5$. The moment of force of the plank AB about A is zero. This means that,

$$(400\text{ N})(2.0\text{ m})\cos\theta + T(4.0\text{ m}\times\sin\theta - 0.60\text{ m}) - R(4.0\text{ m})\cos\theta = 0. \tag{10.2}$$

The reaction of the hinge on the long plank is D. The moment of force of the plank AB about B is zero. This means that,

$$D(4.0\text{ m})\sin\theta - T(0.60\text{ m}) - (400\text{ N})(2.0\text{ m})\cos\theta = 0. \tag{10.3}$$

The moment of force of the plank AC about A is zero. This means that,

$$S(3.0\text{ m})\sin\theta - T(3.0\text{ m}\times\cos\theta - 0.60\text{ m}) - (300\text{ N})(1.5\text{ m})\sin\theta = 0. \tag{10.4}$$

Equations (10.1), (10.2), (10.3), and (10.4) are solved for R, S, T, and D. The results are,

$$R = 326\text{ N},\quad S = 374\text{ N},\quad T = 224\text{ N},\quad D = 323\text{ N}.$$

(b) If a load of 1000 N is suspended at A, Eq. (10.1) becomes,

$$R + S - 400\text{ N} - 300\text{ N} - 1000\text{ N} = 0. \tag{10.5}$$

Equations (10.2), (10.3), and (10.4) are still the same. Solving Eqs. (10.5), (10.2), (10.3), and (10.4) gives,

$$R = 686\text{ N},\quad S = 1014\text{ N},\quad T = 864\text{ N},\quad D = 483\text{ N}.$$

- wxMaxima codes:

```
(%i4)       fpprintprec:5;  ratprint:false;  costheta:4/5;
sintheta: 3/5;
(fpprintprec)      5
(ratprint)  false
(costheta)  4/5
(sintheta)  3/5
(%i5)  eq1: R+S-400-300=0;
(eq1)  S+R-700=0
(%i6)        eq2:   400*2*costheta+T*(4*sintheta-0.6)-R*4*
costheta=0;
(eq2)  1.8*T-(16*R)/5+640=0
```

```
(%i7)  eq3: D*4*sintheta-T*0.6-400*2*costheta=0;
(eq3)  -0.6*T+(12*D)/5-640=0
(%i8)       eq4:   S*3*sintheta-T*(3*costheta-0.6)-300*1.5*
sintheta=0;
(eq4)  -1.8*T+(9*S)/5-270.0=0
(%i10) solve([eq1, eq2, eq3, eq4], [R, S, T, D])$ float(%);
(%o10) [[R=326.0,S=374.0,T=224.0,D=322.67]]
(%i11) eq5: R+S-400-300-1000=0;
(eq5)  S+R-1700=0
(%i13) solve([eq5, eq2, eq3, eq4], [R, S, T, D])$ float(%);
(%o13) [[R=686.0,S=1014.0,T=864.0,D=482.67]]
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign cos θ and sin θ.

(%i5), (%i6), (%i7), (%i8) Assign Eqs. (10.1), (10.2), (10.3), and (10.4) as eq1, eq2, eq3, and eq4, respectively.

(%i10) Solve Eqs. (10.1), (10.2), (10.3), and (10.4) for R, S, T, and D. Part (a).

(%i11) Assign Eq. (10.5) as eq5.

(%i13) Solve Eqs. (10.5), (10.2), (10.3), and (10.4) for R, S, T, and D. Part (b).

Problem 10.7 A metal wire is bent into the letter L and hanged by a nail at A as in Fig. 10.13. The length of the long arm is two times the short one. Calculate the angle, θ.

Solution

Figure 10.14 shows the wire and the forces that act on it. Let the mass per unit length of the wire be ρ. The weight of the short arm is $l\rho g$, the one of the long arm is $2l\rho g$, each acts downward from its center of gravity. The reaction of the nail on the wire is R.

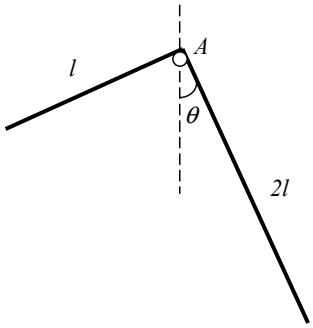

Fig. 10.13 A metal wire bent into letter L, hanged by a nail, Problem 10.7

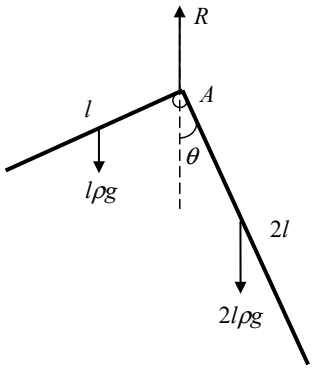

Fig. 10.14 Forces on the metal wire, Problem 10.7

The net force in the y direction is zero. This gives,

$$R - l\rho g - 2l\rho g = 0,$$
$$R - 3l\rho g = 0.$$

The net torque about A is zero. Thus,

$$l\rho g(l/2)\cos\theta - 2l\rho g(l)\sin\theta = 0,$$
$$\tan\theta = \frac{1}{4},$$
$$\theta = \tan^{-1}(1/4) = 0.24 \text{ rad} = 14^\circ.$$

- wxMaxima codes:

```
(%i2)   fpprintprec:5; ratprint:false;
(fpprintprec)       5
(ratprint)   false
(%i3)   solve(l*rho*g*l/2*cos(theta)-2*l*rho*g*l*sin(theta)
=0, sin(theta));
(%o3)   [sin(theta)=cos(theta)/4]
(%i5)   theta:float(atan(1/4)); theta_deg:float (theta/%pi*
180);
(theta)    0.24498
(theta_deg)   14.036
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

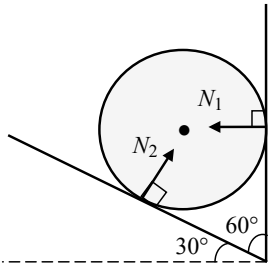

Fig. 10.15 A sphere trapped in a corner, Problem 10.8

(%i3) Solve $l\rho g(l/2)\cos\theta - 2l\rho g(l)\sin\theta = 0$ for $\sin\theta$.

(%i5) Calculate θ and convert the angle to degree.

Problem 10.8 A sphere of radius 0.10 m and mass 10 kg is trapped in a corner that is formed from a vertical wall and an inclined plane at 30° (Fig. 10.15). Friction of the wall and the incline with the sphere is negligible. Calculate the normal forces of the wall and the incline on the sphere, N_1 and N_2.

Solution

Figure 10.16 shows the sphere and the forces acting on it. Here, N_1 is the normal force of the wall on the sphere, N_2 is the normal force of the inclined plane on the sphere, and mg is the weight of the sphere.

The net force in the y direction is zero. It means that,

$$N_2 \cos\ 30^\circ - mg = 0,$$
$$N_2 = \frac{mg}{\cos\ 30^\circ} = \frac{(10\ \text{kg})(9.8\ \text{m/s}^2)}{\cos\ 30^\circ} = 113\ \text{N}.$$

The net force in the x direction is zero. This implies,

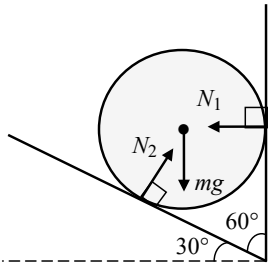

Fig. 10.16 Forces on the sphere, Problem 10.8

$$N_2 \sin 30^\circ - N_1 = 0,$$
$$N_1 = N_2 \sin 30^\circ = (113\ \text{N}) \sin 30^\circ = 56.6\ \text{N}.$$

- wxMaxima codes:

```
(%i5)  fpprintprec: 5; ratprint: false; m:10; g:9.8; angle30:
float(30/180*%pi);
(fpprintprec)      5
(ratprint)   false
(m)    10
(g)    9.8
(angle30)    0.5236
(%i7)    solve([N2*cos(angle30)-m*g=0, N2*sin(angle30)-N1=
0], [N1, N2])$ float(%);
(%o7)  [[N1=56.58,N2=113.16]]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of m, g, and θ.

(%i7) Solve $N_2 \cos\ 30^\circ - mg = 0$ and $N_2 \sin 30^\circ - N_1 = 0$ for N_1 and N_2.

Problem 10.9 A hoist weighing 450 N of length, l, lifts a load of 200 kg (Fig. 10.17). Calculate,

(a) the tension, T, in the cable,
(b) the horizontal, H, and vertical, V, forces of the pivot on the hoist.

Solution

(a) Figure 10.18 shows the hoist and the forces acting on it. The 450 N weight acts downward at the middle of the hoist, the tension in the cable is T, the weight of the load, 200 g, pulls the end of the hoist downward. The horizontal and vertical forces of the pivot on the hoist are H and V. The length of the hoist is l.

The torque about the pivot is zero. This gives the tension, T, in the cable as,

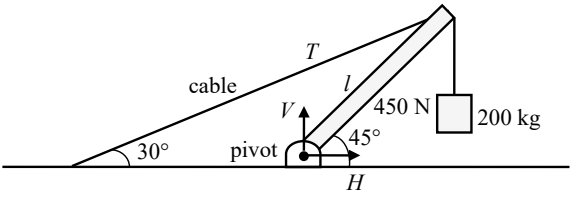

Fig. 10.17 A hoist in action, Problem 10.9

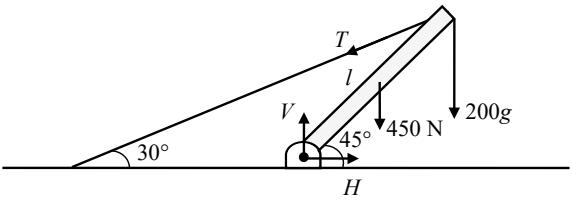

Fig. 10.18 Forces on the hoist, Problem 10.9

$$Tl\sin 15^\circ - (450\text{ N})(l/2)\cos 45^\circ - (200\text{ N})gl\cos 45^\circ = 0, \tag{10.1}$$

$$T = \frac{(225\text{ N})\cos 45^\circ + (200\text{ N})(9.8\text{ m/s}^2)\cos 45^\circ}{\sin 15^\circ} = 5970\text{ N}.$$

(b) The net force in the y direction is zero. This gives the vertical force, V, of the pivot on the hoist as,

$$V - 450\text{ N} - (200\text{ kg})g - T\cos 60^\circ = 0, \tag{10.2}$$

$$V = 5395\text{ N}.$$

The net force in the x direction is zero as well, and this gives the horizontal force, H, of the pivot on the hoist as,

$$H - T\sin 60^\circ = 0, \tag{10.3}$$

$$H = 5170\text{ N}.$$

- wxMaxima codes:

```
(%i3)  fpprintprec:5; ratprint:false; g:9.8;
(fpprintprec)      5
(ratprint)  false
(g)    9.8
(%i5)   solve(T*l*sin(15/180*%pi)-450*l/2*cos(45/180*%pi)-
200 *g*l*cos(45/180*%pi)=0, T)$ float(%);
(%o5)  [T=5969.5]
(%i6)  T: rhs(%[1]);
(T)    5969.5
(%i8)  solve(V-450-200*g-T*cos(60/180*%pi), V)$ float(%);
(%o8)  [V=5394.8]
(%i10)  solve(H-T*sin(60/180*%pi), H)$ float(%);
(%o10)  [H=5169.7]
```

Comments on the codes:

(%i3) Set floating point print precision to 5, internal rational number print to false, assign value of g.

(%i5) Solve $Tl\sin 15^\circ - (450)(l/2)\cos 45^\circ - (200)gl\cos 45^\circ = 0$ for T. Part (a).

(%i6) Assign value of T.

(%i8) Solve $V - 450 - (200)g - T\cos 60^\circ = 0$ for V. Part (b).

(%i10) Solve $H - T\sin 60^\circ = 0$ for H. Part (b).

- Alternative calculation:

```
(%i3)  fpprintprec:5; ratprint:false; g:9.8;
(fpprintprec)      5
(ratprint)  false
(g)   9.8
(%i5)  solve([T*l*sin(15/180*%pi)-450*l/2*cos(45/180*%pi)-
200 *g*l*cos(45/180*%pi)=0, V-450-200*g-T*cos(60/180*%pi),
H-T*sin(60/180*%pi) ], [T, V, H])$ float(%);
(%o5)  [[T=5969.5,V=5394.8,H=5169.8]]
```

Comments on the codes:

(%i3) Set floating point print precision to 5, internal rational number print to false, assign value of g.

(%i5) Solve Eqs. (10.1), (10.2), and (10.3) for T, V, and H.

Problem 10.10 A rectangular metal plate of length 40 cm and mass 3.0 kg is placed on a rough floor and a semi-cylinder of radius 10 cm with negligible friction (Fig. 10.19). Calculate,

(a) the forces acting on the plate at points A and B,

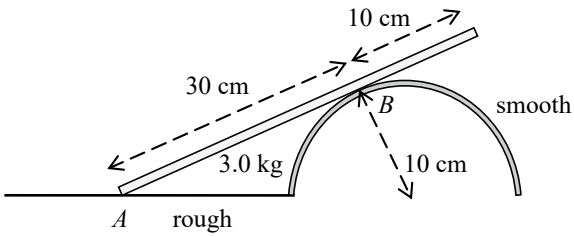

Fig. 10.19 A metal plate resting on a rough floor and a smooth semi-cylinder, Problem 10.10

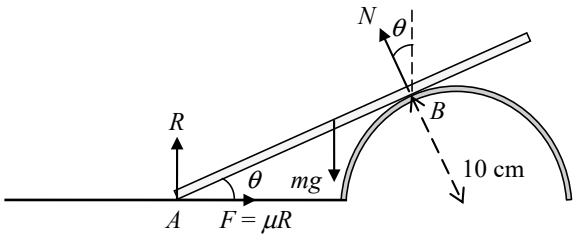

Fig. 10.20 Forces on the metal plate, Problem 10.10

(b) the minimum coefficient of friction, μ, between the floor and the plate so that the plate does not slide down.

Solution

(a) Figure 10.20 shows the forces acting on the rectangular metal plate. Here, N is the normal force of the semi-cylinder on the plate at B, R is the normal force of the floor on the plate at A, $F = \mu R$ is the friction between the plate and the floor, and mg is the weight of the plate acting downward at the center of gravity of the plate.

The equilibrium of forces in the x direction gives,

$$F - N \sin\theta = 0,$$

$$\mu R - N \sin\theta = 0, \tag{10.1}$$

because the friction between the floor and the plate is $F = \mu R$.

From the figure $\tan\theta = 10/30 = 1/3$. This means that $\cos\theta = 3/\sqrt{10}$, $\sin\theta = 1/\sqrt{10}$, and $\theta = 18.4°$.

The equilibrium of forces in the y direction gives,

$$R + N\cos\theta - mg = 0. \tag{10.2}$$

The net torque about point A is zero. Thus,

$$N(0.30\text{ m}) - mg(0.20\text{ m})\cos\theta = 0. \tag{10.3}$$

Solving Eqs. (10.1), (10.2), and (10.3), one obtains,

$$N = 19\text{ N}, \quad R = 12\text{ N}, \quad \mu = 0.50.$$

Thus, the normal force, N, of the semi-cylinder on the plate at B is,

$$N = 19\text{ N}.$$

The normal force, R, of the floor on the plate and the friction, F, between the floor and the plate at A are,

$$R = 12 \text{ N},$$
$$F = \mu R = 5.9 \text{ N}.$$

- wxMaxima codes:

```
(%i4)  fpprintprec:5; ratprint:false; m:3; g:9.8;
(fpprintprec)      5
(ratprint)  false
(m)     3
(g)     9.8
(%i6)  solve([mu*R-N*1/sqrt(10)=0, R+N*3/sqrt(10)-m*g=0,
        N*0.3-m*g*0.2*3/sqrt(10)=0], [N,R,mu])$ float(%);
(%o6)  [[N=18.594,R=11.76,mu=0.5]]
(%i8)  R: rhs(%o6[1][2]); mu: rhs(%o6[1][3]);
(R)     11.76
(mu)    0.5
(%i9)   F: mu*R;
(F)     5.88
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of m and g.

(%i6) Solve $\mu R - N\sin\theta = 0$, $R + N\cos\theta - mg = 0$, and $N(0.3) - mg(0.2)\cos\theta = 0$ for N, R, and μ.

(%i8) Assign values of R and μ.

(%i9) Calculate F.

(b) From part (a), the coefficient of friction, μ, between the floor and the plate is 0.50.

If the coefficient is smaller than 0.5, Eq. (10.1) says that there is a net force on the plate to the left, i.e. the plate would slide to the left.

The moment of force about point B is (Fig. 10.20),

$$mg(0.20 \text{ m})\cos\theta + \mu R(0.30 \text{ m})\sin\theta - R(0.30 \text{ m})\cos\theta = 0. \tag{10.4}$$

If the coefficient is smaller than 0.5, Eq. (10.4) says that the moment of force about point B is negative, i.e. a clockwise motion of the plate about point B which means a slide.

Thus, the minimum coefficient of friction between the floor and the plate so that the plate does not slide down is 0.5.

Problem 10.11

(a) Determine the center of mass of letter F plate (Fig. 10.21).
(b) The plate is suspended at a pivot P and is free to move about the pivot. What is the inclination angle of the plate relative to vertical line to the center of the earth?

Solution

(a) Assume the mass of the plate is proportional to its area. Section the plate into three parts: A, B, and C (Fig. 10.22).

The mass and center of mass of the parts are listed in Table 10.1.

Center of mass of the plate is calculated as follows (Eq. 10.2),

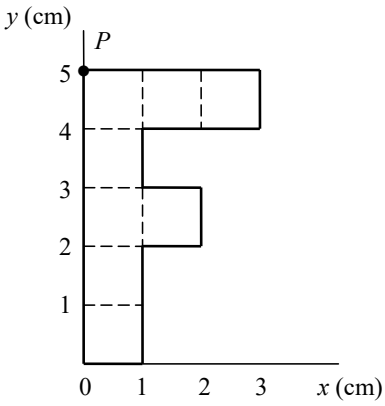

Fig. 10.21 A letter F plate, Problem 10.11

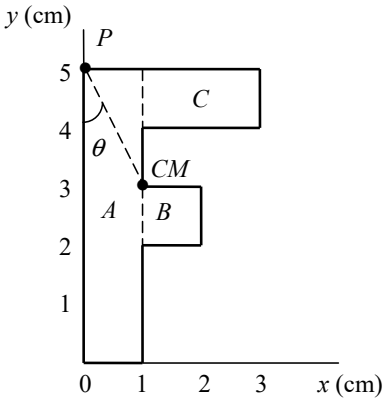

Fig. 10.22 Determining center of mass and angle θ, Problem 10.11

Table 10.1 Center of mass and mass of parts of letter F

Part	x_{CM} (cm)	y_{CM} (cm)	Mass
A	0.5	2.5	5
B	1.5	2.5	1
C	2.0	4.5	2

$$x_{CM} = \frac{\sum m_i x_i}{\sum m_i} = \frac{5(0.50\text{ cm}) + 1(1.5\text{ cm}) + 2(2.0\text{ cm})}{8} = 1.0\text{ cm}$$
$$y_{CM} = \frac{\sum m_i y_i}{\sum m_i} = \frac{5(2.5\text{ cm}) + 1(2.5\text{ cm}) + 2(4.5\text{ cm})}{8} = 3.0\text{ cm}$$

The center of mass of the F plate is (1.0 cm, 3.0 cm). This is marked as *CM* in Fig. 10.22.

(b) If the plate is suspended at point *P*, it will incline at an angle of θ (Fig. 10.22). The *P*-*CM* line is vertical to the center of the earth. From the figure,

$$\tan\theta = \frac{1}{2},$$
$$\theta = \tan^{-1}(1/2) = 0.46\text{ rad} = 27^\circ.$$

- wxMaxima codes:

```
(%i1)   fpprintprec:5;
(fpprintprec)       5
(%i2)   xCM: (5*0.5 + 1.5 + 2*2)/8;
(xCM)   1.0
(%i3)   yCM: (5*2.5 + 2.5 + 2*4.5)/8;
(yCM)   3.0
(%i5)    theta: float(atan(1/2)); theta_deg:float(theta/%pi
*180);
(theta)  0.46365
(theta_deg)  26.565
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i2), (%i3) Calculate x_{CM} and y_{CM}.

(%i5) Calculate θ and convert the angle to degree.

Problem 10.12 A block of mass 2.0 kg is balanced by a uniform bar of length 4.0 m at a distance of 1.0 m by a support (Fig. 10.23). What is the mass, m_{bar}, of the bar?

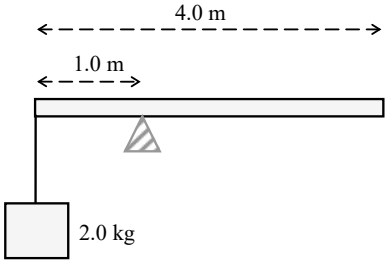

Fig. 10.23 A mass balanced by a uniform bar, Problem 10.12

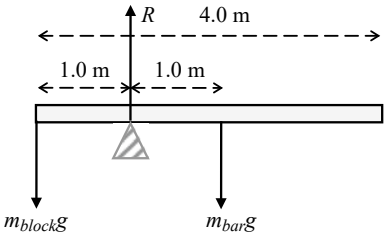

Fig. 10.24 Forces on the bar, Problem 10.12

Solution

The block is balanced by the bar. The center of mass of the bar is 2.0 m from its ends and there the weight of the bar is. Thus, the center of mass is 1.0 m from the support, just like the block. This means that to get a balance, the mass of the bar is 2.0 kg as well.

Figure 10.24 shows the forces acting on the bar. Here, $m_{block}g$ and $m_{bar}g$ are the weights of the block and the bar, respectively, while R is the reaction of the support on the bar.

Applying the second condition of equilibrium, that is, the net torque about the support is zero, the mass of the bar is calculated as follows,

$$\begin{aligned}
&\Sigma\tau = 0,\\
&m_{block}g(1.0\text{ m}) - m_{bar}g(1.0\text{ m}) = 0,\\
&m_{bar} = m_{block} = 2.0\text{ kg}.
\end{aligned}$$

10.3 Summary

A rigid body is in mechanical equilibrium when,

i. the net external force acting on the body is zero, $\Sigma F = 0$,
ii. the net external torque on the body about any point or axis is zero, $\Sigma\tau = 0$.

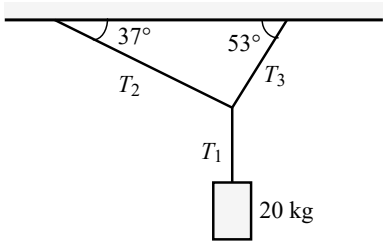

Fig. 10.25 A mass suspended by three cords, Exercise 10.1

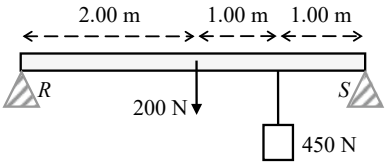

Fig. 10.26 A beam with a hanging load on two supports, Exercise 10.2

10.4 Exercises

Exercise 10.1 A mass of 20 kg hangs from the ceiling by cords as in Fig. 10.25. Find the tensions in the cords, T_1, T_2, and T_3.

(Answer: $T_1 = 196$ N, $T_2 = 118$ N, $T_3 = 157$ N)

Exercise 10.2 Figure 10.26 shows a uniform beam weighing 200 N holding a weight of 450 N on two supports. Determine the forces exerted by the supports, R and S, on the beam.

(Answer: $R = 212$ N, $S = 438$ N)

Exercise 10.3 Figure 10.27 shows a uniform beam of weight 120 N and length of 4.00 m length holding a weight of 400 N. The beam is suspended by two ropes. Find the tensions, T_1 and T_2, in the ropes and angle, θ.

(Answer: $T_1 = 185$ N, $T_2 = 371$ N, $\theta = 14.4°$)

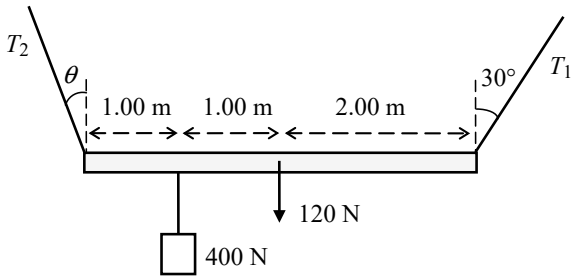

Fig. 10.27 A beam with a load, hanged by two ropes, Exercise 10.3

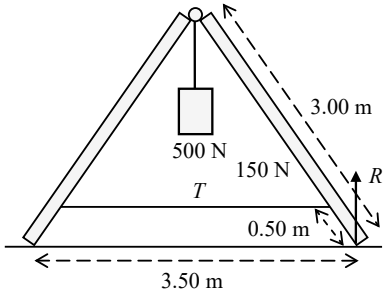

Fig. 10.28 A load is hanged by two planks erected on a smooth floor by means of a hinge and a string, Exercise 10.4

Exercise 10.4

(a) Two uniform planks each of length 3.00 m and weight of 150 N are connected by a hinge and are made to stand on a floor with negligible friction by a rope (Fig. 10.28). A load of 500 N is suspended at the hinge. Determine the tension, T, in the rope and the normal force, R, of the floor on the plank.

(b) The load of 500 N is removed from the hinge. Recalculate the tension, T, in the rope and the normal force, R, of the floor on the plank.

(Answer: (a) $T = 280$ N, $R = 400$ N, (b) $T = 64.6$ N, $R = 150$ N)

Exercise 10.5 Figure 10.29 shows a uniform ladder of length 13.0 m and weight of 300 N rests against a smooth wall at height 12.0 m above a rough floor. Determine the frictional force and the normal reaction of the floor on the ladder.

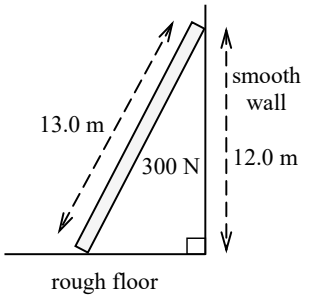

Fig. 10.29 A ladder leaning against a smooth wall, Exercise 10.5

(Answer: 62.5 N, 300 N)

Chapter 11
Oscillation and Simple Harmonic Motion

Abstract This chapter solves problems on simple harmonic motion and oscillation. Simple harmonic motion is defined as a motion whose acceleration is proportional to negative of its displacement. Oscillations of simple pendulum, mass-spring system, and physical pendulum are considered as simple harmonic motions. Both solutions by analytical means and computer calculation by wxMaxima are presented. In wxMaxima, simple harmonic motion is cast as a second order differential equation with its initial values and is solved by predefined functions *ode*2 and *ic*2 or *atvalue* and *desolve*. Simple animations of simple harmonic motion are presented for insight of the motions.

11.1 Basic Concepts and Formulae

(1) A simple harmonic motion (SHM) is a motion with acceleration, a, directly proportional to its displacement, x, and directed to a fixed point called the equilibrium point. The equation of SHM is,

$$a = -\omega^2 x \quad \text{or} \quad \frac{d^2x}{dt^2} = -\omega^2 x, \tag{11.1}$$

where ω is the angular frequency. To get a SHM, a force that is directly proportional and in opposite direction to the displacement is needed. For example, the elastic force of a spring $F = -kx$. Such a force is called a restoring force.

(2) The displacement, x, of a particle that undergoes SHM is a solution of the above equation and is given by, (Eq. 11.2),

$$x = A\cos(\omega t + \phi), \tag{11.2}$$

where A is the amplitude of the oscillation, ω is angular frequency, and ϕ is initial phase constant. The initial phase constant, ϕ, depends on the initial position and velocity of the particle. Other equivalent solution is,

$$x = A\sin(\omega t + \varphi) \tag{11.3}$$

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W. M. S. Wan Hassan, *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-42678-0_11

The time for a complete oscillation or vibration is the period, T, and is defined as, (Eq. 11.4),

$$T = \frac{2\pi}{\omega}. \tag{11.4}$$

(3) The reciprocal of the period is the frequency, f, which is the number of oscillations per second,

$$f = \frac{1}{T} = \frac{\omega}{2\pi}. \tag{11.5}$$

(4) If displacement is given as Eq. (11.2), then velocity of a particle, v, in SHM is,

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi).$$

This means that,

$$\begin{aligned} v^2 &= \omega^2 A^2 \sin^2(\omega t + \phi) \\ &= \omega^2 A^2 [1 - \cos^2(\omega t + \phi)] \\ &= \omega^2 [A^2 - A^2 \cos^2(\omega t + \phi)] \\ &= \omega^2 (A^2 - x^2), \end{aligned}$$

and,

$$| v | = \omega \sqrt{A^2 - x^2}. \tag{11.6}$$

Thus, the maximum speed is $v_{max} = \omega A$ that occurs at $x = 0$. The speed is zero at $x = \pm A$.

(5) The acceleration of a particle in SHM is,

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \phi) = -\omega^2 x. \tag{11.7}$$

The maximum acceleration is $a_{max} = \omega^2 A$ that occurs at $x = \pm A$. The acceleration is zero at $x = 0$. Thus, at the equilibrium point $x = 0$, the speed of the particle is a maximum but the acceleration is zero.

(6) If the initial conditions of the SHM are given, the phase constant, ϕ, and amplitude, A, can be determined. For example, if $x = x_0$ and $v = v_0$ at $t = 0$, and the solution is in the form of $x = A \cos(\omega t + \phi)$, then,

$$\tan \phi = -\frac{v_0}{\omega x_0},$$

and,

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}. \tag{11.8}$$

(7) For a mass-spring system in SHM,

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x,$$

$$\omega = \sqrt{\frac{k}{m}},$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}, \tag{11.9}$$

where m is the mass of the particle attached to the spring with the force constant of k. The period of the oscillation does not depend of the acceleration of gravity.

The kinetic and potential energies are,

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) = \frac{1}{2}k(A^2 - x^2), \tag{11.10}$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t + \phi). \tag{11.11}$$

The total energy is,

$$E = K + U = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2 A^2. \tag{11.12}$$

(8) A simple pendulum in SHM is described by,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta = -\omega^2\theta,$$

$$\omega = \sqrt{\frac{g}{l}},$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}, \tag{11.13}$$

where l is the length of the pendulum and g is the acceleration due to gravity. The period of the oscillation does not depend on the mass of the pendulum. Angular displacement, θ, is small.

(9) A physical pendulum in SHM follows,

$$\frac{d^2\theta}{dt^2} = -\left(\frac{mgd}{I}\right)\theta = -\omega^2\theta,$$

$$\omega = \sqrt{\frac{mgd}{I}},$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}, \tag{11.14}$$

where I is the moment of inertia of the pendulum about a pivot, m is the mass of the pendulum, d is the distance of the center of mass to the pivot, and g is the acceleration due to gravity. Angular displacement, θ, is small.

11.2 Problems and Solutions

Problem 11.1 A spring extends by 3.9 cm when a mass of 10 g is suspended from it. A block of mass 25 g is fixed to one end of the spring and the system is set to oscillate. Determine the period, T, of the oscillations if,

(a) the system oscillates horizontally on a surface with negligible friction as in Fig. 11.1a,
(b) the system oscillates vertically as in Fig. 11.1b.

Solution

(a) First, we calculate the force constant of the spring using the Hooke's law, force = force constant, $k \times$ extension. The force constant of the spring is,

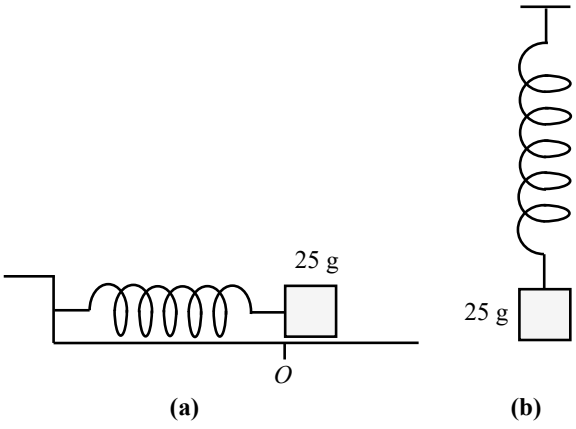


Fig. 11.1 **a** Horizontal and **b** vertical oscillations of a mass-spring system, Problem 11.1

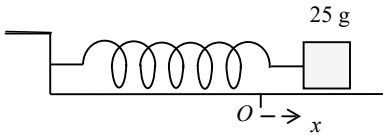


Fig. 11.2 Horizontal oscillations, Problem 11.1

$$k = \frac{\text{force}}{\text{extension}} = \frac{10 \times 10^{-3}\ \text{kg} \times 9.8\ \text{N/kg}}{3.9 \times 10^{-2}\ \text{m}} = 2.5\ \text{N m}^{-1}.$$

Horizontal oscillations: When the block of mass 25 g is displaced by a distance x to the right from the equilibrium point O (Fig. 11.2), the restoring force to the left is,

$$F = -kx.$$

Applying the Newton's second law,

$$F = m\frac{d^2x}{dt^2} = -kx,$$

and,

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{d^2x}{dt^2} + \omega^2 x = 0,$$

or,

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x.$$

The angular frequency of the oscillations is,

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2.5\ \text{N/m}}{0.025\ \text{kg}}} = 10\ \text{rad s}^{-1},$$

and the period of the oscillations is (Eq. 11.4),

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{25 \times 10^{-3}\ \text{kg}}{2.5\ \text{N/m}}} = 0.63\ \text{s}.$$

- wxMaxima codes:

```
(%i5)  fpprintprec:5; extension:3.9e-2; mass:10e-3; m:25e-
3; g:9.8;
(fpprintprec)    5
(extension)   0.039
(mass)   0.01
(m)  0.025
(g)  9.8
(%i6)  force: mass*g;
(force) 0.098
(%i7)  k: force/extension;
(k)  2.5128
(%i8)  omega: sqrt(k/m);
(omega) 10.026
(%i9)  T: 2*float(%pi)/omega;
(T) 0.62671
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of extension, mass m, and g.
(%i6) Calculate force.
(%i7), (%i8), (%i9) Calculate k, ω, and T.

(b) Vertical oscillations: The block of mass 25 g is suspended by the spring at the equilibrium position O as in Fig. 11.3. The block is pulled down and released so that the oscillations begin. Let at an instant the block is at a distance, x, from the equilibrium point. The force that acts on the block by the spring in the upward direction according to the Hooke's law is,

$$F = -kx.$$

Applying the Newton's second law, we have,

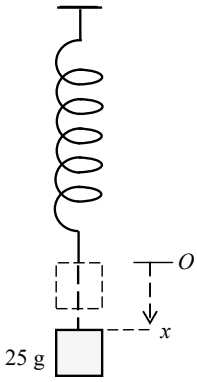


Fig. 11.3 Vertical oscillations, Problem 11.1

$$F = m\frac{d^2x}{dt^2} = -kx,$$

and,

$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{d^2x}{dt^2} + \omega^2 x = 0.$

This is the same result as the one obtained in part (a). The period of oscillation is still 0.63 s.

In conclusion, the frequency and the period of oscillations are the same for vertical or horizontal oscillations of a mass-spring system.

Problem 11.2 A mass-spring system with mass of 4.0 g oscillates with the frequency of 5.0 Hz. What is the force constant, k, of the spring?

Solution

The relationship between the angular frequency ω, frequency f, force constant k, and mass m of a oscillating mass-spring system is (Eq. 11.5),

$$\omega = 2\pi f = \sqrt{\frac{k}{m}}.$$

Using this relationship, the force constant, k, of the spring is written as,

$$k = 4\pi^2 f^2 m = 4\pi^2 (5.0\ \text{s}^{-1})^2 (4.0 \times 10^{-3}\ \text{kg}) = 3.9\ \text{N m}^{-1}.$$

- wxMaxima codes:

```
(%i4)  fpprintprec:5; ratprint:false; m:4e-3; f:5;
(fpprintprec)  5
(ratprint)  false
(m)  0.004
(f)  5
(%i6)  solve(2*%pi*f=sqrt(k/m), k)$ float(%);
(%o6)  [k=3.9478]
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of m and f.
(%i6) Solve $2\pi f = \sqrt{k/m}$ for k.

Problem 11.3 A particle oscillates in a simple harmonic motion. The oscillation frequency is 2.0 Hz. At $t = 0$ s, the displacement is, $x_0 = 2.0$ cm, and velocity is, $v_0 = -24$ cm s^{-1}. Determine,

(a) the maximum speed, v_{max}, and acceleration, a_{max},

(b) the initial phase constant, ϕ,
(c) the displacement, x, velocity, v, and acceleration, a, as functions of time, t. Plot their curves.

Solution

(a) The angular frequency is, (Eq. 11.5),
$\omega = 2\pi f = 2\pi\ (2\ \text{s}^{-1}) = 4\pi\ \text{rad s}^{-1}$.
The amplitude is, (Eq. 11.8),

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} = \sqrt{(2.0\ \text{cm})^2 + \left(\frac{-24\ \text{cm/s}}{4\pi\ \text{rad/s}}\right)^2} = 2.8\ \text{cm}.$$

The maximum speed is, (Eq. 11.6),

$$v_{max} = \omega A = (4\pi\ \text{rad/s})(2.8\ \text{cm}) = 35\ \text{cm s}^{-1}.$$

The maximum acceleration is, (Eq. 11.7),

$$a_{max} = \omega^2 A = (4\pi\ \text{rad/s})^2(2.8\ \text{cm}) = 440\ \text{cm s}^{-2}.$$

- wxMaxima codes:

```
(%i4)  fpprintprec:5; f:2; x0:2; v0:-24;
(fpprintprec)  5
(f)  2
(x0)  2
(v0)  -24
(%i5)  omega: float(2*%pi*f);
(omega) 12.566
(%i6)  A: sqrt(x0^2 + (v0/omega)^2);
(a) 2.7654
(%i7)  vmax: omega*A;
(vmax) 34.751
(%i8)  amax: omega^2*A;
(amax) 436.7
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of f, x_0, and v_0.
(%i5), (%i6), (%i7), (%i8) Calculate ω, A, v_{max}, and a_{max}.

(b) Let the expression for the displacement be,

$$x = A\cos(\omega t + \phi).$$

Differentiating the displacement with respect to time one gets the velocity given by,

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi).$$

At time $t = 0$, the two equations become,

$$x_0 = A\cos\phi,$$
$$v_0 = -\omega A \sin\phi.$$

From these two equations, one obtains,

$$\tan\phi = -\frac{v_0}{\omega x_0} = -\frac{-24 \text{ cm/s}}{(4\pi \text{ rad/s})(2.0 \text{ cm})} = 0.995.$$

Therefore, the initial phase constant, ϕ, is,

$$\phi = \tan^{-1} 0.995 = 0.76\,\text{rad} = 44^\circ.$$

- wxMaxima codes:

```
(%i4)  fpprintprec:5; f:2; x0:2; v0:-24;
(fpprintprec)  5
(f)  2
(x0)  2
(v0)  -24
(%i5)  omega: float(2*%pi*f);
(omega) 12.566
(%i7)     phi:  atan(-v0/(omega*x0));  phi_deg:  float(phi/
%pi*180);
(phi)  0.76235
(phi_deg) 43.679
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of f, x_0, and v_0.
(%i5) Calculate ω.
(%i7) Calculate ϕ and convert the angle to degree.

(c) Using the results of parts (a) and (b), the displacement, x, velocity, v, and acceleration, a, as functions of time, t, are,

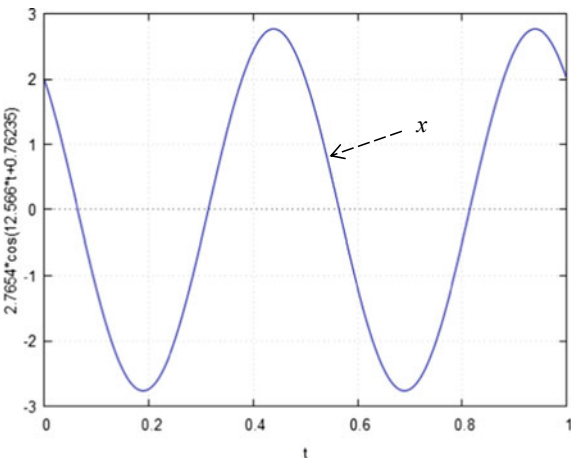
$$x = 2.8\cos(4\pi t + 0.76)\,\text{cm},$$

$$v = -2.8(4\pi)\sin(4\pi t + 0.76)\ \mathrm{cm\,s^{-1}},$$

$$a = -2.8(4\pi)^2\cos(4\pi t + 0.76)\ \mathrm{cm\,s^{-2}}.$$

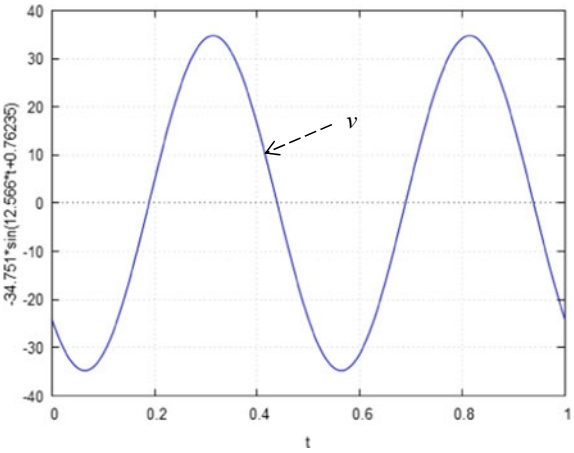
- Curves of displacement, velocity, and acceleration againsts time by wxMaxima:

```
(%i4)  fpprintprec:5; f:2; x0:2; v0:-24;
(fpprintprec)  5
(f)  2
(x0)  2
(v0)  -24
(%i5)  omega: float(2*%pi*f);
(omega) 12.566
(%i6)  A: sqrt(x0^2 + (v0/omega)^2);
(A) 2.7654
(%i7)  phi: atan(-v0/(omega*x0));
(phi)  0.76235
(%i8)  x: A*cos(omega*t + phi);
(x) 2.7654*cos(12.566*t+0.76235)
(%i9)  wxplot2d(x, [t,0,1], grid2d);
```

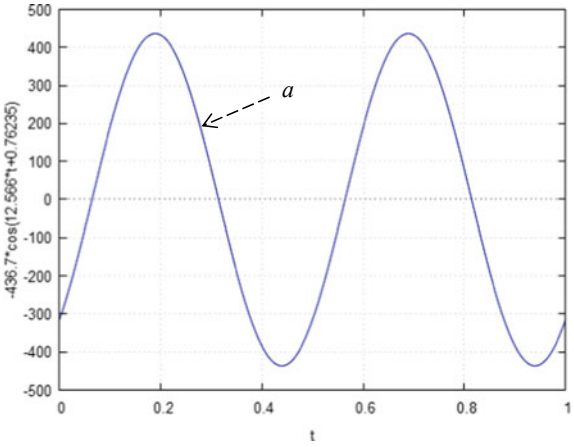



```
(%i10) v: -A*omega*sin(omega*t + phi);
(v) -34.751*sin(12.566*t+0.76235)
```

```
(%i11) wxplot2d(v, [t,0,1], grid2d);
```




```
(%i12) a: -A*omega^2*cos(omega*t + phi);
(a) -436.7*cos(12.566*t+0.76235)
(%i13) wxplot2d(a, [t,0,1], grid2d);
```



Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of f, x_0, and v_0.
(%i5), (%i6), (%i7) Calculate ω, A, and ϕ.
(%i8), (%i9) Define x and plot x against t for $0 \leq t \leq 1$ s with grids.
(%i10), (%i11) Define v and plot v against t for $0 \leq t \leq 1$ s with grids.
(%i12), (%i13) Define a and plot a against t for $0 \leq t \leq 1$ s with grids.

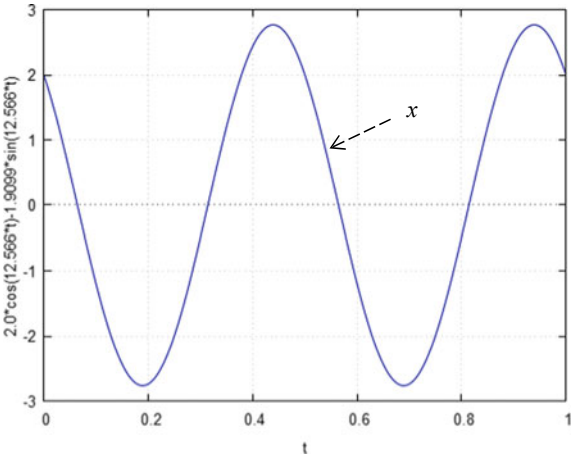
Alternative solution: wxMaxima functions *ode*2 and *ic*2 can be used to solve this problem. See *Solving second order ordinary differential equation* in Appendix A. The ordinary differential equation of the simple harmonic motion is,

$$\frac{d^2x}{dt^2} = -\omega^2 x,$$

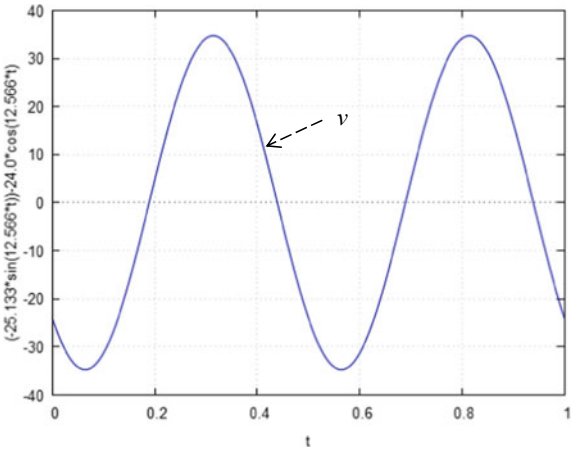
where $\omega = 4\pi$ rad/s, x is displacement (dependent variable) in cm, and t is time (independent variable) in second. The initial conditions are at $t = 0$ s, $x_0 = 2.0$ cm, and $v_0 = -24$ cm/s.

- wxMaxima codes:

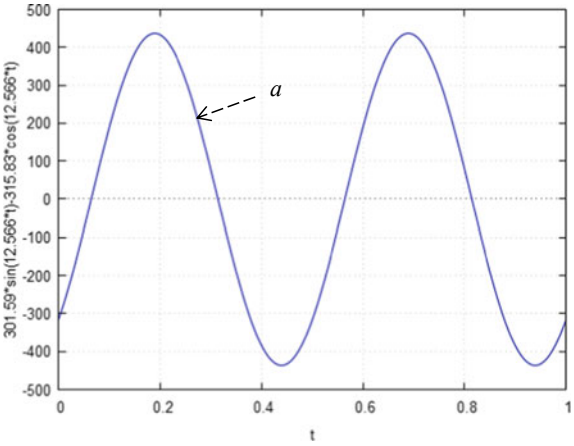
```
(%i4)  fpprintprec:5; omega:4*%pi; x0:2; v0:-24;
(fpprintprec)  5
(omega)  4*%pi
(x0)  2
(v0)  -24
(%i5)  sol: ode2('diff(x,t,2)=-omega^2*x, x, t);
(sol)  x=%k1*sin(4*%pi*t)+%k2*cos(4*%pi*t)
(%i7)  ic2(sol, t=0, x=x0, 'diff(x,t)=v0)$ float(%);
(%o7)  x=2.0*cos(12.566*t)-1.9099*sin(12.566*t)
(%i8)  x: rhs(%);
(x)  2.0*cos(12.566*t)-1.9099*sin(12.566*t)
(%i9)  wxplot2d(x, [t,0,1], grid2d);
```




```
(%i10) v: diff(x,t);
(v) -25.133*sin(12.566*t)-24.0*cos(12.566*t)
(%i11) wxplot2d(v, [t,0,1], grid2d);
```




```
(%i12) a: diff(v,t);
(a) 301.59*sin(12.566*t)-315.83*cos(12.566*t)
(%i13) wxplot2d(a, [t,0,1], grid2d);
```



Comments on the codes:

(%i4) Set floating point print precision to 5, assign value of ω, x_0, and v_0.
(%i5) Get a general solution of $d^2x/dt^2 = -\omega^2 x$.
(%i7) Set the initial conditions and get a particular solution.
(%o7) The solution.
(%i8), (%i9) Assign x and plot x against t for $0 \le t \le 1$ s.
(%i10), (%i11) Calculate v and plot v against t for $0 \le t \le 1$ s.
(%i12), (%i13) Calculate a and plot a against t for $0 \le t \le 1$ s.

The wxMaxima codes say that the displacement, x, velocity, v, and acceleration, a, as functions of time, t, are,

$$x = 2.0\cos(12.566t) - 1.9099\sin(12.566t)\ \text{cm},$$

$$v = -25.133\sin(12.566t) - 24.0\cos(12.566t)\ \text{cm s}^{-1},$$

$$a = 301.59\sin(12.566t) - 315.83\cos(12.566t)\ \text{cm s}^{-2}.$$

These x, v, and a are the same as those in part (c).
Further question: Show that,

$$2.8\cos(4\pi t + 0.76) = 2.0\cos(12.566t) - 1.9\sin(12.566t).$$

Answer:

$$\begin{aligned} 2.8\cos(4\pi t + 0.76) &= 2.8\,[\cos(4\pi t)\cos(0.76) - \sin(4\pi t)\sin(0.76)] \\ &= 2.8\,[\cos(12.566t)\cos(0.76) - \sin(12.566t)\sin(0.76)] \end{aligned}$$

$$= 2.0\cos(12.566t) - 1.9\sin(12.566t).$$

- wxMaxima codes:

```
(%i1)  fpprintprec:5;
(fpprintprec)  5
(%i4)  2.8*(cos(4*%pi*t)*cos(0.76)-sin(4*%pi*t)*sin(0.76))$
float(%); expand(%);
(%o3)  2.8*(0.72484*cos(12.566*t)-0.68892*sin(12.566*t))
(%o4)  2.0295*cos(12.566*t)-1.929*sin(12.566*t)
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i4) Calculate 2.8 $[\cos(4\pi t)\cos(0.76) - \sin(4\pi t)\sin(0.76)]$.
(%o4) The result.

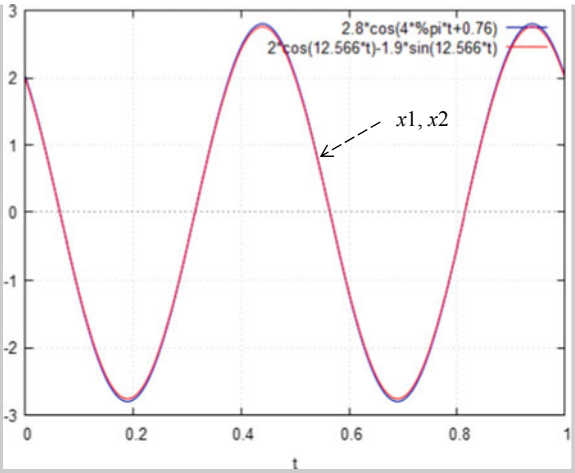
Another way to show $2.8\cos(4\pi t + 0.76) = 2.0\cos(12.566t) - 1.9\sin(12.566t)$ is by plotting their curves on a graph and see if they coincide. Let the curves be,

$$x1 = 2.8\cos(4\pi t + 0.76),$$

$$x2 = 2.0\cos(12.566t) - 1.9\sin(12.566t).$$

- wxMaxima codes:

```
(%i2)   x1: 2.8*cos(4*%pi*t + 0.76); x2: 2*cos(12.566*t) -
1.9*sin(12.566*t);
(x1)  2.8*cos(4*%pi*t+0.76)
(x2)  2*cos(12.566*t)-1.9*sin(12.566*t)
(%i3)  wxplot2d([x1,x2], [t,0,1], grid2d);
```



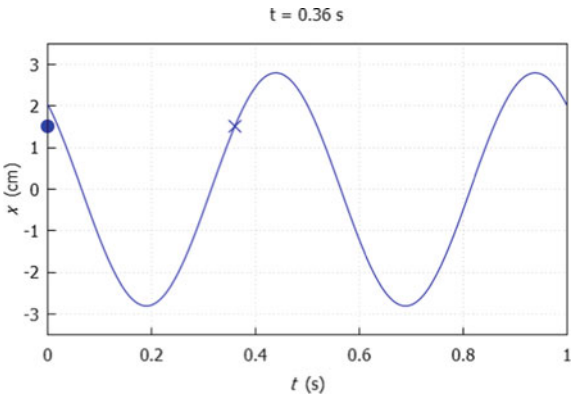
Comments on the codes:

(%i2) Define *x1* and *x*2.
(%i3) Plot $x1$ and $x2$ against t for $0 \leq t \leq 1$.

Indeed, the curves coincide with each other, showing that they are the same.

- Animation of the simple harmonic motion of the particle, $x = 2.8\cos(4\pi t + 0.76)$ cm, by wxMaxima:

```
(%i1)  fpprintprec:3;
(fpprintprec)  3
(%i2)  with_slider_draw(
t, makelist(i,i,0,1,0.04),
title=concat("t = ",t," s"),
point_size=2, point_type=7,
points([0], [2.8*cos(4*%pi*t + 0.76)]),
point_size=2, point_type=2,
points([t], [2.8*cos(4*%pi*t + 0.76)]),
explicit(2.8*cos(4*%pi*x + 0.76 ), x, 0, 1),
grid=true, yrange=[-3.5,3.5],
xlabel="{/Helvetica-Italic t} (s)",
ylabel="{/Helvetica-Italic x} (cm)");
```



Comments on the codes:

To run the animation, copy the codes to the wxMaxima command window; press <shift> and <enter> keys simultaneously to run the codes; right click the graphic that appears and choose *Start Animation*. The large dot represents the particle.

Problem 11.4 An object of mass 0.070 kg attached to a spring is in simple harmonic motion with amplitude of 0.13 m. At $t = 0$, the displacement is $x_0 = 0.050$ m and the velocity is $v_0 = 0.24$ m s^{-1}.

(a) What is the period, T, of the oscillations?
(b) If the displacement is $x = 0.13\cos(\omega t + \phi)$, determine ω and ϕ.
(c) What is the total energy, E, of the system?
(d) Plot displacement, x, velocity, v, and acceleration, a, of the object against time, t.

Solution

(a) The period of the oscillations is calculated as follows (Eqs. 11.6 and 11.4),

$$v_0 = \omega\sqrt{A^2 - x_0^2} = \frac{2\pi}{T}\sqrt{A^2 - x_0^2},$$

$$\omega = \frac{v_0}{\sqrt{A^2 - x_0^2}} = \frac{0.24 \text{ m/s}}{\sqrt{(0.13 \text{ m})^2 - (0.050 \text{ m})^2}} = 2.0 \text{ rad s}^{-1},$$

$$T = \frac{2\pi}{v_0}\sqrt{A^2 - x_0^2} = \frac{2\pi}{0.24 \text{ m/s}}\sqrt{(0.13 \text{ m})^2 - (0.050 \text{ m})^2} = 3.1 \text{ s}.$$

- wxMaxima codes:

```
(%i5)  fpprintprec:5; m:0.07; A:0.13; x0:0.05; v0:0.24;
(fpprintprec)  5
(m)  0.07
(a) 0.13
(x0)  0.05
(v0)  0.24
(%i6)  omega: float(v0/sqrt(A^2-x0^2));
(omega) 2.0
(%i7)  T: float(2*%pi/v0*sqrt(A^2 - x0^2));
(t) 3.1416
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of m, A, x_0, and v_0.
(%i6), (%i7) Calculate ω and T.

(b) From part (a), the angular frequency is (Eq. 11.4),

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{3.1\text{ s}} = 2.0\text{ rad s}^{-1}.$$

The displacement and velocity are,

$$x = 0.13\cos(2t + \phi),$$

$$v = \frac{dx}{dt} = -2(0.13)\sin(2t + \phi).$$

The initial conditions at $t = 0$ yields,

$$0.050 = 0.13\cos\phi,$$

$$0.24 = -2(0.13)\sin\phi.$$

The two equations enable the initial phase constant, ϕ, to be calculated,

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{0.24}{-2(0.13)} \times \frac{0.13}{0.05} = -2.4,$$
$$\phi = \tan^{-1}(-2.4) = -1.2\text{ rad} = -67°.$$

- wxMaxima codes:

```
(%i1)  fpprintprec:5;
(fpprintprec)  5
(%i3)  phi:atan(-2.4); phi_deg:float(phi*180/%pi);
(phi)  -1.176
(phi_deg) -67.38
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i3) Calculate ϕ and convert the angle to degree.

(c) The total energy, E, of the system is (Eq. 11.12),

$$E = \frac{1}{2}m\omega^2 A^2 = \frac{1}{2}(0.070\text{ kg})(2.0\text{ rad/s})^2(0.13\text{ m})^2$$
$$= 0.0024\text{ J}.$$

- wxMaxima codes:

```
(%i1)  fpprintprec:5;
(fpprintprec)  5
(%i2)  E: 1/2*0.07*2^2*0.13^2;
(E) 0.002366
```

Comments on the codes:

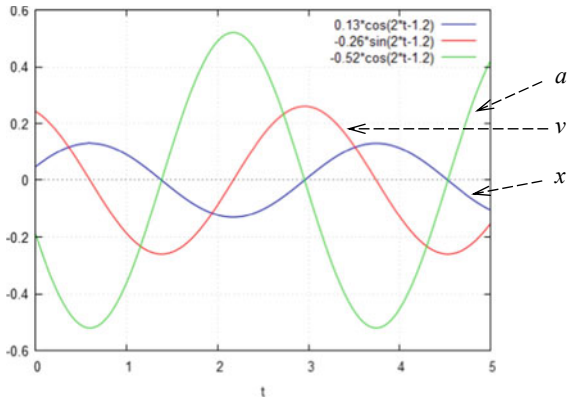
(%i1) Set floating point print precision to 5.
(%i2) Calculate E.

(d) We have the amplitude of $A = 0.13$ m, angular frequency of $\omega = 2.0\text{ rad s}^{-1}$, and phase constant of $\phi = -1.2$ rad. This means that the displacement, x, velocity, v, and acceleration, a, of the object are,

$$x = A\cos(\omega t + \phi) = 0.13\cos(2t - 1.2)\text{ m},$$
$$v = -A\omega\sin(\omega t + \phi) = -0.26\sin(2t - 1.2)\text{ m s}^{-1},$$
$$a = -A\omega^2\cos(\omega t + \phi) = -0.52\sin(2t - 1.2)\text{ m s}^{-2}.$$

- Plots of displacement, x, velocity, v, and acceleration, a, againsts time, t, by wxMaxima:

```
(%i3)     x:0.13*cos(2*t  -1.2);  v:-0.26*sin(2*t-1.2);  a:-
0.52*cos(2*t-1.2);
(x)  0.13*cos(2*t-1.2)
(v)  -0.26*sin(2*t-1.2)
(a)  -0.52*cos(2*t-1.2)
(%i4)  wxplot2d([x,v,a], [t,0,5], grid2d);
```



Comments on the codes:

(%i3) Define x, v, and a in terms of t.
(%i4) Plot x, v, and a against t for $0 \le t \le 5$ s with grids.

Alternative solution: This problem is solved by functions *ode*2 and *ic*2 of wxMaxima. See *Solving second order ordinary differential equation* in Appendix A. The ordinary differential equation of this problem is,

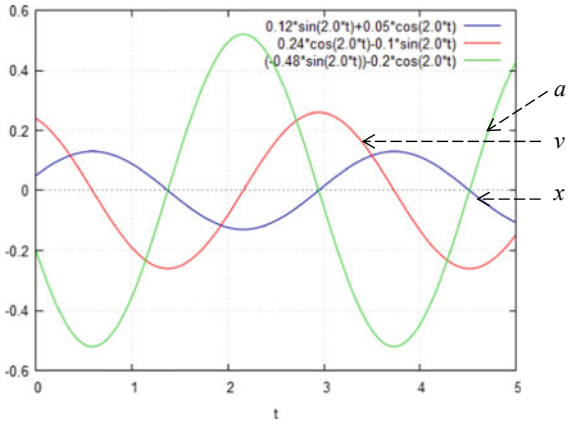
$$\frac{d^2x}{dt^2} = -\omega^2 x,$$

where $\omega = 2.0$ rad/s, x is displacement (dependent variable) in meter, and t is time (independent variable) in second. The initial conditions are at $t = 0$ s, $x_0 = 0.050$ m, and $v_0 = 0.24$ m/s.

- wxMaxima codes:

```
(%i5)  fpprintprec:5; ratprint:false; omega:2; x0:0.05; v0:0.24;
(fpprintprec)  5
(ratprint)  false
```

```
(omega)  2
(x0)  0.05
(v0)  0.24
(%i6)  sol: ode2('diff(x,t,2)=-omega^2*x, x, t);
(sol)  x=%k1*sin(2*t)+%k2*cos(2*t)
(%i8)  ic2(sol, t=0, x=x0, 'diff(x,t)=v0)$ float(%);
(%o8)  x=0.12*sin(2.0*t)+0.05*cos(2.0*t)
(%i9)  x: rhs(%);
(x)  0.12*sin(2.0*t)+0.05*cos(2.0*t)
(%i10) v: diff(x,t);
(v)  0.24*cos(2.0*t)-0.1*sin(2.0*t)
(%i11) a: diff(v,t);
(a)  -0.48*sin(2.0*t)-0.2*cos(2.0*t)
(%i12) wxplot2d([x,v,a], [t,0,5], grid2d);
```



Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of ω, x_0, and v_0.
(%i6) Get a general solution of $d^2x/dt^2 = -\omega^2 x$.
(%i8) Set initial conditions and get a particular solution.
(%o8) The solution.
(%i9), (%i10), (%i11) Assign the solution to x, calculate v and a.
(%i12) Plot x, v, and a against t for $0 \le t \le 5$ s.

According to the codes, the displacement, velocity, and acceleration of the object are,

$$x = 0.05\cos(2t) + 0.12\sin(2t) \text{ m},$$

$$v = 0.24\cos(2t) - 0.1\sin(2t)\ \text{m s}^{-1},$$
$$a = -0.2\cos(2t) - 0.48\sin(2t)\ \text{m s}^{-2}.$$

These displacement, velocity, and acceleration are the same as those in part (d). Further question: Show that,

$$0.13\cos(2t - 1.2) = 0.05\cos(2t) + 0.12\sin(2t).$$

Answer:

$$\begin{aligned} 0.13\cos(2t - 1.2) &= 0.13\ [\cos(2t)\cos(1.2) + \sin(2t)\sin(1.2)] \\ &= 0.05\cos(2t) + 0.12\sin(2t). \end{aligned}$$

- wxMaxima codes:

```
(%i2)  fpprintprec:5; ratprint:false;
(fpprintprec)  5
(ratprint)  false
(%i4)  0.13*(cos(2*t)*cos(1.2)+sin(2*t)*sin(1.2)); expand(%);
(%o3)  0.13*(0.93204*sin(2*t)+0.36236*cos(2*t))
(%o4)  0.12117*sin(2*t)+0.047107*cos(2*t)
```

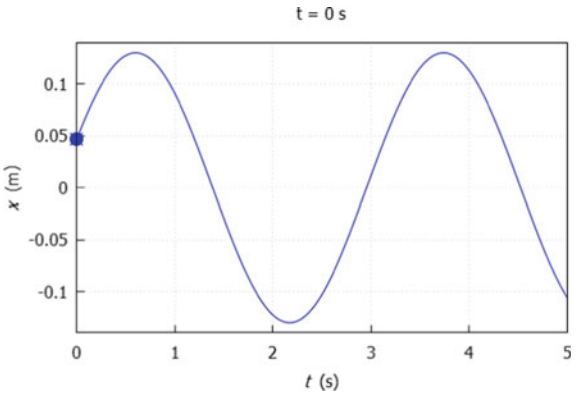
Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Calculate $0.13\ [\cos(2t)\cos(1.2) + \sin(2t)\sin(1.2)]$.

- Animation of the simple harmonic motion of the object, $x = 0.13\cos(2t - 1.2)$ m, by wxMaxima:

```
(%i1)  fpprintprec:3;
(fpprintprec)  3
(%i2)  with_slider_draw(
t, makelist(i,i,0,5,0.25),
title=concat("t = ",t," s"),
point_size=2, point_type=7,
points([0], [0.13*cos(2*t -1.2)]),
point_size=2, point_type=2,
points([t], [0.13*cos(2*t -1.2)]),
explicit(0.13*cos(2*x -1.2), x, 0, 5),
```

```
grid=true, yrange=[-0.14,0.14],
xlabel="{/Helvetica-Italic t}(s)",
ylabel="{/Helvetica-Italic x}(m)");
```



Comments on the codes:

To run the animation, copy the codes to the wxMaxima command window; press <shift> and <enter> keys simultaneously to run the codes; right click the graphic that appears and choose *Start Animation*. The large dot represents the object.

Problem 11.5 A particle of mass 3.0 kg oscillates according to a simple harmonic motion with displacement given by,

$$x = 5.0 \cos\left(\frac{\pi}{3}t - \frac{\pi}{4}\right),$$

where x is in meter and t in second. What are the values of,

(a) amplitude, A,
(b) period, T,
(c) frequency, f,
(d) phase constant, ϕ?

Solution

By comparing the equation with that of general displacement equation (Eq. 11.2), we have,

$$x = 5.0 \cos\left(\frac{\pi}{3}t - \frac{\pi}{4}\right),$$

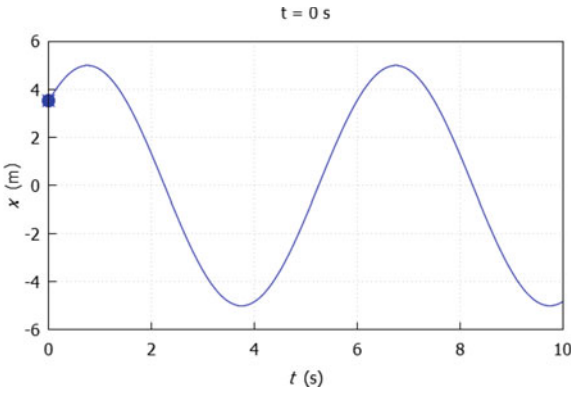
$$x = A\cos(\omega t + \phi).$$

We find that,

(a) the amplitude is $A = 5.0$ m,
(b) the angular frequency is $\omega = 2\pi/T = \pi/3$. Therefore, the period is $T = 6.0$ s,
(c) the frequency is $f = 1/T = 1/6$ Hz,
(d) the phase constant is $\phi = -\pi/4$ rad.

- Animation of the simple harmonic motion of the particle, $x = 5.0\cos\left(\frac{\pi}{3}t - \frac{\pi}{4}\right)$, by wxMaxima:

```
(%i1)  fpprintprec:3;
(fpprintprec)  3
(%i2)  with_slider_draw(
t, makelist(i,i,0,10,0.5),
title=concat("t = ",t," s"),
point_size=2, point_type=7,
points([0], [5*cos(%pi/3*t -%pi/4)]),
point_size=2, point_type=2,
points([t], [5*cos(%pi/3*t -%pi/4)]),
explicit(5*cos(%pi/3*x -%pi/4), x, 0, 10),
grid=true, yrange=[-6,6],
xlabel="{/Helvetica-Italic t} (s)",
ylabel="{/Helvetica-Italic x} (m)");
```



Comments on the codes:

To run the animation, copy the codes to the wxMaxima command window; press <shift> and <enter> keys simultaneously to run the codes; right click the graphic that appears and choose *Start Animation.* The big dot represents the particle.

Problem 11.6 An object of mass 2.0 kg is in simple harmonic motion with the displacement given by,

$$x = 3.0\cos(5t + 2),$$

where x is in meter and t in second. Calculate the,

(a) velocity, v, of the object as a function of time,
(b) maximum speed, v_{max}, of the object,
(c) maximum kinetic energy, K_{max}, of the object,
(d) total energy, E.

Plot the displacement, x, velocity, v, and kinetic energy, K, of the object against time, t.

Solution

(a) The velocity, v, of the object is the time derivative of the displacement, x,

$$v = \frac{dx}{dt} = -15\sin(5t + 2)\ \text{m s}^{-1}.$$

- wxMaxima codes:

```
(%i1)  x: 3*cos(5*t + 2);
(x)  3*cos(5*t+2)
(%i2)  v: diff(x,t);
(v)  -15*sin(5*t+2)
```

Comments on the codes:

(%i1) Define x.
(%i2) Calculate v.

(b) From the result of part (a), the maximum speed is $v_{\text{max}} = 15\ \text{m s}^{-1}$, because the maximum of a sine function is 1.
(c) The maximum kinetic energy, K_{max}, is at the maximum speed of the object,

$$K_{max} = \frac{1}{2}mv_{max}^2 = \frac{1}{2}(2.0\ \text{kg})(15\ \text{m/s})^2 = 225\ \text{J}.$$

(d) The total energy, E, is the sum of kinetic energy, K, and the potential energy, U, that is, $E = K + U$. When $K = K_{max}$, $U = 0$. The total energy, E, is,

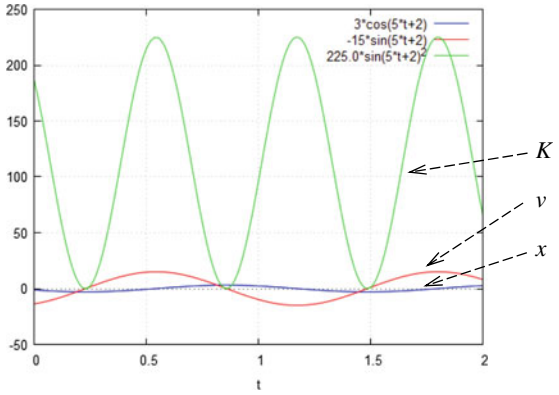
$$E = K_{\max} = 225\,\text{J}.$$

The displacement, x, velocity, v, and kinetic energy, K, of the object are,

$$x = 3\cos(5t + 2)\ \text{m},$$
$$v = -15\sin(5t + 2)\ \text{m s}^{-1},$$
$$K = \frac{1}{2}mv^2 = \frac{1}{2}(2.0\ \text{kg})(-15\ \text{m/s})^2\sin^2(5t + 2) = 225\sin^2(5t + 2)\ \text{J}.$$

- Plots of the displacement, x, velocity, v, and kinetic energy, K, againsts time, t, by wxMaxima:

```
(%i3)  x:3*cos(5*t+2); v:-15*sin(5*t+2);
K:0.5*2*(-15)^2*sin(5*t+2)*sin(5*t+2);
(x)  3*cos(5*t+2)
(v)  -15*sin(5*t+2)
(K)  225.0*sin(5*t+2)^2
(%i4)  wxplot2d([x,v,K], [t,0,2], grid2d);
```

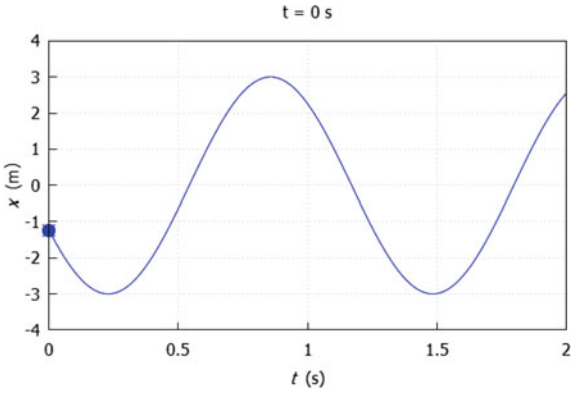


Comments on the codes:

(%i3) Define x, v, and K as functions of t.
(%i4) Plot x, v, and K against t for $0 \le t \le 2$ s with grids.

- Animation of the simple harmonic motion of the object, $x = 3.0\cos(5t + 2)$ m:

```
(%i1)  fpprintprec:3;
(fpprintprec)  3
(%i2)  with_slider_draw(
t, makelist(i,i,0,2,0.1),
title=concat("t = ",t," s"),
point_size=2, point_type=7,
points([0], [3*cos(5*t +2)]),
point_size=2, point_type=2,
points([t], [3*cos(5*t +2)]),
explicit(3*cos(5*x +2), x, 0, 2),
grid=true, yrange=[-4,4],
xlabel="{/Helvetica-Italic t} (s)",
ylabel="{/Helvetica-Italic x} (m)");
```



Comments on the codes:

To run the animation, copy the codes to the wxMaxima command window; press <shift> and <enter> keys simultaneously to run the codes; right click the graphic that appears and choose *Start Animation.* The large dot represents the object.

Problem 11.7 A block of mass 50 g attached to a spring with a force constant of 0.10 N m^{-1} is in a simple harmonic motion on a horizontal floor with negligible friction. The block is pulled by a distance of 10 cm from the equilibrium point and released from rest at $t = 0$ s. Determine the,

(a) displacement, x, velocity, v, and acceleration, a, of the block as fuctions of time, t, and plot their curves,

(b) maximum speed, v_{max}, and acceleration, a_{max},
(c) time, t, taken to reach $x = 5.0$ cm for the first time,
(d) displacement, x, velocity, v, and acceleration, a, of the block at $t = \frac{\pi}{3\sqrt{2}}$ s,
(e) force, F, acting on the block at positions $x = 10$ cm, $x = 5.0$ cm, and $x = 0$,
(f) potential energy, U, kinetic energy, K, and total energy, E, as functions of time.

Solution

(a) First, we calculate the angular frequency (Eq. 11.9),

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.10 \text{ N/m}}{50 \times 10^{-3} \text{ kg}}} = \sqrt{2} \text{ rad s}^{-1}.$$

The amplitude is $A = 0.10$ m. Thus, the displacement, x, is,

$$x = A \cos(\omega t + \phi) = 0.1 \cos(\sqrt{2t} + \phi) \text{ m}.$$

The first and second derivatives are the velocity, v, and the acceleration, a, of the block,

$$v = \frac{dx}{dt} = -0.1\sqrt{2} \sin(\sqrt{2t} + \phi) \text{ m s}^{-1},$$
$$a = \frac{dv}{dt} = -0.2 \cos(\sqrt{2t} + \phi) \text{ m s}^{-2}.$$

At $t = 0$, the initial conditions give,

$$x = 0.1 \text{ m} = 0.1 \cos \phi \text{ m},$$

$$v = 0 \text{ m/s} = -0.1\sqrt{2} \sin \phi \text{ m s}^{-1}.$$

The two equations give,

$$\tan \phi = 0,$$

and,

$$\phi = 0.$$

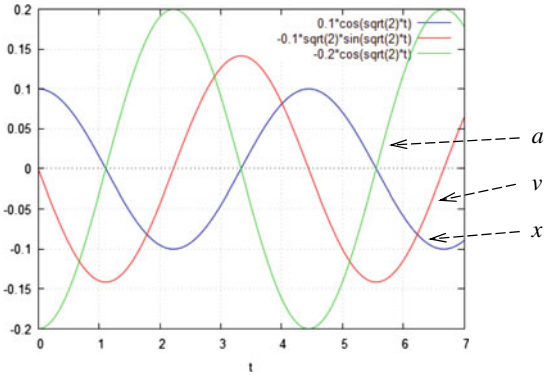
Therefore, the displacement, x, velocity, v, and acceleration, a, of the block are,

$$x = 0.1 \cos\left(\sqrt{2t}\right) \text{ m},$$
$$v = -0.1\sqrt{2} \sin\left(\sqrt{2t}\right) \text{ m s}^{-1},$$

$$a = -0.2\cos\left(\sqrt{2}t\right)\ \text{m s}^{-2}.$$

- Plots of the displacement, x, velocity, v, and acceleration, a, of the block as fuctions of time, t, by wxMaxima:

```
(%i3)  x:0.1*cos(sqrt(2)*t); v:-0.1*sqrt(2)*sin(sqrt(2)*t);
a:-0.2*cos(sqrt(2)*t);
(x)  0.1*cos(sqrt(2)*t)
(v)  -0.1*sqrt(2)*sin(sqrt(2)*t)
(a)  -0.2*cos(sqrt(2)*t)
(%i4)  wxplot2d([x,v,a], [t,0,7], grid2d);
```



Comments on the codes:

(%i3) Define x, v, and a as functions of t.
(%i4) Plot x, v, and a against t for $0 \le t \le 7$ s with grids.

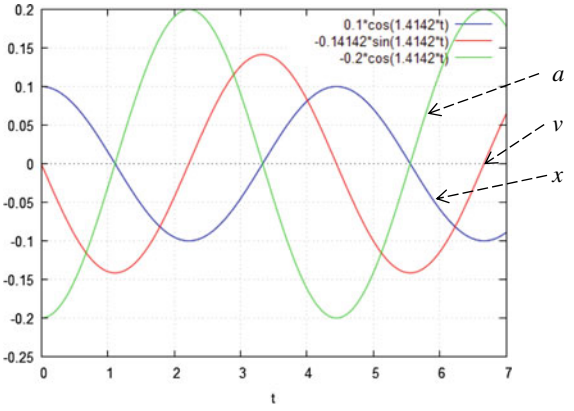
Alternative solution: This problem can be solved by functions *ode*2 and *ic*2 of wxMaxima. See *Solving second order ordinary differential equation* in Appendix A. The ordinary differential equation which is the SHM equation is,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$

where $\omega = \sqrt{2}$ rad s^{-1}, x is displacement (dependent variable) in meter, t is time (independent variable) in second, and the initial conditions are at $t = 0$ s, displacement is 0.10 m and velocity is zero.

- wxMaxima codes:

```
(%i3)  fpprintprec:5; ratprint:false; omega:sqrt(2);
(fpprintprec)  5
(ratprint)  false
(omega)  sqrt(2)
(%i4)  sol: ode2('diff(x,t,2)+omega^2*x=0, x, t);
(sol)  x=%k1*sin(sqrt(2)*t)+%k2*cos(sqrt(2)*t)
(%i6)  ic2(sol, t=0, x=0.1, 'diff(x,t)=0)$ float(%);
(%o6)  x=0.1*cos(1.4142*t)
(%i7)  x: rhs(%);
(x)  0.1*cos(1.4142*t)
(%i8)  v: diff(x,t);
(v)  -0.14142*sin(1.4142*t)
(%i9)  a: diff(v,t);
(a)  -0.2*cos(1.4142*t)
(%i10) wxplot2d([x,v,a], [t,0,7], grid2d);
```



Comments on the codes:

(%i3) Set floating point print precision to 5, internal rational number print to false, assign value of ω.
(%i4) Get a general solution of $d^2x/dt^2 + \omega^2 x = 0$.
(%i6) Set the initial conditions and get a particular solution.
(%o6) The solution.
(%i7), (%i8), (%i9) Assign the solution to x, calculate v and a.
(%i10) Plot x, v, and a against t for $0 \leq t \leq 7$ s.

The codes say that the displacement, x, velocity, v, and acceleration, a, of the block are,

$$\begin{aligned} x &= 0.1\ \cos\ (1.4142t)\ \text{m}, \\ v &= -1.4142\ \sin\ (1.4142t)\ \text{m s}^{-1}, \\ a &= -0.2\ \cos\ (1.4142t)\ \text{m s}^{-2}. \end{aligned}$$

(b) Using results of part (a), the values of maximum speed and acceleration are,

$$v_{max} = 0.1\sqrt{2}\ \text{m s}^{-1} = 0.14\ \text{m s}^{-1},$$

$$a_{max} = 0.2\ \text{m s}^{-2}.$$

(c) Using the displacement equation (Eq. 11.2), we can calculate t if x is known,

$$\begin{aligned} x &= 0.1\cos(\sqrt{2}t), \\ 5.0 \times 10^{-2}\ \text{m} &= (0.1\ \text{m})\ \cos(\sqrt{2}t), \\ \cos(\sqrt{2}t) &= 0.5, \\ \sqrt{2}t &= \cos^{-1}(0.5) = \frac{\pi}{3}, \\ t &= \frac{\pi}{3\sqrt{2}}\ \text{s} = 0.74\ \text{s}. \end{aligned}$$

Thus, the time taken to reach $x = 5.0$ cm for the first instance is 0.74 s.

- wxMaxima codes:

```
(%i2)  fpprintprec:5; ratprint:false;
(fpprintprec)  5
(ratprint)  false
(%i4)  solve(sqrt(2)*t=acos(0.5), t)$ float(%);
(%o4)  [t=0.74048]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $\sqrt{2}t = \cos^{-1}(0.5)$ for t.

(d) Using results of part (a), displacement, velocity, and acceleration at $t = \frac{\pi}{3\sqrt{2}}$ s are,

$$x = 0.1\cos(\sqrt{2}t) = 0.1\cos(\sqrt{2}\cdot\frac{\pi}{3\sqrt{2}}) = 0.05 \text{ m},$$

$$v = -0.1\sqrt{2}\sin(\sqrt{2}t) = -0.1\sqrt{2}\sin(\sqrt{2}\cdot\frac{\pi}{3\sqrt{2}}) = -0.12 \text{ m s}^{-1},$$

$$a = -0.2\cos(\sqrt{2}t) = -0.2\cos(\sqrt{2}t\cdot\frac{\pi}{3\sqrt{2}}) = -0.10 \text{ m s}^{-2}.$$

- wxMaxima codes:

```
(%i1)  fpprintprec:5;
(fpprintprec)  5
(%i2)  x(t):=0.1*cos(sqrt(2)*t);
(%o2)  x(t):=0.1*cos(sqrt(2)*t)
(%i3)  v(t):="(diff(x(t), t));
(%o3)  v(t):=-0.1*sqrt(2)*sin(sqrt(2)*t)
(%i4)  a(t):="(diff(v(t), t));
(%o4)  a(t):=-0.2*cos(sqrt(2)*t)
(%i5)  t: %pi/(3*sqrt(2));
(t)  %pi/(3*sqrt(2))
(%i7)  x(t)$ float(%);
(%o7)  0.05
(%i9)  v(t)$ float(%);
(%o9)  -0.12247
(%i11) a(t)$ float(%);
(%o11) -0.1
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2), (%i3), (%i4) Define $x(t)$, $v(t)$, and $a(t)$.
(%i5) Assign t.
(%i7), (%i9), (%i11) Calculate $x(t)$, $v(t)$, and $a(t)$.

- Alternative wxMaxima calculation:

```
(%i1)  fpprintprec:5;
(fpprintprec)  5
(%i2)  x: 0.1*cos(sqrt(2)*t);
(x)  0.1*cos(sqrt(2)*t)
(%i3)  v:diff(x, t);
(v)  -0.1*sqrt(2)*sin(sqrt(2)*t)
(%i4)  a: diff(v, t);
(a)  -0.2*cos(sqrt(2)*t)
```

```
(%i6)  subst(t=%pi/(3*sqrt(2)), [x,v,a])$ float(%);
(%o6)  [0.05,-0.12247,-0.1]
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2), (%i3), (%i4) Assign x, v, and a as functions of t.
(%i6) Substitute t to get x, v, and a.

(e) The force that acts on the block is $F = -kx$. Thus,

$$\text{at } x = 10 \text{ cm}, F = -(0.10 \text{ N/m})(10 \times 10^{-2} \text{ m}) = 0.010 \text{ N},$$
$$\text{at } x = 5.0 \text{ cm}, F = -(0.10 \text{ N/m})(5.0 \times 10^{2} \text{ m}) = 0.0050 \text{ N},$$
$$\text{at } x = 0 \text{ cm}, F = 0 \text{ N}.$$

(f) The elastic potential energy is (Eq. 11.11),

$$U = \frac{1}{2}kx^2 = \frac{1}{2}(0.10 \text{ N/m})(0.10)^2\cos^2(\sqrt{2}t) \text{ m}^2$$
$$= 5.0 \times 10^{-4}\cos^2(\sqrt{2}t) \text{ J}.$$

The kinetic energy is (Eq. 11.10),

$$K = \frac{1}{2}mv^2 = \frac{1}{2}(50 \times 10^{-3}\text{kg})(0.10)^2(2)\sin^2(\sqrt{2}t) \text{ m}^2/\text{s}^2$$
$$= 5.0 \times 10^{-4}\sin^2(\sqrt{2}t) \text{ J}.$$

The total energy is (Eq. 11.12),

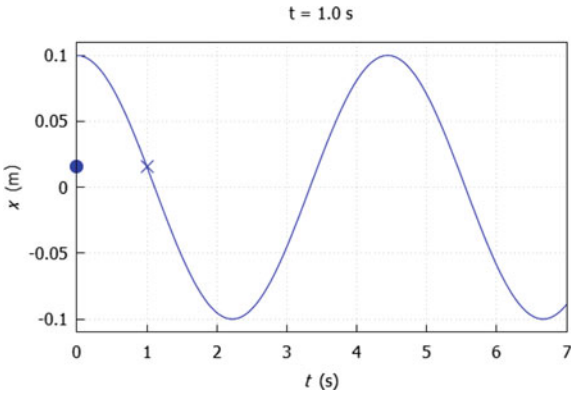
$$E = U + K = 5.0 \times 10^{-4}[\cos^2(\sqrt{2}t) + \sin^2(\sqrt{2}t)]$$
$$= 5.0 \times 10^{-4} \text{ J}.$$

The because $\cos^2(\sqrt{2}t) + \sin^2(\sqrt{2}t) = 1$.

- Animation of the simple harmonic motion of the block, $x = 0.1 \cos(\sqrt{2}t)$ m, by wxMaxima:

```
(%i1)  fpprintprec:3;
(fpprintprec)  3
(%i2)  with_slider_draw(
t, makelist(i,i,0,7,0.5),
title=concat("t = ",t," s"),
```

```
point_size=2, point_type=7,
points([0], [0.1*cos(sqrt(2)*t)]),
point_size=2, point_type=2,
points([t], [0.1*cos(sqrt(2)*t)]),
explicit(0.1*cos(sqrt(2)*x), x, 0, 7),
grid=true, yrange=[-0.11,0.11],
xlabel="{/Helvetica-Italic t} (s)",
ylabel="{/Helvetica-Italic x} (m)");
```



Comments on the codes:

To run the animation, copy the codes to the wxMaxima command window; press <shift> and <enter> keys simultaneously to run the codes; right click the graphic that appears and choose *Start Animation*. The large dot represents the block.

Problem 11.8 A meter stick is pivoted at one of its end and is free to oscillate in simple harmonic motion about the pivot. Calculate its period of oscillation, T.

Solution

The oscillation period of a physical pendulum is (Eq. 11.13),

$$T = 2\pi\sqrt{\frac{I}{Mgd}},$$

where I is the moment of inertia of the pendulum about the pivot, d is the distance between the center of mass of the pendulum and the pivot, M is the pendulum mass, and g is the acceleration of gravity.

For this problem, the mass of the stick is M, length is $l = 1.00$ m, and the distance between the center of mass (CM) and the pivot is $d = l/2 = 0.50$ m (Fig. 11.4).

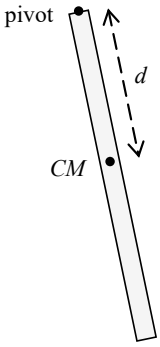


Fig. 11.4 A meter stick pendulum, Problem 11.8

For the stick, the moment of inertia about its center of mass is (Fig. 9.1a),

$$I_{CM} = \frac{1}{12} M l^2.$$

Applying the parallel axes theorem, the moment of inertia about the pivot is (Fig. 9.1j),

$$I = I_{CM} + Md^2 = \frac{1}{12} M l^2 + M\left(\frac{l}{2}\right)^2 = \frac{M l^2}{3}.$$

The period, T, of oscillations of the meter stick pendulum is,

$$\begin{aligned} T &= 2\pi \sqrt{\frac{I}{Mgd}} = 2\pi \sqrt{\frac{Ml^2/3}{Mg(l/2)}} = 2\pi \sqrt{\frac{2l}{3g}} \\ &= 2\pi \sqrt{\frac{2(1.00\ \text{m})}{3(9.8\ \text{m/s}^2)}} \\ &= 1.64\ \text{s}. \end{aligned}$$

- wxMaxima codes:

```
(%i1)  fpprintprec:5;
(fpprintprec)  5
(%i5)  d:l/2; ICM:(1/12)*M*l^2; I:ICM+M*d^2; T:2*%pi*sqrt(I/
(M*g*d));
(d)  l/2
(ICM)  (M*l^2)/12
(I)  (M*l^2)/3
(T)  (2^(3/2)*%pi*sqrt(l/g))/sqrt(3)
```

```
(%i7)  subst([l=1, g=9.8], T)$ float(%);
(%o7)  1.6388
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i5) Assign d, I_{CM}, I, and T.
(%i7) Substitute values of l and g and calculate T.

Problem 11.9 An object of mass 36 g oscillates in a simple harmonic motion along the x axis with the origin as the equilibrium point. The amplitude, A, and period, T, of the oscillations are 13 cm and 12 s, respectively. At $t = 0$ its displacement is 13 cm and velocity is zero.

(a) What is the object speed, v, at $x = 5.0$ cm?
(b) Determine the equation for the displacement, x, and plot the displacement, x, against time, t.
(c) What is the force, F, acting on the object at $t = 2.0$ s?
(d) Calculate the shortest time taken by the object to move from $x = 6.5$ cm to $x = -6.5$ cm.

Solution

(a) First, we calculate the angular frequency from the period of the oscillations (Eq. 11.9),

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{12} \text{ rad s}^{-1} = \frac{\pi}{6} \text{ rad s}^{-1}.$$

The speed of the object at $x = 5.0$ cm is (Eq. 11.6),

$$v = \omega\sqrt{A^2 - x^2} = \frac{\pi}{6}\sqrt{13^2 - 5.0^2} \text{ cm s}^{-1} = 6.3 \text{ cm s}^{-1}.$$

(b) Let the expression for the displacement be (Eq. 11.2),

$$x = A\cos(\omega t + \phi).$$

Differentiating this with respect to time gives the velocity of the object,

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi).$$

At $t = 0$, the equations for displacement and velocity become,

$$13 = A\cos\phi,$$

$$0 = -\omega A \sin\phi.$$

These two equations give,

$$\tan\phi = 0.$$

The initial phase constant is,

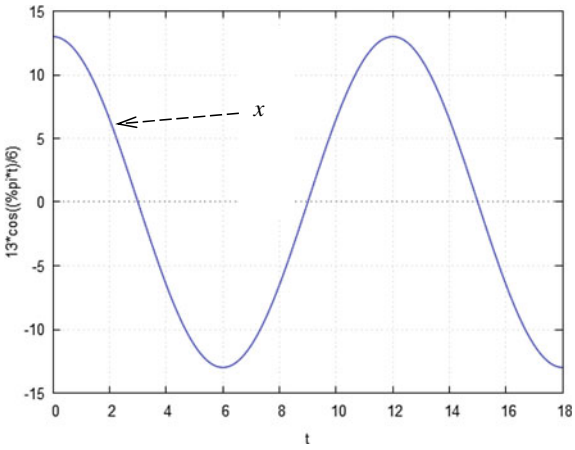
$$\phi = 0.$$

The equation of the displacement of the object is,

$$x = 13\cos(\frac{\pi}{6}t)\,\text{cm} = 13\cos(0.52t)\text{cm}.$$

- Plot of displacement against time by wxMaxima:

```
(%i1)  x: 13*cos(%pi/6*t);
(x)  13*cos((%pi*t)/6)
(%i2)  wxplot2d(x, [t,0,18], grid2d);
```



Comments on the codes:

(%i1) Define x in terms of t.
(%i2) Plot x against t for $0 \leq t \leq 18$ s with grids.

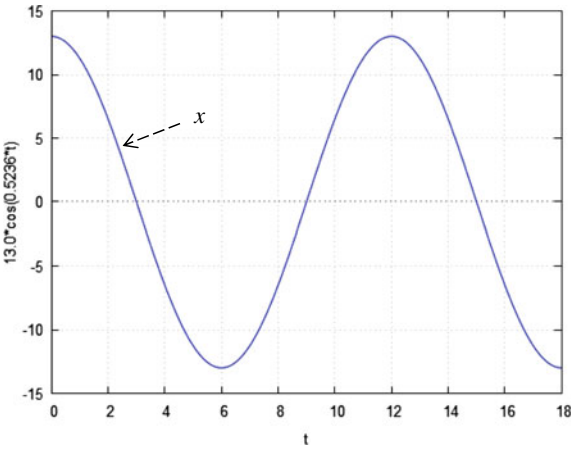
Alternative solution: The simple harmonic motion equation (Eq. 11.1) is a second order ordinary differential equation. To solve the equation, functions *ode*2 and *ic*2 of wxMaxima can be used. See *Solving second order ordinary differential equation* in Appendix A. The related ordinary differential equation is,

$$\frac{d^2x}{dt^2} = -\omega^2 x,$$

where $\omega = \pi/6$ rad/s, x is displacement (dependent variable) in cm, t is time (independent variable) in second, and at $t = 0$ s, displacement is 13 cm and velocity is zero.

- wxMaxima codes:

```
(%i3)  fpprintprec:5; ratprint:false; omega:%pi/6;
(fpprintprec)  5
(ratprint)  false
(omega)  %pi/6
(%i4)  soln: ode2('diff(x,t,2)=-omega^2*x, x, t);
(soln) x=%k1*sin((%pi*t)/6)+%k2*cos((%pi*t)/6)
(%i6)  ic2(soln, t=0, x=13, 'diff(x,t)=0); float(%);
(%o5)  x=13*cos((%pi*t)/6)
(%o6)  x=13.0*cos(0.5236*t)
(%i7)  x: rhs(%);
(x)  13.0*cos(0.5236*t)
(%i8)  wxplot2d(x, [t,0,18], grid2d);
```



Comments on the codes:

(%i3) Set floating point print precision to 5, internal rational number print to false, assign value of ω.
(%i4) Get a general solution of $d^2x/dt^2 = -\omega^2 x$.
(%i6) Set the initial conditions and get a particular solution.
(%o5), (%o6) The solution.
(%i7) Assign the solution to x.
(%8) Plot x against t for $0 \le t \le 18$ s.

The codes say that the displacement of the object is,

$$x = 13\cos(\frac{\pi}{6}t) = 13\cos(0.5236t)\text{ cm}.$$

(c) At $t = 2.0$ s, the displacement, x, and the force, F, on the object are,

$$\begin{aligned} x &= (13\text{ cm})\cos(\frac{\pi}{6}\text{ rad/s} \times 2.0\text{ s}) = (13\text{ cm})\cos(\frac{\pi}{3}) = 6.5\text{ cm}, \\ F &= ma = -m\omega^2 x = -(36 \times 10^{-3}\text{ kg})(\frac{\pi}{6}\text{ rad/s})^2(6.5 \times 10^{-2}\text{ m}) \\ &= -6.4 \times 10^{-4}\text{ N}. \end{aligned}$$

(d) Using the equation of displacement, we have,

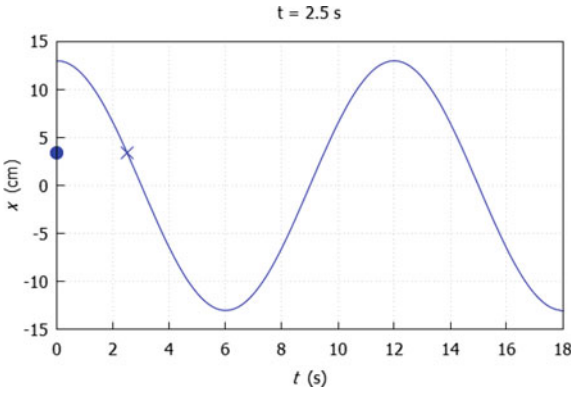
$$\begin{aligned} x &= 13\cos(\frac{\pi}{6}t), \\ 6.5\text{ cm} &= (13\text{ cm})\cos(\frac{\pi}{6}t_1) \Rightarrow t_1 = 2.0\text{ s}, \\ -6.5\text{ cm} &= (13\text{ cm})\cos(\frac{\pi}{6}t_2) \Rightarrow t_2 = 4.0\text{ s}, \\ t_2 - t_1 &= 2.0\text{ s}. \end{aligned}$$

The shortest time taken by the object to move from $x = 6.5$ cm to $x = -6.5$ cm is 2.0 s.

- Animation of the simple harmonic motion of the object, $x = 13\cos(\frac{\pi}{6}t)$ cm, by wxMaxima:

```
(%i1)  fpprintprec:3;
(fpprintprec)  3
(%i2)  with_slider_draw(
t, makelist(i,i,0,18,0.5),
title=concat("t = ",t," s"),
point_size=2, point_type=7,
points([0], [13*cos(%pi/6*t)]),
```

```
    point_size=2, point_type=2,
    points([t], [13*cos(%pi/6*t)]),
    explicit(13*cos(%pi/6*x), x, 0, 18),
    grid=true, yrange=[-15,15],
    xlabel="{/Helvetica-Italic t} (s)",
    ylabel="{/Helvetica-Italic x} (cm)");
```



Comments on the codes:

To run the animation, copy the codes to the wxMaxima command window; press <shift> and <enter> keys simultaneously to run the codes; right click the graphic that appears and choose *Start Animation*. The large dot represents the object.

Problem 11.10 A mass-spring system oscillates in a simple harmonic motion with ampitude A and total energy E.

- (a) Plot the curves of kinetic, potential, and total energies against the displacement
- (b) Calculate the kinetic and potential energies when the displacement are 0, $A/2$, and A.
- (c) What is the displacement when the kinetic energy becomes equal to the potential energy?
- (d) What is the displacement when the kinetic energy is twice the potential energy?

Solution

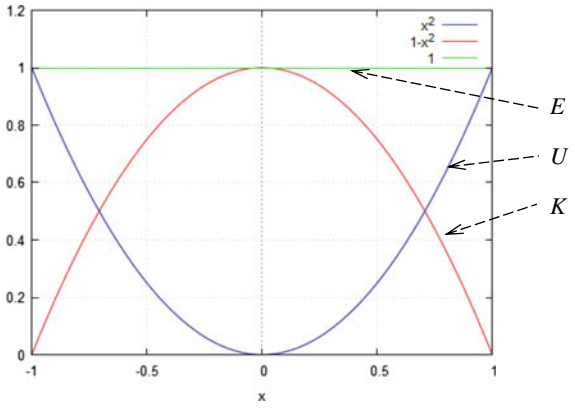
(a) The kinetic, potential, and total energies as functions of displacement of a mass-spring system are given by Eqs. (11.10), (11.11), and (11.12), respectively,

$$K = \frac{1}{2}k(A^2 - x^2),$$

$$U = \frac{1}{2}kx^2,$$
$$E = K + U = \frac{1}{2}kA^2.$$

- Curves of K, U, and E against x plotted by wxMaxima:

```
(%i2)  k:2; A:1;
(k)  2
(a) 1
(%i5)  U:1/2*k*x^2; K:1/2*k*(A^2-x^2); E:U + K;
(U)  x^2
(K)  1-x^2
(E) 1
(%i6)  wxplot2d([U,K,E], [x,-1,1], [y,0,1.2], grid2d);
```



Comments on the codes:

(%i2) Assign values of $k = 2$ and $A = 1$.
(%i5) Define U, K, and E in terms of x.
(%i6) Plot U, K, and E against x for $0 \le x \le 1$ with grid.

(b) At $x = 0$, the kinetic and potential energies are,

$$K = \frac{1}{2}k(A^2 - x^2) = \frac{1}{2}kA^2 = E,$$
$$U = 0.$$

At $x = A/2$, the kinetic and potential energies are,

$$K = \frac{1}{2}k(A^2 - x^2) = \frac{1}{2}k(A^2 - \frac{A^2}{4}) = \frac{3}{8}kA^2 = \frac{3}{4}E,$$
$$U = \frac{1}{2}kx^2 = \frac{1}{2}k\frac{A^2}{4} = \frac{1}{8}kA^2 = \frac{1}{4}E.$$

At $x = A$, the kinetic and potential energies are,

$$K = 0,$$
$$U = \frac{1}{2}kA^2 = E.$$

(c) When the kinetic energy is equal to the potential energy, the displacements are,

$$K = U,$$
$$\frac{1}{2}k(A^2 - x^2) = \frac{1}{2}kx^2,$$
$$x = \pm\frac{A}{\sqrt{2}}.$$

(d) When the kinetic energy is twice the potential energy, the displacements are,

$$K = 2U,$$
$$\frac{1}{2}k(A^2 - x^2) = kx^2,$$
$$x = \pm\frac{A}{\sqrt{3}}.$$

Problem 11.11 The period, T, of a simple pendulum is 2.5 s. Calculate the length, l, of the string.

Solution

The length, l, of the string of a simple pendulum is calculated as follows (Eq. 11.3),

$$T = 2\pi\sqrt{\frac{l}{g}},$$
$$2.5\text{ s} = 2\pi\sqrt{\frac{l}{9.8\text{ m/s}^2}},$$
$$l = 1.6\text{ m}.$$

- wxMaxima codes:

```
(%i4)  fpprintprec:5; ratprint:false; T:2.5; g:9.8;
(fpprintprec)  5
(ratprint)  false
(t) 2.5
(g)  9.8
(%i6)  solve(T=2*%pi*sqrt(l/g), l)$ float(%);
(%o6)  [l=1.5515]
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of T and g.
(%i6) Solve $T = 2\pi\sqrt{\frac{l}{g}}$ for l.

Problem 11.12 Figure 11.5 shows a displacement, x, versus time, t, curve of an object undergoing simple harmonic motion. What are the signs (positive or negative) of velocity, v, and acceleration, a, at time t_1 and t_2?

Solution

At time t_1, the velocity of the object is negative (the gradient of the x-t curve is negative). The acceleration of the object is negative (the acceleration for SHM is proportional to the negative of the displacement). The curve is concave down meaning that the gradient is getting more negative and hence negative acceleration.

At time t_2, the velocity of the object is negative (the gradient of the x-t curve is negative). The acceleration of the object is positive (the acceleration for SHM is proportional to the negative of the displacement). The curve is concave up meaning that the gradient is getting less negative and hence positive acceleration. Table 11.1 summarizes the results.

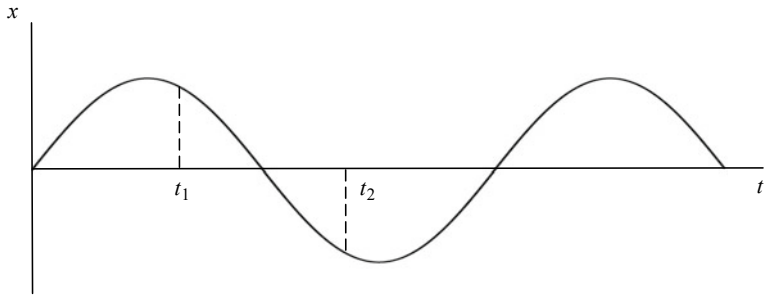


Fig. 11.5 Displacement against time curve of a simple harmonic motion, Problem 11.12

Table 11.1 Signs of displacement, velocity, and acceleration at two different times

Time, t	Displacement, x	Velocity, v	Acceleration, a
t_1	+	−	−
t_2	−	−	+

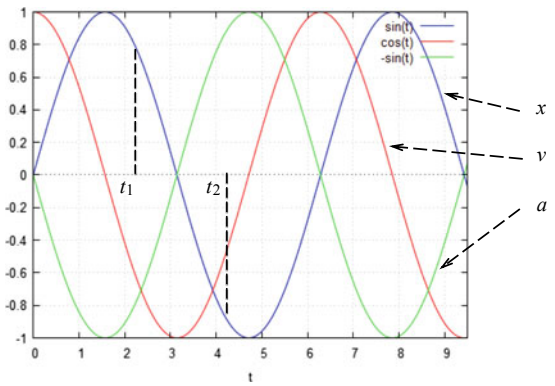
Let us verify this by doing some plots. Take the displacement, velocity, and acceleration of the object to be,

$$x = A\sin(\omega t) = \sin t,$$
$$v = A\omega\cos(\omega t) = \cos t,$$
$$a = -\omega^2 x = -\sin t,$$

where we set $A = 1$ and $\omega = 1$, to make the plots simple.

- Plots of displacement, velocity, and acceleration against time by wxMaxima:

```
(%i2)  A:1; omega:1;
(a) 1
(omega)  1
(%i5)    x: A*sin(omega*t); v: A*omega*cos(omega*t); a: -
omega^2*x;
(x)  sin(t)
(v)  cos(t)
(a)  -sin(t)
(%i6)   wxplot2d([x,v,a], [t,0,9.5], grid2d);
```



We can see clearly that at t_1, displacement is positive, velocity is negative, and acceleration is negative. At t_2, displacement is negative, velocity is negative, and acceleration is positive.

11.3 Summary

- A simple harmonic motion (SHM) is a motion with acceleration, a, directly proportional to its displacement, x, and directed to a fixed point called the equilibrium point. The equation of SHM is,

$$a = -\omega^2 x \quad \text{or} \quad \frac{d^2x}{dt^2} = -\omega^2 x,$$

  where ω is the angular frequency. To get a SHM, a force that is directly proportional and
  opposite in direction to the displacement called a restoring force is needed.
- A general solution of the SHM equation for a mass-spring system is,

$$x = A\cos(\omega t + \phi),$$

  where A is the amplitude of the oscillations, $\omega = \sqrt{k/m}$ is the angular frequency, and ϕ is the phase constant. The period of the oscillations is $T = 2\pi/\omega = 2\pi\sqrt{m/k}$.
- The velocity and acceleration of the vibrating mass are,

$$v = \frac{dx}{dt} = -A\omega\sin(\omega t + \varphi), \quad v_{\max} = A\omega,$$
$$a = \frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \varphi), \quad a_{\max} = A\omega^2.$$

- The potential energy, kinetic energy, and total energy are,

$$U = \frac{1}{2}kx^2,$$
$$K = \frac{1}{2}k(A^2 - x^2),$$
$$E = U + K = \frac{1}{2}kA^2.$$

11.4 Exercises

Exercise 11.1

(a) A spring stretches by 4.0 cm when a mass of 0.050 kg is hung on it. What is the force constant of the spring?
(b) A mass of 0.15 kg is hung from the spring and vertical oscillations are started. What is the period of the oscillations?

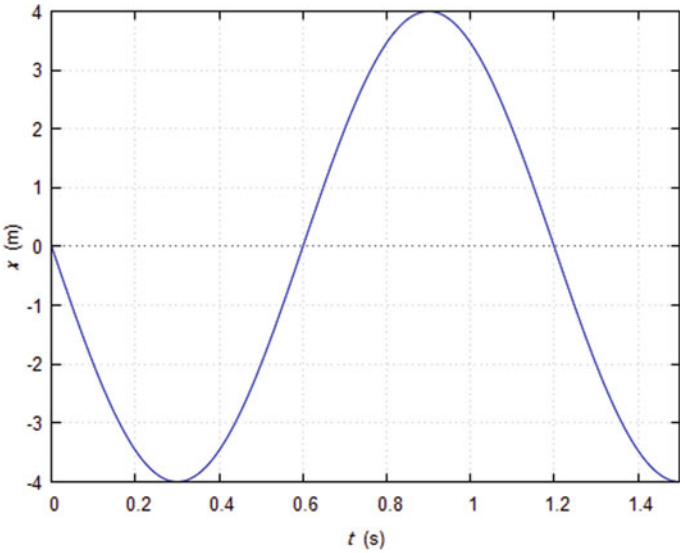


Fig. 11.6 Displacement x against time t curve of a simple harmonic motion, Exercise 11.2

(Answer: (a) 12 N m^{-1} (b) 0.70 s)

Exercise 11.2 Figure 11.6 shows the displacement, x, against time, t, curve of a simple harmonic oscillations.

(a) Find amplitude, A, period, T, and frequency, f, of the oscillations.
(b) Write an equation of the form $x = A\cos(\omega t + \phi)$ for the oscillations.

(Answer: (a) $A = 4.0$ m, $T = 1.2$ s, $f = 0.83$ Hz (b) $x = 4.0\cos(5.2t + \pi/2)$)

Exercise 11.3 The equation of the displacement of an object in a simple harmonic motion is,

$$x = 0.30\cos(0.50t),$$

where x is in meter and t in second. Calculate the displacement, velocity, and acceleration of the object at $t = 3.0$ s.

(Answer: $x = 0.021$ m, $v = -0.150$ m s^{-1}, $a = -0.0053$ m s^{-2})

Exercise 11.4 A meterstick is hung as a compound pendulum with the pivot at its 90 cm mark. Calculate the period of oscillation of the pendulum.

(Answer: 1.57 s)

Exercise 11.5 Figure 11.7 shows a rectangular block of wood of mass, m, floating in a liquid of density, ρ. The area of bottom face of the block is A. The block is

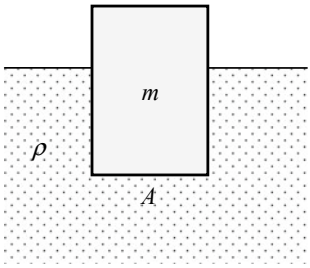


Fig. 11.7 A block of wood floating in a liquid, Exercise 11.5

depressed down a bit and let to oscillate up and down in the liquid. Show that the period of oscillations is,

$$T = \frac{2\pi}{\sqrt{\rho A g/m}},$$

where g is the acceleration of gravity.

Chapter 12
General Laws of Gravity

12.1 Basic Concepts and Formulae

(1) Newton's universal law of gravity states that the attractive gravitational force between two particles of masses m_1 and m_2 separated by a distance, r, is,

$$F = G\frac{m_1 m_2}{r^2}, \tag{12.1}$$

where $G = 6.672 \times 10^{-11}$ N m^2 kg^{-2}is the universal gravitational constant.

(2) Kepler's planetary laws state that,

i. All planets move in elliptical orbits with the sun as one of the focal points.

ii. The radial vector from the sun to the planet sweeps an equal area in equal time intervals.

iii. The square of the orbital period of a planet is directly proportional to the cube of semi major axis of its elliptical orbit.

The Kepler's second law (ii) is a consequence of gravity is a central force, that is, the force that is directed to a fixed point. This implies the conservation of angular momentum of the planet-sun system. The Kepler's third law (iii) is consistent with the Newton's law of gravity. The Newton's second law of motion and the Newton's law of universal gravity can be used to show that the period, T, and orbital radius, r, of a planet around the sun is according to,

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right)r^3, \tag{12.2}$$

where M_s is the mass of the sun. Orbits of most planets are almost a circle. For planets with elliptical orbits, the equation is still applicable if the radius, r, is replaced by the semi major axis, a, of the ellipse.

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W. M. S. Wan Hassan, *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-42678-0_12

(3) Because gravity is a conservative force, a potential energy function, U, can be defined. The gravitational potential energy of two particles with masses m_1 and m_2 separated by a distance, r, is,

$$U = -G\frac{m_1 m_2}{r}, \tag{12.3}$$

where $U \to 0$ as $r \to \infty$. The total potential energy of a number of particles is obtained by summing the potential energy of each pair of particles.

(4) For an isolated system consisting of two particles one of mass m and the other M, where $M >> m$, the total energy of the system is,

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}, \tag{12.4}$$

where M is assumed to be at rest, while m moves at a speed of v around M. The first term is the kinetic energy of the small mass and the second term is the gravitational potential energy of both masses. The total energy is a constant. If m moves in circular orbit of radius, r, about M, the total energy of the system is,

$$E = -\frac{GMm}{2r}. \tag{12.5}$$

The total energy is negative. The total energy is negative for all bound system, that is, the system with a closed orbit.

(5) The potential energy of gravitation attraction between a point mass m and a body of mass M is,

$$U = -Gm\int \frac{dM}{r}, \tag{12.6}$$

where dM is the elementary mass of the body and r is the distance between the point and the elementary masses.

(6) For a particle that is located outside a uniform spherical shell or a solid sphere with spherically symmetric distribution of mass, the sphere attracts the particle as if the mass of the sphere is centered at the center of the sphere.

(7) The gravitational force between a particle inside a uniform spherical shell and the spherical shell itself is zero.

(8) The gravitational force on a particle in a homogenous solid sphere is directly proportional to the distance of the particle from the center of the sphere and points to the center of the sphere,

$$F = \frac{GMm}{R^3}r, \qquad r < R, \tag{12.7}$$

where m is the mass of the particle, M is the mass of the sphere, R is the radius of the sphere, and r is the distance between the particle and the center of the sphere.

(9) The gravitational field at a distance of R from a body of mass M is,

$$g = \frac{GM}{R^2}. \tag{12.8}$$

12.2 Problems and Solutions

Problem 12.1 Two particles of masses 100 kg and 200 kg are separated by a distance of 5.0 m (Fig. 12.1).

(a) Calculate the gravitational force on the particle of mass 100 kg by the particle of mass 200 kg.
(b) A third particle of mass 300 kg is placed at point P. What is the net gravitational force on the third particle?

Solution

(a) The gravitational force on the particle of mass 100 kg by the particle of mass 200 kg is (Eq. 12.1),

$$F = G\frac{m_1 m_2}{r^2} = \left(6.67 \times 10^{-11}\frac{\text{N m}^2}{\text{kg}^2}\right)\frac{(100\,\text{kg})(200\,\text{kg})}{(5.0\,\text{m})^2} = 5.3 \times 10^{-8}\text{N}.$$

The force, F, is to the right and is shown in Fig. 12.2a.

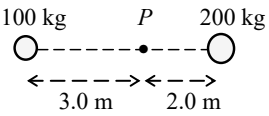


Fig. 12.1 Two particles separated by a distance of 5.0 m, Problem 12.1

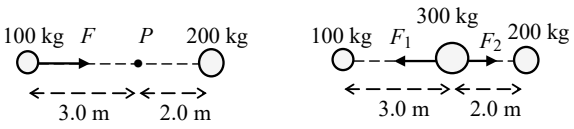


Fig. 12.2 **a** Two particles separated by a distance of 5.0 m, **b** a third particle is placed at point P, Problem 12.1

- wxMaxima codes:

```
(%i4) fpprintprec:5; G:6.67e-11; m1:100; m2:200;
(fpprintprec) 5
(G) 6.67*10^-11
(m1) 100
(m2) 200
(%i5) F: G*m1*m2/5^2;
(F) 5.336*10^-8
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of G, m_1, and m_2.
(%i5) Calculate F.

(b) The force on the particle of mass 300 kg by the particle of mass 100 kg is (Fig. 12.2b),

$$F_1 = \left(6.67 \times 10^{-11}\ \frac{\text{N m}^2}{\text{kg}^2}\right)\frac{(100\ \text{kg})(300\ \text{kg})}{(3.0\ \text{m})^2} = 2.2 \times 10^{-7}\ \text{N}, \quad \text{to the left.}$$

The force on the particle of mass 300 kg by the particle of mass 200 kg is,

$$F_2 = \left(6.67 \times 10^{-11}\ \frac{\text{N m}^2}{\text{kg}^2}\right)\frac{(300\ \text{kg})(200\ \text{kg})}{(2.0\ \text{m})^2} = 1.0 \times 10^{-6}\ \text{N}, \quad \text{to the left.}$$

The net gravitational force on the particle of mass 300 kg is,

$$F_2 - F_1 = 1.0 \times 10^{-6}\text{N} - 2.2 \times 10^{-7}\,\text{N} = 7.8 \times 10^{-7}\,\text{N}, \quad \text{to the left.}$$

- wxMaxima codes:

```
(%i5) fpprintprec:5; G:6.67e-11; m1:100; m2:200; m3:300;
(fpprintprec) 5
(G) 6.67*10^-11
(m1) 100
(m2) 200
(m3) 300
(%i6) F1: G*m1*m3/3^2;
(F1) 2.2233*10^-7
(%i7) F2: G*m3*m2/2^2;
(F2) 1.0005*10^-6
(%i8) F2-F1;
```

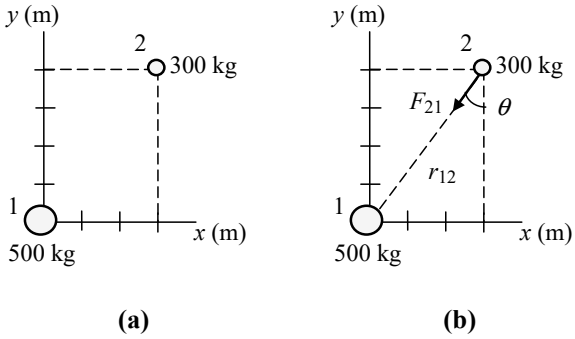


Fig. 12.3 **a** Two particles, one at the origin, the other at (3, 4) m, **b** determining gravitational force on the particle, Problem 12.2

```
(%o8) 7.7817*10^-7
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of G, m_1, m_2, and m_3.
(%i6), (%i7) Calculate F_1 and F_2.
(%i8) Calculate $F_2 - F_1$.

Problem 12.2 A particle of mass 500 kg (1) is located at the origin and another particle of mass 300 kg (2) is located at (3, 4) m (Fig. 12.3a). What is the gravitational force on the particle of mass 300 kg?

Solution

Figure 12.3b shows the gravitational force, F_{21}, on particle 2 due to particle 1. The universal gravitational law in vector form is (Eq. 12.1),

$$\boldsymbol{F}_{21} = -G\frac{m_1 m_2}{r_{12}^2}\hat{\boldsymbol{r}}_{12},$$

where $\boldsymbol{F}_{21}$ is the force on the particle 2 by the particle 1, r_{12} is the distance between the particles, $\hat{\boldsymbol{r}}_{12}$ is the unit vector from the particle 1 to particle 2, while m_1 and m_2 are masses of particle 1 and 2, respectively. For this problem,

$$\begin{aligned}
\boldsymbol{r}_{12} &= 3\mathbf{i} + 4\mathbf{j}\ \text{m},\\
r_{12}^2 &= (3\,\text{m})^2 + (4\,\text{m})^2 = 25\,\text{m}^2,\\
r_{12} &= 5\ \text{m},\\
\hat{\boldsymbol{r}}_{12} &= \frac{\boldsymbol{r}_{12}}{r_{12}} = \frac{1}{5}(3\mathbf{i} + 4\mathbf{j}).
\end{aligned}$$

The gravitational force on the particle of mass 300 kg is,

$$\begin{aligned}\boldsymbol{F}_{21} &= -\left(6.67\times10^{-11}\frac{\mathrm{N\,m^2}}{\mathrm{kg^2}}\right)\frac{(500\,\mathrm{kg})(300\,\mathrm{kg})}{25\,\mathrm{m^2}}\left(\frac{1}{5}\right)(3\,\mathbf{i}+4\,\mathbf{j})\\ &= -2.4\times10^{-7}\mathbf{i}-3.2\times10^{-7}\mathbf{j}\,\mathrm{N}.\end{aligned}$$

The magnitude of this gravitational force is,

$$\begin{aligned}F_{21} &= \sqrt{(-2.4\times10^{-7})^2+(-3.2\times10^{-7})^2}\,\mathrm{N}\\ &= 4.0\times10^{-7}\,\mathrm{N}.\end{aligned}$$

Angle θ is (Fig. 12.3b),

$$\tan\theta=\frac{3}{4},\quad \theta=\tan^{-1}\left(\frac{3}{4}\right)=0.64\,\mathrm{rad}=37^\circ.$$

The gravitational force F_{21} and the angle θ are shown in Fig. 12.3b.

- wxMaxima codes:

```
(%i5) fpprintprec:5; G:6.67e-11; m1:500; m2:300; r:5;
(fpprintprec) 5
(G) 6.67*10^-11
(m1) 500
(m2) 300
(r) 5
(%i6) F21x: -G*m1*m2/r^2*(3/5);
(F21x) -2.4012*10^-7
(%i7) F21y: -G*m1*m2/r^2*(4/5);
(F21y) -3.2016*10^-7
(%i8) F21vec: [F21x, F21y];
(F21vec) [-2.4012*10^-7,-3.2016*10^-7]
(%i9) F21: sqrt(F21x^2 + F21y^2);
(F21) 4.002*10^-7
(%i10) theta: float(atan(3/4));
(theta) 0.6435
(%i11) theta_deg: float(theta*180/%pi);
(theta_deg) 36.87
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of G, m_1, m_2, and r.
(%i6), (%i7) Calculate F_{21x} and F_{21y}.
(%i8) Assign vector $\boldsymbol{F}_{21}$.
(%i9) Calculate magnitude F_{21}.
(%i10), (%i11) Calculate θ and convert the angle to degree.

Alternative solution: The distance between the particles is,

$$r_{12} = \sqrt{(3\text{ m})^2 + (4\text{ m})^2} = 5\text{ m}.$$

The magnitude of gravitational force on the particle of mass 300 kg is,

$$\begin{aligned} F_{21} = G\frac{m_1 m_2}{r_{12}^2} &= \left(6.67 \times 10^{-11}\frac{\text{N m}^2}{\text{kg}^2}\right)\frac{(500\text{ kg})(300\text{ kg})}{(5.0\text{ m})^2} \\ &= 4.0 \times 10^{-7}\text{ N}. \end{aligned}$$

- wxMaxima codes:

```
(%i5) fpprintprec:5; G:6.67e-11; m1:500; m2:300; r:5;
(fpprintprec) 5
(G) 6.67*10^-11
(m1) 500
(m2) 300
(r) 5
(%i6) F21: G*m1*m2/r^2;
(F21) 4.002*10^-7
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of G, m_1, m_2, and r.
(%i6) Calculate F_{21}.

Problem 12.3 Four particles of different masses are fixed at corners of a rectangle as in Fig. 12.4 (a). What is the gravitational force on particle A?

Solution

The gravitational forces on the particle A are shown in Fig. 12.4b. They are $\boldsymbol{F}_{AB}$, $\boldsymbol{F}_{AC}$, and $\boldsymbol{F}_{AD}$. The gravitational force on the particle A by the particle B is,

$$\begin{aligned} \boldsymbol{F}_{AB} = -G\frac{m_A m_B}{r_{BA}^2}\hat{\boldsymbol{r}}_{BA} &= -\left(6.67 \times 10^{-11}\frac{\text{N m}^2}{\text{kg}^2}\right)\left(\frac{500\text{ kg} \times 600\text{ kg}}{(3\text{ m})^2}\right)(-\mathbf{i}) \\ &= 2.2 \times 10^{-6}\,\mathbf{i}\text{ N}. \end{aligned}$$

The gravitational force on the particle A by the particle C is,

$$\begin{aligned} \boldsymbol{F}_{AC} = -G\frac{m_A m_C}{r_{CA}^2}\hat{\boldsymbol{r}}_{CA} &= -\left(6.67 \times 10^{-11}\frac{\text{N m}^2}{\text{kg}^2}\right)\left(\frac{500\text{ kg} \times 600\text{ kg}}{(3\text{ m})^2 + (4\text{ m})^2}\right)\frac{(-3\mathbf{i} - 4\mathbf{j})}{\sqrt{3^2 + 4^2}} \\ &= 2.4 \times 10^{-7}\,\mathbf{i} + 3.2 \times 10^{-7}\,\mathbf{j}\text{ N}. \end{aligned}$$

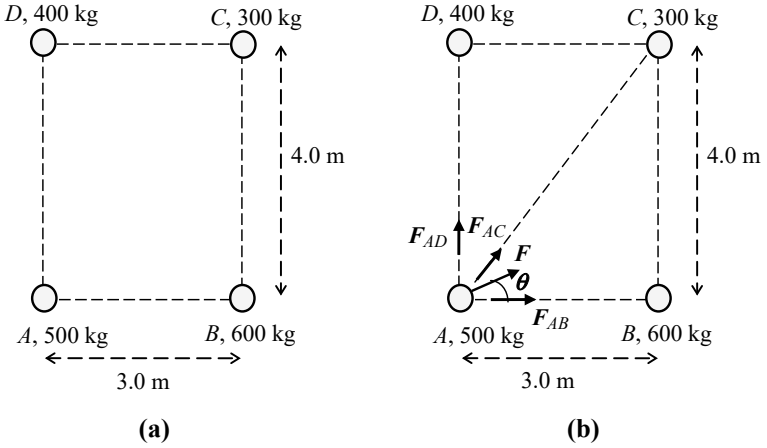


Fig. 12.4 **a** Four particles of different masses fixed at corners of a rectangle, **b** gravitational force on the particle at A, Problem 12.3

The gravitational force on the particle A by the particle D is,

$$\begin{aligned}\boldsymbol{F}_{AD} &= -G\frac{m_A m_D}{r_{DA}^2}\hat{\boldsymbol{r}}_{DA} = -(6.67\times 10^{-11}\frac{\text{N m}^2}{\text{kg}^2})\left(\frac{500\text{ kg}\times 400\text{ kg}}{(4\text{ m})^2}\right)(-\mathbf{j})\\ &= 8.3\times 10^{-7}\,\mathbf{j}\text{ N}.\end{aligned}$$

The net gravitational force on the particle A is the vector sum of these three forces,

$$\boldsymbol{F} = \boldsymbol{F}_{AB} + \boldsymbol{F}_{AC} + \boldsymbol{F}_{AD} = 2.5\times 10^{-6}\,\mathbf{i} + 1.2\times 10^{-6}\,\mathbf{j}\text{ N}.$$

The magnitude of the net gravitational force on the particle A is,

$$\begin{aligned}F &= \sqrt{(2.5\times 10^{-6})^2 + (1.2\times 10^{-6})^2}\text{ N}\\ &= 2.7\times 10^{-6}\text{ N}.\end{aligned}$$

The direction of the net gravitational force on the particle A is,

$$\tan\theta = \frac{1.2\times 10^{-6}}{2.5\times 10^{-6}},\quad \theta = \tan^{-1}\left(\frac{1.2\times 10^{-6}}{2.5\times 10^{-6}}\right) = 0.44\text{ rad} = 25^\circ.$$

The force, $\boldsymbol{F}$, and the angle, θ, are shown in Fig. 12.4b.

- wxMaxima codes:

```
(%i9) fpprintprec:5; G:6.67e-11; mA:500; mB:600; mC:300;
mD:400; rBA:3; rCA:5; rDA:4;
(fpprintprec) 5
(G) 6.67*10^-11
(mA) 500
(mB) 600
(mC) 300
(mD) 400
(rBA) 3
(rCA) 5
(rDA) 4
(%i10) FABvec: [-G*mA*mB/rBA^2*(-1), 0];
(FABvec) [2.2233*10^-6,0]
(%i11) FACvec: [-G*mA*mC/rCA^2*(-3)/rCA, -G*mA*mC/rCA^2*(-
4)/rCA];
(FACvec) [2.4012*10^-7,3.2016*10^-7]
(%i12) FADvec: [0, -G*mA*mD/rDA^2*(-1)];
(FADvec) [0,8.3375*10^-7]
(%i13) Fvec: FABvec + FACvec + FADvec;
(Fvec) [2.4635*10^-6,1.1539*10^-6]
(%i14) F: sqrt(Fvec[1]^2 + Fvec[2]^2);
(F) 2.7203*10^-6
(%i15) theta: atan(Fvec[2]/Fvec[1]);
(theta) 0.43806
(%i16) theta_deg: float(theta*180/%pi);
(theta_deg) 25.099
```

Comments on the codes:

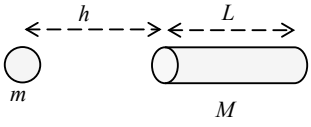
(%i9) Set floating point print precision to 5, assign values of G, m_A, m_B, m_C, m_D, r_{BA}, r_{CA}, and r_{DA}.
(%i10), (%i11), (%i12) Assign vectors $\boldsymbol{F}_{AB}$, $\boldsymbol{F}_{AC}$, and $\boldsymbol{F}_{AD}$.
(%i13), (%i14) Calculate vector $\boldsymbol{F}$ and its magnitude F.
(%i15), (%i16) Calculate θ and convert the angle to degree.

Problem 12.4 A particle of mass, m, and a rod of length, L, and mass, M, are arranged as in Fig. 12.5. What is the gravitational force, F, on the particle?

Solution

Figure 12.6 shows the particle, the rod, and the coordinates needed to calculate the force. An elementary mass of the rod is $dM = (dx/L)M$. The distance between the elementary mass and the particle is x.

Fig. 12.5 Problem 12.4



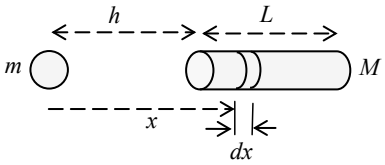


Fig. 12.6 Determining gravitational force on the particle due to the rod, Problem 12.4

The gravitational force on the particle by the elementary mass is,

$$dF = G\frac{m\,dM}{x^2} = G\frac{mM}{L}\frac{dx}{x^2},$$

directed to the right. The force, F, on the particle by the whole rod is obtained by integration,

$$\begin{aligned} F &= \int dF = \int_{x=h}^{h+L} G\frac{mM}{L}\frac{dx}{x^2} = \left[\frac{GmM}{L}\left(-\frac{1}{x}\right)\right]_h^{h+L} \\ &= \frac{GmM}{L}\left(-\frac{1}{h+L}+\frac{1}{h}\right) \\ &= \frac{GmM}{h(h+L)}. \end{aligned}$$

The force, F, points to the right.

- wxMaxima codes:

```
(%i2) assume (L>0); assume (h>0);
(%o1) [L>0]
(%o2) [h>0]
(%i3) F: integrate( G*m*M/(L*x^2), x, h, h+L );
(F) (G*M*(1/h-1/(h+L))*m)/L
(%i4) ratsimp(%);
(%o4) (G*M*m)/(h^2+L*h)
```

Comments on the codes:

(%i2) Assume $L > 0$ and $h > 0$.
(%i3), (%i4) Integrate $F = \int_{x=h}^{h+L} G\frac{mM}{L}\frac{dx}{x^2}$ and simplify the result.

Problem 12.5 The radius and mass of a planet are $R = 2000$ km and $M = 6.0 \times 10^{22}$ kg, respectively.

(a) What is the acceleration of gravity on the surface of the planet?

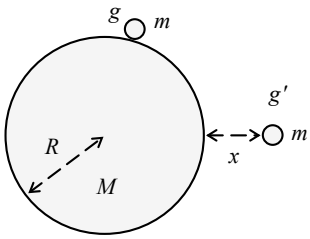


Fig. 12.7 An object on the surface and at altitude x from the planet, Problem 12.5

(b) Calculate the acceleration of gravity of the planet at an altitude of 1000 km.

Solution

(a) For an object of mass m located at the surface of the planet, the weight of the object equals the gravitational attraction between the object and the planet (Fig. 12.7). Using (Eq. 12.8), one has,

$$mg = \frac{GmM}{R^2},$$

where g is the acceleration of gravity at the planet surface, M is the mass. and R is the radius of the planet.

The acceleration of gravity g at the surface of the planet is,

$$g = \frac{GM}{R^2} = (6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}) \frac{(6.0 \times 10^{22} \text{ kg})}{(2000 \times 10^3 \text{ m})^2} = 0.44 \text{ m s}^{-2}.$$

(b) For an object of mass, m, at altitude, x, away from the planet surface, the weight of the object is the gravitational force between the object and the planet (Fig. 12.7),

$$mg' = \frac{GmM}{(x+R)^2},$$

where g' is the acceleration of gravity at an altitude of x. The acceleration of gravity, g', of the planet at altitude, x, is,

$$g' = \frac{GM}{(x+R)^2} = \left(6.67 \times 10^{-11} \frac{\text{Nm}^2}{\text{kg}^2}\right) \frac{(6.0 \times 10^{22} \text{ kg})}{(3000 \times 10^3 \text{ m})^2} = 0.44 \text{ m s}^{-2}$$

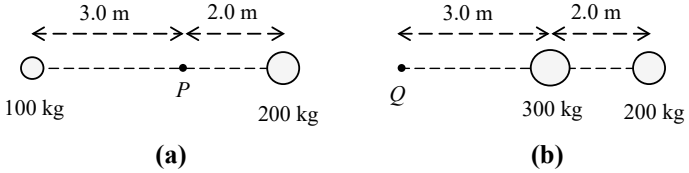


Fig. 12.8 Determining gravitational fields at points **a** *P* and **b** *Q*, Problem 12.6

- wxMaxima codes:

```
(%i5) fpprintprec:5; G:6.67e-11; R:2000e3; M:6e22; x:1000e3;
(fpprintprec) 5
(G) 6.67*10^-11
(R) 2.0*10^6
(M) 6.0*10^22
(x) 1.0*10^6
(%i6) g: G*M/R^2;
(g) 1.0005
(%i7) gprime: G*M/(R+x)^2;
(gprime) 0.44467
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of *G*, *R*, *M*, and *x*.
(%i6), (%i7) Calculate *g* and *g'*.

Problem 12.6 Calculate the gravitational fields at point *P* of Fig. 12.8a and at point *Q* of Fig. 12.8b.

Solution

(a) The gravitational field at a point *R* away from a body of mass *M* is $g = GM/R^2$ (Eq. 12.8). The gravitational field at point *P* by the particle of mass 100 kg is (Fig. 12.8a),

$$\boldsymbol{g}_1 = (6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}) \frac{(100 \text{ kg})}{(3.0 \text{ m})^2} (-\mathbf{i}) = -7.4 \times 10^{-10} \mathbf{i} \text{ N kg}^{-1},$$

while the gravitational field at point *P* by the particle of mass 200 kg is,

$$\boldsymbol{g}_2 = (6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}) \frac{(200 \text{ kg})}{(2.0 \text{ m})^2} \mathbf{i} = 3.3 \times 10^{-9} \mathbf{i} \text{ N kg}^{-1}.$$

The effective gravitational field at point *P* is the vector sum of the two gravitational fields,

$$\boldsymbol{g}_P = \boldsymbol{g}_1 + \boldsymbol{g}_2 = 2.6 \times 10^{-9}\mathbf{i}\ \text{N kg}^{-1}.$$

The direction of the gravitational field is to the positive x direction.

- wxMaxima codes:

```
(%i6) fpprintprec:5; m1:100; r1:3; m2:200; r2:2; G:6.67e-11;
(fpprintprec) 5
(m1) 100
(r1) 3
(m2) 200
(r2) 2
(G) 6.67*10^-11
(%i7) g1: -G*m1/r1^2;
(g1) -7.4111*10^-10
(%i8) g2: G*m2/r2^2;
(g2) 3.335*10^-9
(%i9) gP: g1 + g2;
(gP) 2.5939*10^-9
```

Comments on the codes:

(%i6) Set floating point print precision to 5, assign values of m_1, r_1, m_2, r_2, and G.
(%i7), (%i8), (%i9) Calculate g_1, g_2, and g_P.

(b) The gravitational field at point Q by the particle of mass 300 kg is (Fig. 12.8b),

$$\boldsymbol{g}_3 = (6.67 \times 10^{-11}\frac{\text{N m}^2}{\text{kg}^2})\frac{(300\ \text{kg})}{(3.0\ \text{m})^2}\mathbf{i} = 2.2 \times 10^{-9}\mathbf{i}\ \text{N kg}^{-1},$$

while the gravitational field at point Q by the particle of mass 200 kg is,

$$\boldsymbol{g}_4 = (6.67 \times 10^{-11}\frac{\text{N m}^2}{\text{kg}^2})\frac{(200\ \text{kg})}{(5.0\ \text{m})^2}\mathbf{i} = 5.3 \times 10^{-10}\mathbf{i}\ \text{N kg}^{-1}.$$

The effective gravitational field at point Q is the vector sum of the two gravitational fields.

$$\boldsymbol{g}_Q = \boldsymbol{g}_3 + \boldsymbol{g}_4 = 2.8 \times 10^{-9}\mathbf{i}\ \text{N kg}^{-1}.$$

The direction of the gravitational field is to the positive x direction.

- wxMaxima codes:

```
(%i6) fpprintprec:5; m3:300; r3:3; m4:200; r4:5; G:6.67e-11;
(fpprintprec) 5
(m3) 300
(r3) 3
(m4) 200
(r4) 5
(G) 6.67*10^-11
(%i7) g3: G*m3/r3^2;
(g3) 2.2233*10^-9
(%i8) g4: G*m4/r4^2;
(g4) 5.336*10^-10
(%i9) gQ: g3 + g4;
(gQ) 2.7569*10^-9
```

Comments on the codes:

(%i6) Set floating point print precision to 5, assign values of m_3, r_3, m_4, r_4, and G.
(%i7), (%i8), (%i9) Calculate g_3, g_4, and g_Q.

Problem 12.7 What is the gravitational field at the origin of Fig. 12.9a?

Solution

Figure 12.9b shows the gravitational fields at the origin due to three particles. The gravitational field at a point R away from a body of mass M is $g = GM/R^2$ (Eq. 12.8). The gravitational field at the origin by the particle of mass 600 kg is,

$$\boldsymbol{g}_1 = (6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}) \frac{(600 \text{ kg})}{(3.0 \text{ m})^2} \mathbf{j} = 4.4 \times 10^{-9} \mathbf{j} \,\text{N kg}^{-1}.$$

The gravitational field at the origin by the particle of mass 400 kg is,

$$\boldsymbol{g}_2 = (6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}) \frac{(400 \,\text{kg})}{(4.0 \,\text{m})^2} \mathbf{j} = 1.7 \times 10^{-9} \mathbf{j} \,\text{N kg}^{-1}.$$

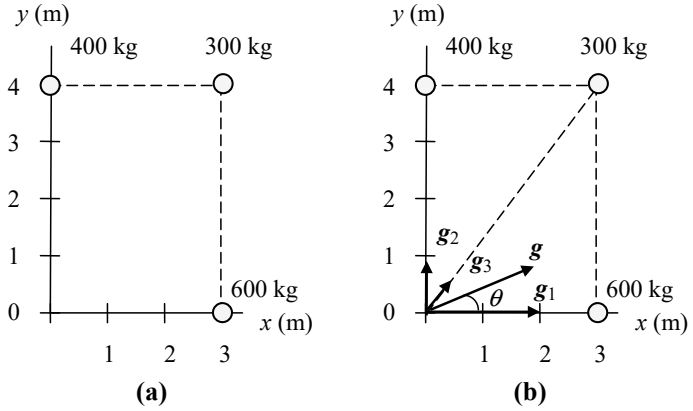


Fig. 12.9 **a** A three-particle system, **b** determining the gravitational field at the origin, Problem 12.7

The magnitude of the gravitational field at the origin by the particle of mass 300 kg is,

$$(6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}) \frac{(300 \text{ kg})}{(5.0 \text{ m})^2} = 8.0 \times 10^{-10} \text{ N kg}^{-1}.$$

The x and y components of this gravitational field are,

$$\frac{3.0 \text{ m}}{5.0 \text{ m}} \times 8.0 \times 10^{-10} \, \mathbf{i} \text{ N kg}^{-1} = 4.8 \times 10^{-10} \, \mathbf{i} \text{ N kg}^{-1},$$

$$\frac{4.0 \text{ m}}{5.0 \text{ m}} \times 8.0 \times 10^{-10} \, \mathbf{j} \text{ N kg}^{-1} = 6.4 \times 10^{-10} \, \mathbf{j} \text{ N kg}^{-1}.$$

Thus, the gravitational field at the origin by the particle of mass 300 kg is,

$$g_3 = 4.8 \times 10^{-10} \text{ N kg}^{-1} \, \mathbf{i} + 6.4 \times 10^{-10} \text{ N kg}^{-1} \, \mathbf{j}.$$

The gravitational field, $\boldsymbol{g}$, at the origin is the vector sum of fields by each particle,

$$g = g_1 + g_2 + g_3 = 4.9 \times 10^{-9} \, \mathbf{i} + 2.3 \times 10^{-9} \, \mathbf{j} \text{ N kg}^{-1}.$$

The magnitude, g, of the gravitational field at the origin is,

$$g = \sqrt{(4.9 \times 10^{-9})^2 + (2.3 \times 10^{-9})^2} \text{ N kg}^{-1} = 5.4 \times 10^{-9} \text{ N kg}^{-1}.$$

The angle, θ, of the field with respect to the x axis is (Fig. 12.9b),

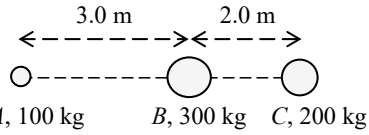


Fig. 12.10 A three-particle system, Problem 12.10

$$\tan\theta = \frac{2.3}{4.9}, \quad \theta = \tan^{-1}(2.3/4.9) = 0.44 \text{ rad} = 25°.$$

The gravitational field $\boldsymbol{g}$ and the angle θ are shown in Fig. 12.9b.

- wxMaxima codes:

```
(%i2) fpprintprec:5; G:6.67e-11;
(fpprintprec) 5
(G) 6.67*10^-11
(%i3) g1vec: [G*600/3^2, 0];
(g1vec) [4.4467*10^-9,0]
(%i4) g2vec: [0, G*400/4^2];
(g2vec) [0,1.6675*10^-9]
(%i5) g3vec: [G*300/5^2*(3/5), G*300/5^2*(4/5)];
(g3vec) [4.8024*10^-10,6.4032*10^-10]
(%i6) gvec: g1vec + g2vec + g3vec;
(gvec) [4.9269*10^-9,2.3078*10^-9]
(%i7) g: sqrt(gvec[1]^2 + gvec[2]^2);
(g) 5.4406*10^-9
(%i8) theta: atan(gvec[2]/gvec[1]);
(theta) 0.43806
(%i9) theta_deg: float(theta*180/%pi);
(theta_deg) 25.099
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and assign value of G.
(%i3), (%i4), (%i5) Assign vectors $\boldsymbol{g_1}$, $\boldsymbol{g_2}$, and $\boldsymbol{g_3}$.
(%i6), (%i7) Calculate vector, $\boldsymbol{g}$, and its magnitude, g.
(%i8), (%i9) Calculate θ and convert the angle to degree.

Problem 12.8 The gravitational field in a region is $3\,\mathbf{i} + 5\,\mathbf{j}$ N kg^{-1}. A particle of mass 20 kg is placed in the region. Calculate,

(a) the gravitational force, F, on the particle,
(b) the acceleration, a, of the particle.

Solution

(a) The force on the particle in the gravitational field is,

$$\boldsymbol{F} = m\boldsymbol{g} = (20\,\text{kg})(3\,\mathbf{i} + 5\,\mathbf{j})\ \text{N/kg} = 60\,\mathbf{i} + 100\,\mathbf{j}\ \text{N}.$$

(b) The acceleration of the particle is,

$$\boldsymbol{a} = \boldsymbol{F}/m = \frac{(60\,\mathbf{i} + 100\mathbf{j})\ \text{N}}{20\,\text{kg}} = 3\mathbf{i} + 5\,\mathbf{j}\,\text{m}\,\text{s}^{-2}$$

- wxMaxima codes:

```
(%i3) fpprintprec:5; m:20; g:[3, 5];
(fpprintprec) 5
(m) 20
(g) [3,5]
(%i4) F: m*g;
(F) [60,100]
(%i5) a: F/m;
(a) [3,5]
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of m and $\boldsymbol{g}$.
(%i4), (%i5) Calculate $\boldsymbol{F}$ and $\boldsymbol{a}$.

Problem 12.9 What is the gravitational potential energy, U_g, of a particle of mass, m, at the surface of the earth?

Solution

The gravitational potential energy of two bodies of masses M and m at a distance r apart is (Eq. 12.3),

$$U_g = -\frac{GMm}{r}.$$

For this problem,

$$U_g = -\frac{GMm}{R},$$

where M is the mass of the earth, R is the radius of the earth, and m is the mass of the particle. The gravitational field of the earth is (Eq. 12.8),

$$g = \frac{GM}{R^2} \Rightarrow GM = gR^2.$$

Thus, the gravitational energy of the particle is,

$$U_g = -\frac{GMm}{R} = -\frac{(gR^2)m}{R} = -mgR.$$

Problem 12.10 What is the gravitational energy, U, of three-particle system shown in Fig. 12.10?

Solution

The gravitational potential energy of a two-particle system X and Y is (Eq. 12.3),

$$U_{XY} = -G\frac{m_X m_Y}{r_{XY}},$$

where m_X and m_Y are the masses of the particles and r_{XY} is their separation distance.

For three-particle system, the gravitational potential energy is the potential energies of all pairs of particles,

$$\begin{aligned} U &= U_{AB} + U_{BC} + U_{AC} \\ &= -G\left(\frac{m_A m_B}{r_{AB}} + \frac{m_B m_C}{r_{BC}} + \frac{m_A m_C}{r_{AC}}\right) \\ &= -6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}\left(\frac{(100\text{ kg})(300\text{ kg})}{3.0\text{ m}} + \frac{(300\text{ kg})(200\text{ kg})}{2.0\text{ m}}\right. \\ &\quad \left. + \frac{(100\text{ kg})(200\text{ kg})}{5.0\text{ m}}\right) \\ &= -2.9 \times 10^{-6}\text{ J}. \end{aligned}$$

- wxMaxima codes:

```
(%i8) fpprintprec:5; mA:100; mB:300; mC:200; rAB:3; rBC:2;
rAC:5;
G:6.67e-11;
(fpprintprec) 5
(mA) 100
(mB) 300
(mC) 200
(rAB) 3
```

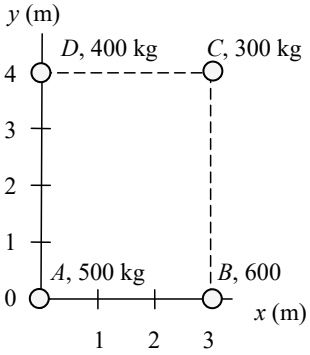


Fig. 12.11 A four-particle system, Problem 12.11

```
(rBC) 2
(rAC) 5
(G) 6.67*10^-11
(%i9) U: -G*(mA*mB/rAB + mB*mC/rBC + mA*mC/rAC);
(U) -2.9348*10^-6
```

Comments on the codes:

(%i8) Set floating point print to 5, assign values of m_A, m_B, m_C, r_{AB}, r_{BC}, r_{AC}, and G.
(%i9) Calculate U.

Problem 12.11 Calculate the gravitational potential energy, U, of the system of particles as shown in Fig. 12.11.

Solution

The gravitational potential energy of a pair of particles X and Y is (Eq. 12.3),

$$U_{XY} = -G\frac{m_X m_Y}{r_{XY}},$$

where m_X and m_Y are the masses of the particles and r_{XY} is their separation distance.

The gravitational potential energy of the four-particle system is the sum of potential energies of all pairs of particles,

$$\begin{aligned} U &= U_{AB} + U_{AC} + U_{AD} + U_{BC} + U_{BD} + U_{CD} \\ &= -G\left(\frac{m_A m_B}{r_{AB}} + \frac{m_A m_C}{r_{AC}} + \frac{m_A m_D}{r_{AD}} + \frac{m_B m_C}{r_{BC}} + \frac{m_B m_D}{r_{BD}} + \frac{m_C m_D}{r_{CD}}\right) \\ &= -6.67 \times 10^{-11}\frac{\text{N m}^2}{\text{kg}^2}\left(\frac{(500\text{ kg})(600\text{ kg})}{3.0\text{ m}} + \frac{(500\text{ kg})(300\text{ kg})}{5.0\text{ m}}\right. \\ &\quad \left. + \frac{(500\text{ kg})(400\text{ kg})}{4.0\text{ m}} + \frac{(600\text{ kg})(300\text{ kg})}{4.0\text{ m}}\right. \end{aligned}$$

$$+\frac{(600\text{ kg})(400\text{ kg})}{5.0\text{ m}}+\frac{(300\text{ kg})(400\text{ kg})}{3.0\text{ m}}\Bigg)$$
$$=-2.1\times10^{-5}\text{ J}.$$

- wxMaxima codes:

```
(%i12) fpprintprec:5; mA:500; mB:600; mC:300; mD:400; rAB:3;
rAC:5; rAD:4; rBC:4; rBD:5; rCD:3; G:6.67e-11;
(fpprintprec) 5
(mA) 500
(mB) 600
(mC) 300
(mD) 400
(rAB) 3
(rAC) 5
(rAD) 4
(rBC) 4
(rBD) 5
(rCD) 3
(G) 6.67*10^-11
(%i13) U: -G*(mA*mB/rAB + mA*mC/rAC + mA*mD/rAD + mB*mC/rBC
+ mB*mD/rBD + mC*mD/rCD);
(U) -2.0877*10^-5
```

Comments on the codes:

(%i12) Set floating point print to 5, assign values of m_A, m_B, m_C, m_D, r_{AB}, r_{AC}, r_{AD}, r_{BC}, r_{BD}, r_{CD}, and G.
(%i13) Calculate U.

Problem 12.12 An object at rest is released from an altitude of 100 km. At what speed, v, it hits the earth? Neglect air resistance. The radius of the earth is $R = 6.4 \times 10^6$ m.

Solution

Figure 12.12 shows the object in the initial and final instances. Here, M and R are the mass and the radius of the earth, while m is the mass and h is the altitude of the object. In addition, U_{init}, K_{init}, U_{final}, K_{final} are the potential and kinetic energies in the initial and final instances, respectively.

Applying the conservation of energy, that is, the sum of gravitational potential energy and kinetic energy at initial position $U_{init} + K_{init}$ equals the ones at final position $U_{final} + K_{final,}$ we have,

$$U_{init}+K_{init}=U_{final}+K_{final},$$
$$-\frac{GmM}{R+h}+0=-\frac{GmM}{R}+\frac{1}{2}mv^2,$$

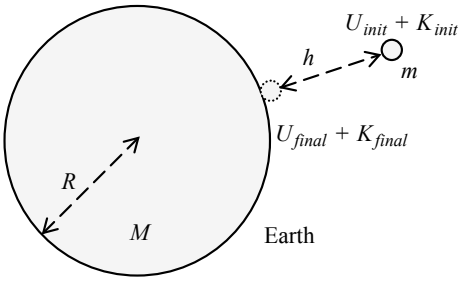


Fig. 12.12 A object released at altitude h from the Earth, Problem 12.12

$$-\frac{gR^2m}{R+h} + 0 = -\frac{gR^2m}{R} + \frac{1}{2}mv^2,$$

where v is the speed the object hits the earth and we have used the fact that $GM = gR^2$ (Eq. 12.8). By solving the last equation for v, the object hits the earth at the speed of,

$$\begin{aligned} v &= \sqrt{2gR^2\left(\frac{1}{R} - \frac{1}{R+h}\right)} = \sqrt{2g\left(R - \frac{R^2}{R+h}\right)} \\ &= \sqrt{2(9.8\ \text{m/s}^2)\left(6.4\times10^6\ \text{m} - \frac{(6.4\times10^6\ \text{m})^2}{(6.4\times10^6 + 100\times10^3)\text{m}}\right)} \\ &= 1.4\times10^3\ \text{m s}^{-1}. \end{aligned}$$

- wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; g:9.8; G:6.67e-11; R:6.4e6;
h:100e3;
(fpprintprec) 5
(ratprint) false
(g) 9.8
(G) 6.67*10^-11
(R) 6.4*10^6
(h) 1.0*10^5
(%i8) solve(-g*R^2*m/(R+h)=-g*R^2*m/R+0.5*m*v^2, v)$ float(%);
(%o8) [v=-1389.2,v=1389.2]
(%i9) v: sqrt(2*g*(R-R^2/(R+h)));
(v) 1389.2
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of g, G, R, and h.

$$\text{(\%i8)}\quad \text{Solve} -\frac{gR^2m}{R+h} = -\frac{gR^2m}{R} + \frac{1}{2}mv^2 \text{for } v.$$

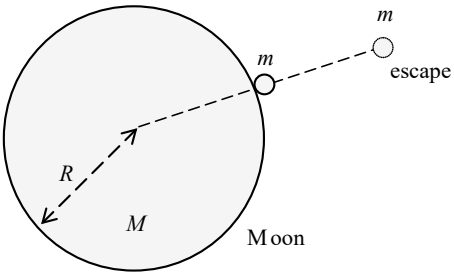


Fig. 12.13 Determining escape velocity of the Moon, Problem 12.13

(%i9) Direct calculation of v.

Problem 12.13 The acceleration of gravity on the moon is 1.6 m s^{-2} and the radius of the moon is 1740 km. Calculate the escape velocity for the moon.

Solution

Figure 12.13 shows the moon and a particle on the surface of the moon. The particle that has escaped from the moon is also shown. The mass of the moon is M, its radius is R, and the mass of the particle is m.

The sum of gravitational potential energy and kinetic energy of the particle on the moon and the one after escape are the same. We have,

$$U_{moon} + K_{moon} = U_{escape} + K_{escape},$$
$$-\frac{GmM}{R} + \frac{1}{2}mv^2 = 0 + 0.$$

But $g = GM/R^2$ and $GM = gR^2$, and the equation becomes,

$$-gmR + \frac{1}{2}mv^2 = 0.$$

The escape velocity from the moon is,

$$v = \sqrt{2gR} = \sqrt{2(1.6 \text{ m/s}^2)(1740 \times 10^3 \text{ m})} = 2400 \text{ m s}^{-1}.$$

- wxMaxima codes:

```
(%i3)fpprintprec:5; g:1.6; R:1740e3;
(fpprintprec) 5
(g) 1.6
(R) 1.74*10^6
```

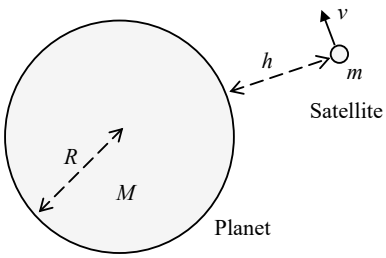


Fig. 12.14 A satellite orbiting a planet at altitude h, Problem 12.14

```
(%i4) v: sqrt(2*g*R);
(v) 2359.7
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of g and R.
(%i4) Calculate the escape velocity, v.

Problem 12.14 The radius and mass of a planet are 2000 km and 6.0×10^{22} kg, respectively. What are the period, T, and speed, v, of a satellite in a circular orbit at an altitude of 2000 km around the planet?

Solution

Figure 12.14 shows the satellite orbiting the planet. In the figure, M and R are the mass and radius of the planet, while m, v, and h are the mass, speed, and altitude of the satellite, respectively.

The gravitational attraction force between the satellite and the planet is (Eq. 12.1),

$$\frac{GMm}{(R+h)^2}.$$

The centripetal force that causes the satellite to move in circular orbit is (Eq. 6.2)

$$\frac{mv^2}{(R+h)}.$$

Both forces are equal in magnitude,

$$\frac{GMm}{(R+h)^2} = \frac{mv^2}{(R+h)}.$$

This gives the speed, v, of the satellite as,

$$v = \sqrt{\frac{GM}{R+h}} = \sqrt{\frac{(6.67 \times 10^{-11}\ \text{N m}^2/\text{kg}^2)(6.0 \times 10^{22}\ \text{kg})}{4000 \times 10^3\ \text{m}}}$$
$$= 1.0 \times 10^3\ \text{m s}^{-1}.$$

The period, T, of the satellite in the circular orbit is,

$$T = \frac{2\pi(R+h)}{v} = \frac{2\pi(4000 \times 10^3\ \text{m})}{1000\ \text{m/s}} = 2.5 \times 10^4\ \text{s}.$$

- wxMaxima codes:

```
(%i5) fpprintprec:5; G:6.67e-11; M:6e22; R:2000e3; h:2000e3;
(fpprintprec) 5
(G) 6.67*10^-11
(M) 6.0*10^22
(R) 2.0*10^6
(h) 2.0*10^6
(%i6) v: sqrt(G*M/(R+h));
(v) 1000.2
(%i7) T: float(2*%pi*(R+h)/v);
(T) 2.5126*10^4
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of G, M, R, and h.
(%i6), (%i7) Calculate v and T.

Problem 12.15 Calculate the work, W, done by the gravitational force on a body of mass 80 kg as the body is displaced by 12.74×10^6 m from the earth surface. Radius of earth $= 6.37 \times 10^6$ m, universal gravitational constant $= 6.67 \times 10^{-11}$ N m^2 kg^{-2}, and mass of earth $= 5.98 \times 10^{24}$ kg.

Solution

Figure 12.15 shows the earth of mass, $M = 5.98 \times 10^{24}$ kg and the displaced body of mass, $m = 80$ kg. The radius of the earth is $R_1 = 6.37 \times 10^6$ m and $R_2 = 6.37 \times 10^6$ m $+ 12.74 \times 10^6$ m $= 1.911 \times 10^7$ m.

The work done is,

$$W = \int F \cdot dr = \int_{R_1}^{R_2} \frac{-GMm}{r^2} dr = \frac{GMm}{r}\Big|_{R_1}^{R_2} = GMm\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

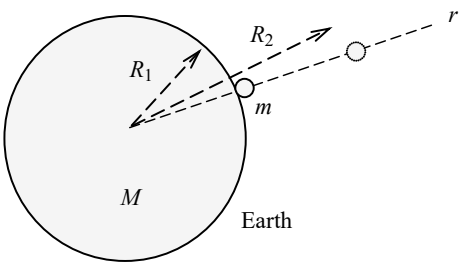


Fig. 12.15 A body displaced from R_1 to R_2 from the center of the Earth, Problem 12.15

$$= \left(6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}\right)(5.98 \times 10^{24}\ \text{kg})(80\ \text{kg})$$
$$\times \left(\frac{1}{(6.37 \times 10^6 + 12.74 \times 10^6)\text{m}} - \frac{1}{6.37 \times 10^6\ \text{m}}\right)$$
$$= -3.3 \times 10^9\ \text{J}.$$

- wxMaxima codes:

```
(%i7) fpprintprec:5; ratprint:false; G:6.67e-11; M:5.98e24;
m:80; R1:6.37e6; R2:6.37e6+12.74e6;
(fpprintprec) 5
(ratprint) false
(G) 6.67*10^-11
(M) 5.98*10^24
(m) 80
(R1) 6.37*10^6
(R2) 1.911*10^7
(%i8) F: -G*M*m/r^2;
(F) -(3.1909*10^16)/r^2
(%i9) W: integrate( F, r, R1, R2);
(W) -3.3395*10^9
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of G, M, m, R_1, and R_2.
(%i8) Assign F.
(%i9) Calculate work done, $W = \int_{R_1}^{R_2} \frac{-GMm}{r^2}\, dr$.

Problem 12.16 A rocket is launched with initial speed of 1.11×10^4 m s^{-1} from the earth and moves up to a distance of $3R$ from the earth center, where R is the radius of the earth. Neglecting air resistance and rotation of the earth, what is the final speed v of the rocket? Radius of the earth, $R = 6.37 \times 10^6$ m, universal gravitational constant, $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, and mass of the earth, $M = 5.98 \times 10^{24}$ kg.

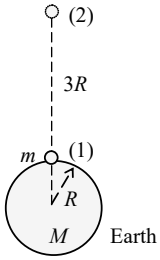


Fig. 12.16 A rocket moves from R to $3R$ from the center of the Earth, Problem 12.16

Solution

Figure 12.16 shows the earth and the rocket at initial (1) and final (2) positions.

The gravitational potential energy of the masses M and m at a distance r apart is (Eq. 12.3),

$$U = -\frac{GMm}{r}.$$

Applying the conservation of energy at position (1) and (2), we have,

$$U_1 + K_1 = U_2 + K_2,$$
$$-\frac{GMm}{R} + \frac{1}{2}mv_1^2 = -\frac{GMm}{3R} + \frac{1}{2}mv_2^2.$$

Here, $U_1 = -GMm/R$ and $K_1 = mv_1{}^2/2$ are gravitational potential energy and kinetic energy of the rocket at initial position (1), while $U_2 = -GMm/(3R)$ and $K_2 = mv_2{}^2/2$ are gravitational potential energy and kinetic energy of the rocket at final position (2), respectively. From the last equation, the speed, v_2, of the rocket at the final position (2) is,

$$\begin{aligned} v_2 &= \left(-\frac{4GM}{3R} + v_1^2\right)^{1/2} \\ &= \left(-4(6.67 \times 10^{-11}\frac{\text{N m}^2}{\text{kg}^2})\frac{(5.98 \times 10^{24}\ \text{kg})}{3(6.37 \times 10^6\ \text{m})} + (1.11 \times 10^4\ \text{m/s})^2\right)^{1/2} \\ &= 6300\ \text{m s}^{-1}. \end{aligned}$$

- wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; G:6.67e-11; M:5.98e24;
R:6.37e6; v1:1.11e4;
(fpprintprec) 5
```

```
(ratprint) false
(G) 6.67*10^-11
(M) 5.98*10^24
(R) 6.37*10^6
(v1) 1.11*10^4
(%i8) solve(-G*M*m/R+1/2*m*v1^2 = -G*M*m/(3*R)+1/2*m*v2^2,
v2)$ float(%);
(%o8) [v2=-6302.5,v2=6302.5]
(%i9) v2: sqrt(-4*G*M/(3*R)+v1^2);
(v2) 6302.5
```

Commends on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of G, M, R, and v_1.
(%i8) Solve $-\frac{GMm}{R} + \frac{1}{2}mv_1^2 = -\frac{GMm}{3R} + \frac{1}{2}mv_2^2$ for v_2.
(%i9) Calculate v_2.

Problem 12.17 Calculate the gravitational potential energy, U, of the earth. Mass and radius of the earth are $M = 6.0 \times 10^{24}$ kg and $R = 6.4 \times 10^6$ m, respectively.

Solution

The density of the earth is,

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}.$$

where M and R are mass and radius of the earth, and we assumed the density is uniform.

Consider a solid sphere of radius, r, and a spherical shell of radius, r, with thickness, dr (Fig. 12.17). Both sphere and the spherical shell have the same density, ρ.

The mass of the sphere is,

$$\frac{4}{3}\pi r^3 \rho = \frac{4}{3}\pi r^3 \frac{3M}{4\pi R^3} = \frac{Mr^3}{R^3}.$$

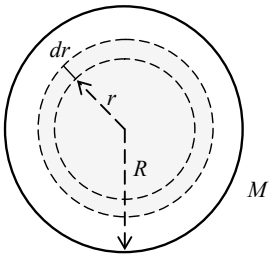


Fig. 12.17 Determining gravitational potential energy of the Earth, Problem 12.17

The mass of the spherical shell is,

$$4\pi r^2 dr\ \rho = 4\pi r^2 dr \frac{3M}{4\pi R^3} = \frac{3Mr^2 dr}{R^3}.$$

The gravitational energy of the sphere and the spherical shell is (Eq. 12.3),

$$dU = -\frac{G\frac{Mr^3}{R^3}\frac{3Mr^2 dr}{R^3}}{r} = -\frac{3GM^2}{R^6} r^4 dr.$$

The total gravitational energy, U, for a sphere of radius, R, is,

$$\begin{aligned} U &= \int dU = -\frac{3GM^2}{R^6}\int_0^R r^4 dr = -\frac{3GM^2}{R^6}\frac{r^5}{5}\Big|_0^R \\ &= -\frac{3GM^2}{5R}. \end{aligned}$$

Numerically,

$$\begin{aligned} U &= -\frac{3(6.67\times 10^{-11}\ \text{N m}^2/\text{kg}^2)(6.0\times 10^{24}\ \text{kg})^2}{5(6.4\times 10^6\ \text{m})} \\ &= -2.3\times 10^{32}\ \text{J}. \end{aligned}$$

- wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec) 5
(ratprint) false
(%i3) U: -3*G*M^2/R^6*integrate(r^4, r, 0, R);
(U) -(3*G*M^2)/(5*R)
(%i6) G: 6.67e-11; M:6e24; R:6.4e6;
(G) 6.67*10^-11
(M) 6.0*10^24
(R) 6.4*10^6
(%i7) U: -3*G*M^2/R^6*integrate(r^4, r, 0, R);
(U) -2.2511*10^32
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3) Calculate $U = -\frac{3GM^2}{R^6}\int_0^R r^4 dr$ in symbols.
(%i6) Assign values of G, M, and R.
(%i7) Recalculate $U = -\frac{3GM^2}{R^6}\int_0^R r^4 dr$ to get a numerical value.

12.3 Summary

- Newton's universal law of gravity states that the attractive gravitational force between two particles of masses, m_1 and m_2, separated by a distance, r, is,

$$F = G\frac{m_1 m_2}{r^2},$$

- where $G = 6.672 \times 10^{-11}$ N m^2 kg^{-2} is the universal gravitational constant.
- The gravitational potential energy of two particles with masses, m_1 and m_2, separated by a distance, r, is,

$$U = -G\frac{m_1 m_2}{r}.$$

- The gravitational field at a distance, R, form a body of mass, M, is,

$$g = \frac{GM}{R^2}.$$

12.4 Exercises

Exercise 12.1 Calculate the gravitational attraction force between two metal spheres of masses 100 kg whose centers are 50 cm apart.

(Answer: 2.7×10^{-6} N)

Exercise 12.2 Three masses, A of 4.0 kg, B of 3.0 kg, and C of 5.0 kg are fixed in a plane as shown in Fig. 12.18.

(a) Calculate the gravitational force on mass A.
(b) What is the gravitational field at point P?
(c) Calculate the gravitational potential energy of the three particles.

(Answer: (a) $\boldsymbol{F} = \left(1.5 \times 10^{-10}\mathbf{i} + 2.0 \times 10^{-10}\mathbf{j}\right)$N
(b) $\boldsymbol{g} = \left(-3.9 \times 10^{-11}\mathbf{i} - 9.5 \times 10^{-11}\mathbf{j}\right)$N kg^{-1}
(c) $U = -1.1 \times 10^{-9}$ J)

Exercise 12.3 Given that the radius of Earth is 6370 km, acceleration of gravity near the Earth's surface is 9.8 m s^{-2}, and the universal gravitational constant is 6.67×10^{-11} N m^2 kg^{-2}, calculate the Earth's mass.

(Answer: 6.0×10^{24} kg)

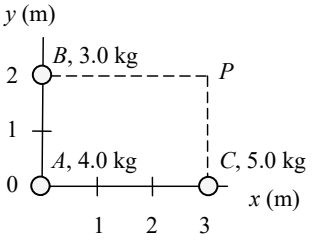


Fig. 12.18 A three-mass system, Exercise 12.2

Exercise 12.4 A satellite orbits Earth in a circular path of radius r with orbital period of T. Show that,

$$T = \frac{2\pi r^{3/2}}{\sqrt{GM}},$$

where M is mass of Earth and G is the universal gravitational constant.

Exercise 12.5 Given that masses of Earth and Moon are 6.0×10^{24} kg and 7.3×10^{22} kg, respectively, and the Earth-Moon distance is 3.8×10^5 km, calculate the energy needed to separate the Moon from Earth by infinite distance.

(Answer: 3.8×10^{28} J)

Chapter 13
Elastic Properties

13.1 Basic Concepts and Formulae

(1) Hooke's law states that the force F of a wire is proportional to its extension x, or,

$$F = -kx, \tag{13.1}$$

where k is a proportional constant called the force constant. The extension x is the increase in length of the wire due to the force. The negative sign signifies that the direction of the force is opposite to that of the extension. The units of k are N m^{-1}.

The elastic potential energy of a stretched wire is given by the following formula,

$$W = \frac{1}{2}Fx = \frac{1}{2}kx^2. \tag{13.2}$$

(2) Elastic properties can be described by stress and strain. Stress is a quantity proportional to the deformation-causing force. Strain is a quantitative measure of the deformation. The elastic modulus is calculated by dividing stress by strain,

$$\text{elastic modulus} = \frac{\text{stress}}{\text{strain}}. \tag{13.3}$$

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W. M. S. Wan Hassan, *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-42678-0_13

The SI units of stress and elastic modulus are N m^{-2}, while strain has no unit.

(3) Extension, shear, and volume change are the three major deformations. There are three elastic moduli corresponding to the three deformations.

(a) Young's modulus Y is defined as follows,

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{\Delta l/l}, \tag{13.4}$$

where F is the force applied to the wire, A is the cross-sectional area, Δl is the extension, and l is the original length of the wire. The SI units of F, A, Δl, and l are N, m^2, m, and m respectively. Thus the SI units of Young's modulus and stress are N m^{-2}. Strain has no unit.

(b) Shear modulus S is defined as,

$$S = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F/A}{\Delta x/h}, \tag{13.5}$$

where F is the force which acts on area A, Δx is the transverse displacement and h is the initial length, Fig. 13.4. The SI units of shear modulus and shear stress are N m^{-2}. Shear strain has no unit.

(c) Bulk modulus B is defined as,

$$B = \frac{\text{volume stress}}{\text{volume strain}} = -\frac{F/A}{\Delta V/V} = -\frac{\Delta p}{\Delta V/V}, \tag{13.6}$$

where Δp is the change in pressure of original volume V and ΔV is the change in volume. The SI units of bulk modulus and volume stress are N m^{-2}. Volume strain has no unit.

The reciprocal of B represents the compressibility K, that is, $K = 1/B$. The SI unit of compressibility is m^2 N^{-1}.

(4) The isothermal bulk modulus of a gas is the ratio of the change in the pressure to the fractional change in the volume at constant temperature. For an ideal gas,

$$pV = \mu RT = \text{constant},$$

where p, V, μ, R, and T are the pressure, volume, number of mol, universal gas constant, and temperatue of the gas, respectively. Thus, for an isothermal process of a gas,

$$p\,dV + V\,dp = 0,$$
$$-\frac{dp}{dV/V} = p,$$

$$B_{isothermal} = -\frac{\Delta p}{\Delta V/V} = p.$$

Isothermal bulk modulus of a gas is its pressure,

$$B_{isothermal} = p. \tag{13.7}$$

For adiabatic process of a gas,

$$pV^{\gamma} = \text{constant},$$

where p and V are the pressure and volume of the gas, respectively, and $\gamma = c_p/c_v$ is the ratio of specific heat capacity of the gas at constant pressure c_p to the one at constant volume c_v and p is the pressure of the gas. Thus,

$$\begin{aligned} p\,\gamma V^{\gamma-1}dV + V^{\gamma}\,dp &= 0, \\ -\frac{dp}{dV/V} &= \gamma p, \\ B_{adiabatic} = -\frac{\Delta p}{\Delta V/V} &= \gamma p. \end{aligned}$$

Adiabatic bulk modulus of a gas is,

$$B_{adiabatic} = \gamma p. \tag{13.8}$$

(5) Poisson ratio σ of a wire is defined as,

$$\sigma = \frac{\text{change in diameter/original diameter}}{\text{extension/original length}}. \tag{13.9}$$

13.2 Problems and Solutions

Problem 13.1 When a 225 kg load is suspended by an iron rod of length 4.0 m and cross-sectional area 0.50 cm^2, the rod stretches by 1.0 mm. Determine (a) the Young's modulus, and (b) the force constant of the iron rod.

Solution

(a) Figure 13.1 shows the rod before and after the load suspended. The length of the rod is l and the extension is Δl.

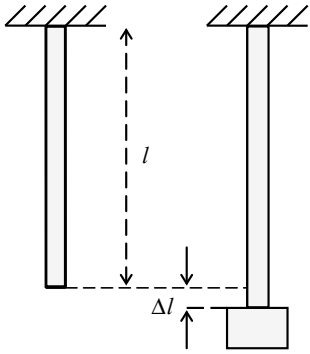

Fig. 13.1 A load suspended by an iron rod, Problem 13.1

The Young's modulus is (Eq. 13.4),

$$Y = \frac{F/A}{\Delta l/l} = \frac{Fl}{A\Delta l} = \frac{(225\text{ kg} \times 9.8\text{ N/kg})(4.0\text{ m})}{(0.50 \times 10^{-4}\text{ m}^2)(1.0 \times 10^{-3}\text{ m})} = 1.8 \times 10^{11}\text{ N m}^{-2}.$$

(b) From the Hooke's law, $F = -kx$ (Eq. 13.1), the force constant is,

$$k = \left|\frac{F}{\Delta l}\right| = \frac{225\text{ kg} \times 9.8\text{ N/kg}}{1.0 \times 10^{-3}\text{ m}} = 2.2 \times 10^{6}\text{ N m}^{-1}.$$

- wxMaxima codes:

```
(%i5)    fpprintprec:5; l:4; A:0.5e-4; dl:1e-3; F:225*9.8;
(fpprintprec)         5
(l)        4
(A)       5.0*10^-5
(dl)       0.001
(F)       2205.0
(%i6)    Y: (F/A)/(dl/l);
(Y)       1.764*10^11
(%i7)    k: F/dl;
(k)       2.205*10^6
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of l, A, Δl, and F.
(%i6), (%i7) Calculate Y and k.

Problem 13.2 As shown in Fig. 13.2, two steel wires of 80 cm in length and 1.0 mm in radius are used to suspend weights of 50 and 30 N. The steel wire has a Young's modulus of 2.0×10^{11} N m^{-2}. Calculate (a) the strain of each wire, (b) stress of each wire, (c) extension of each wire, and (d) stored energy in the wires.

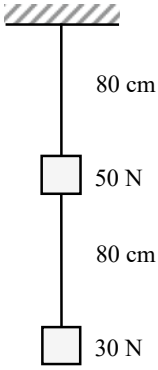

Fig. 13.2 Two identical steel wires suspending 50 and 30 N loads, Problem 13.2

Solution

(a) From the definition of Young's modulus (Eq. 13.4), the strain is,

$$Y = \frac{F/A}{\Delta l/l},$$
$$\frac{\Delta l}{l} = \frac{F}{AY}.$$

The strain of the upper wire is,

$$\frac{\Delta l}{l} = \frac{F}{AY} = \frac{50\ \text{N} + 30\ \text{N}}{\pi(1.0 \times 10^{-3}\ \text{m})^2(2.0 \times 10^{11}\ \text{N/m}^2)} = 1.3 \times 10^{-4}.$$

The strain of the lower wire is,

$$\frac{\Delta l}{l} = \frac{F}{AY} = \frac{30\ \text{N}}{\pi(1.0 \times 10^{-3}\ \text{m})^2(2.0 \times 10^{11}\ \text{N/m}^2)} = 4.8 \times 10^{-5}.$$

- wxMaxima codes:

```
(%i5)      fpprintprec:5; Y:2e11; r:1e-3; A:float(%pi*r^2);
l:0.8;
(fpprintprec)         5
(Y)       2.0*10^11
(r)       0.001
(A)       3.1416*10^-6
(l)        0.8
(%i6)    Upper_wire_strain: (50+30)/(A*Y);
```

```
(Upper_wire_strain) 1.2732*10^-4
(%i7)   Lower_wire_strain: 30/(A*Y);
(Lower_wire_strain) 4.7746*10^-5
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of Y, r, A, and l.
(%i6), (%i7) Calculate upper and lower wire strains.

(b) The stress of the upper wire is,

$$\frac{F}{A} = \frac{50\ \text{N} + 30\ \text{N}}{\pi(1.0 \times 10^{-3}\ \text{m})^2} = 2.6 \times 10^7\ \text{N m}^{-2}.$$

The stress of the lower wire is,

$$\frac{F}{A} = \frac{30\ \text{N}}{\pi(1.0 \times 10^{-3}\ \text{m})^2} = 9.6 \times 10^6\ \text{N m}^{-2}.$$

- wxMaxima codes:

```
(%i5)     fpprintprec:5; Y:2e11; r:1e-3; A:float(%pi*r^2);
l:0.8;
(fpprintprec)       5
(Y)      2.0*10^11
(r)      0.001
(A)     3.1416*10^-6
(l)      0.8
(%i6)  Upper_wire_stress: (50+30)/A;
(Upper_wire_stress) 2.5465*10^7
(%i7)  Lower_wire_stress: 30/A;
(Lower_wire_stress) 9.5493*10^6
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of Y, r, A, and l.
(%i6), (%i7) Calculate upper and lower wire stresses.

(c) The extension of the upper wire is its strain times its original length. From part (a),

$$\frac{\Delta l_{upper}}{l} = 1.3 \times 10^{-4},$$
$$\Delta l_{upper} = (1.3 \times 10^{-4})l,$$

$$\Delta l_{upper} = (1.3 \times 10^{-4})(80 \times 10^{-2}\ \text{m}) = 1.0 \times 10^{-4}\ \text{m}.$$

The extension of the lower wire is its strain times its original length. From part (a),

$$\begin{aligned}
\frac{\Delta l_{lower}}{l} &= 4.8 \times 10^{-5}, \\
\Delta l_{lower} &= (4.8 \times 10^{-5})l, \\
\Delta l_{lower} &= (4.8 \times 10^{-5})(80 \times 10^{-2}\ \text{m}) = 3.8 \times 10^{-5}\ \text{m}.
\end{aligned}$$

- wxMaxima codes:

```
(%i5)     fpprintprec:5; Y:2e11; r:1e-3; A:float(%pi*r^2);
l:0.8;
(fpprintprec)       5
(Y)     2.0*10^11
(r)      0.001
(A)     3.1416*10^-6
(l)      0.8
(%i6)   Upper_wire_strain: (50+30)/(A*Y);
(Upper_wire_strain) 1.2732*10^-4
(%i7)   Upper_wire_extension: Upper_wire_strain*l;
(Upper_wire_extension) 1.0186*10^-4
(%i8)    Lower_wire_strain: 30/(A*Y);
(Lower_wire_strain) 4.7746*10^-5
(%i9)    Lower_wire_extension: Lower_wire_strain*l;
(Lower_wire_extension) 3.8197*10^-5
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of Y, r, A, and l.
(%i7), (%i9) Calculate upper and lower wire extensions.

(d) The energy stored in the wires is,

$$\begin{aligned}
U_{upper} + U_{lower} &= \frac{1}{2}F_{upper}\Delta l_{upper} + \frac{1}{2}F_{lower}\Delta l_{lower} \\
&= \frac{1}{2}(80\ \text{N})(1.0 \times 10^{-4}\ \text{m}) + \frac{1}{2}(30\ \text{N})(3.8 \times 10^{-5}\ \text{m}) \\
&= 4.6 \times 10^{-3}\ \text{J}.
\end{aligned}$$

- wxMaxima codes:

```
(%i5)      fpprintprec:5; Y:2e11; r:1e-3; A:float(%pi*r^2);
l:0.8;
(fpprintprec)          5
(Y)        2.0*10^11
(r)         0.001
(A)        3.1416*10^-6
(l)         0.8
(%i6)      Upper_wire_strain: (50+30)/(A*Y);
(Upper_wire_strain)       1.2732*10^-4
(%i7)      Upper_wire_extension: Upper_wire_strain*l;
(Upper_wire_extension)      1.0186*10^-4
(%i8)      Lower_wire_strain: 30/(A*Y);
(Lower_wire_strain)      4.7746*10^-5
(%i9)      Lower_wire_extension: Lower_wire_strain*l;
(Lower_wire_extension)     3.8197*10^-5
(%i10)    Uupper: (1/2)*80*Upper_wire_extension;
(Uupper)     0.0040744
(%i11)    Ulower: (1/2)*30*Lower_wire_extension;
(Ulower)     5.7296*10^-4
(%i12)    Uupper + Ulower;
(%o12)    0.0046473
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of Y, r, A, and l.
(%i10), (%i11) Calculate energy stored in upper and lower wires.
(%i12) Calculate total energy stored in the wires.

Problem 13.3 Estimate the density of water in a lake at the depth of 200 m and the pressure of 21 atm. The bulk modulus of water is 2.2×10^9 Pa. A pressure of 1 atm $= 1.01 \times 10^5$ N m^{-2}.

Solution

The relationship between change in density and change in volume is as follows,

$$\rho = \frac{m}{V},$$
$$\frac{\Delta\rho}{\rho} = -\frac{\Delta V}{V}.$$

Using the definition of bulk modulus (Eq. 13.6) we have,

$$B = -\frac{\Delta p}{\Delta V/V} = \frac{\Delta p}{\Delta\rho/\rho}.$$

The change in density is,

$$\Delta\rho = \frac{\rho\ \Delta p}{B} = \frac{(1000\ \text{kg/m}^3)(21-1)(1.01\times 10^5\ \text{N/m}^2)}{2.2\times 10^9\ \text{N/m}^2}$$
$$= 0.92\ \text{kg m}^{-3}.$$

We have used the conversion 1 atm $= 1.01 \times 10^5$ N m^{-2} in the calculation. The density of water at 200 m depth is,

$$\rho + \Delta\rho = 1000\ \text{kg m}^{-3} + 0.92\ \text{kg m}^{-3}$$
$$= 1001\ \text{kg m}^{-3}.$$

- wxMaxima codes:

```
(%i4)       fpprintprec:5; delta_p:(21−1)*1.01e5; B:2.2e9;
rho:1000;
(fpprintprec)        5
(delta_p)      2.02*10^6
(B) 2.2*10^9
(rho) 1000
(%i5) delta_rho: rho*delta_p/B;
(delta_rho)   0.91818
(%i6) water_density: rho + delta_rho;
(water_density) 1000.9
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of Δp, B, and ρ.
(%i5) Calculate change in density of water.
(%i6) Calculate density of water.

Problem 13.4 When a wire is stretched, its length increases, its diameter decreases, while its volume remains the same. Show that the Poisson ratio of the wire is ½.

Solution

Figure 13.3 shows the wire before and after the stretch. The diameter d decreases, the length l increases, and the volume V remains the same in the deformation.

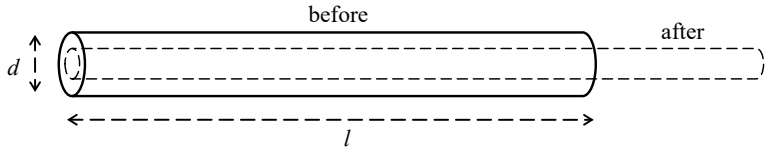

Fig. 13.3 A wire, before and after the stretch, Problem 13.4

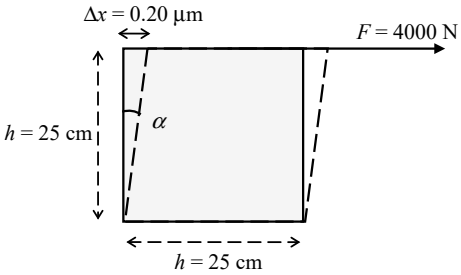

Fig. 13.4 A tangential force acting on the surface of a cube, Problem 13.5

The volume of the wire is,

$$V = \pi\left(\frac{d}{2}\right)^2 l = \frac{\pi}{4}d^2 l.$$

Using the calculus, the change in V is,

$$\Delta V = \frac{\pi}{4}l(2d)\Delta d + \frac{\pi}{4}d^2\Delta l.$$

But in the deformation, the volume remains the same, that is, $\Delta V = 0$. This implies,

$$2l\,\Delta d = -d\,\Delta l$$
$$\frac{\Delta d}{d} = -\frac{1}{2}\frac{\Delta l}{l}.$$

Thus, the Poisson ratio of the wire is,

$$\sigma = \frac{\text{change in diameter/original diameter}}{\text{extension/original length}} = \left|\frac{\Delta d/d}{\Delta l/l}\right| = \frac{1}{2}.$$

Problem 13.5 A tangential force of $F = 4000$ N acts on the surface of a cube with $h = 25$ cm sides, as shown in Fig. 13.4. The cube is sheared, and the upper surface is moved $\Delta x = 0.20\ \mu$m with respect to the lower surface. Calculate the cube's shear modulus.

Solution

From Fig. 13.4,

$$\text{shear strain} = \alpha \approx \tan\alpha = \frac{\Delta x}{h} = \frac{0.20 \times 10^{-6}\ \text{m}}{25 \times 10^{-2}\ \text{m}} = 8.0 \times 10^{-7}.$$

The shear modulus of the cube is,

$$S = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F/A}{\alpha} = \frac{F/A}{\Delta x/h} = \frac{4000\ \text{N}/(25 \times 10^{-2}\ \text{m})^2}{0.20 \times 10^{-6}\ \text{m}/25 \times 10^{-2}\ \text{m}}$$
$$= 8.0 \times 10^{10}\ \text{Pa}.$$

- wxMaxima codes:

```
(%i4)      fpprintprec:5; F:4000; A:(25e-2)^2; alpha:0.2e-6/
25e-2;
(fpprintprec)          5
(F)        4000
(A)        0.0625
(alpha) 8.0*10^-7
(%i5)     S:(F/A)/alpha;
(S) 8.0*10^10
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of F, A, and shear strain α.
(%i5) Calculate shear modulus S.

Problem 13.6 A load of 3.0 kg is suspended from a 2.0 m long, 1.0 mm diameter steel wire. Steel has a Young's modulus of 2.0×10^{11} Pa. Determine (a) the extension, and (b) the amount of energy stored in the wire.

Solution

(a) Using the definition of the Young's modulus (Eq. 13.4), the extension of the wire can be calculated. Let $\Delta l = e$,

$$Y = \frac{F/A}{e/l},$$
$$e = \frac{Fl}{AY} = \frac{(3.0\ \text{kg} \times 9.8\ \text{m/s}^2)(2.0\ \text{m})}{\pi(\frac{1.0\times10^{-3}}{2}\ \text{m})^2(2.0 \times 10^{11}\ \text{N/m}^2)} = 3.7 \times 10^{-4}\ \text{m}.$$

(b) The energy stored in the stretched wire is (Eq. 13.2),

$$U = \frac{1}{2}Fe = \frac{1}{2}(3.0\ \text{kg} \times 9.8\ \text{m/s}^2)(3.7 \times 10^{-4}\ \text{m}) = 5.5 \times 10^{-3}\ \text{J}.$$

- wxMaxima codes:

```
(%i7)   fpprintprec:5; m:3; g:9.8; l:2; d:1e-3; Y:2e11; F:m*g;
(fpprintprec)         5
(m)       3
(g)       9.8
(l)        2
(d)       0.001
(Y)       2.0*10^11
(F)       29.4
(%i9)    e: F*l/(%pi*(d/2)^2*Y)$ float(%);
(%o9)   3.7433*10^-4
(%i11) U: 0.5*F*e$ float(%);
(%o11) 0.0055027
```

Comments on the codes:

(%i7) Set floating point print precision to 5, assign values of m, g, l, d, Y, and F.
(%i9) Calculate the extension e, part (a).
(%i11) Calculate the energy stored U, part (b).

Problem 13.7 A copper wire with a length of 1.1 m and a cross-sectional area of 1.4×10^{-6} m^2 is joined to an iron wire with identical dimensions. A mass of m is suspended from the wires, and an extension of 0.01 m is observed. Copper and iron have Young's moduli of 1.3×10^{11} and 2.1×10^{11} Pa, respectively. Determine (a) the extension ratio between copper and iron, (b) the extension of each wire, and (c) the mass m.

Solution

(a) Figure 13.5 shows the two wires and the suspended load. The stresses of the two wires are the same, because the weight acts along both wires. This means that the same pulling force acts on each wire, We write (Eq. 13.4),

$$\frac{F}{A} = \frac{Ye}{l} = \frac{Y_{copper}\ e_{copper}}{l} = \frac{Y_{ironi}\ e_{iron}}{l} \tag{13.1}$$

$$\frac{e_{copper}}{e_{iron}} = \frac{Y_{iron}}{Y_{copper}} = \frac{2.1}{1.3} = 1.62. \tag{13.2}$$

The ratio of extension of copper to that of iron is 1.62.

(b) The total extension of both wires is,

$$e_{copper} + e_{iron} = 0.01 \text{ m}. \tag{13.3}$$

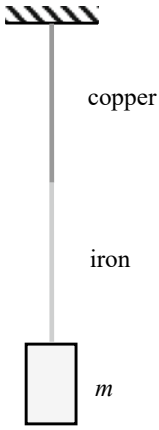

Fig. 13.5 A load suspended by copper and iron wires, Problem 13.7

Equations (13.1), (13.2), and (13.3) can be solved simultaneously to give the extensions of copper, iron, and the force. The results are,

$$e_{copper} = 0.0062 \text{ m},$$
$$e_{iron} = 0.0038 \text{ m},$$
$$F = 1022 \text{ N}.$$

(c) The mass m is,

$$m = \frac{F}{g} = \frac{1022 \text{ N}}{9.8 \text{ m/s}^2} = 104 \text{ kg}.$$

- wxMaxima codes:

```
(%i8)       fpprintprec:5; ratprint:false; l:1.1; A:1.4e-6;
e:0.01; Ycopper:1.3e11; Yiron:2.1e11; g:9.8;
(fpprintprec)           5
(ratprint)       false
(l)          1.1
(A)         1.4*10^-6
(e)          0.01
(Ycopper)        1.3*10^11
(Yiron)        2.1*10^11
(g)          9.8
(%i9)       ecopper_over_eiron: Yiron/Ycopper;
(ecopper_over_eiron)    1.6154
(%i11)   solve([F/A=Ycopper*ecopper/l, ecopper/eiron=Yiron/
Ycopper, ecopper+eiron=0.01], [ecopper,eiron,F])$ float(%);
(%o11)   [[ecopper=0.0061765,eiron=0.0038235,F=1021.9]]
```

```
(%i12)    F: rhs(%[1][3]);
(F)  1021.9
(%i13     m: F/g;
(m)  104.28
```

Comments on the codes:

(%i8) Set floating point print precision to 5, internal rational number print to false, assign values of l, A, e, Y_{copper}, Y_{iron}, and g.
(%i9) Calculate e_{copper}/e_{iron}, part (a).
(%i11) Solve $\frac{F}{A} = \frac{Y_{copper}\, e_{copper}}{l}$, $\frac{e_{copper}}{e_{iron}} = \frac{Y_{iron}}{Y_{copper}}$, and $e_{copper} + e_{iron} = 0.01$ for e_{copper}, e_{iron}, and F, part (b).
(%i12) Assign F.
(%i13) Calculate m, part (c).

Problem 13.8 Fig. 13.6 depicts a 100 cm long light bar suspended by two wires, one copper wire with cross-sectional area of 1.0 mm^2 and the other steel wire with a cross-sectional area of 2.0 mm^2. Copper and steel have Young's modulus of 1.3×10^{11} and 2.3×10^{11} Pa, respectively. Where should a load of weight W be placed so that,

(a) each wire experiences the same amount of stress?
(b) the strain in each wire is identical?

Solution

(a) Figure 13.7 shows the forces acting on the bar. Here, F_{copper} and F_{steel} are the forces of copper and steel wires, W is the weight of the load, and x is the position where W is suspended.

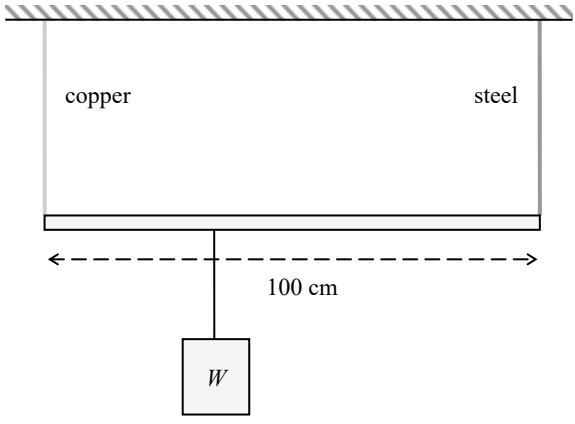

Fig. 13.6 A light bar with a load, suspended by copper and steel wires, Problem 13.8

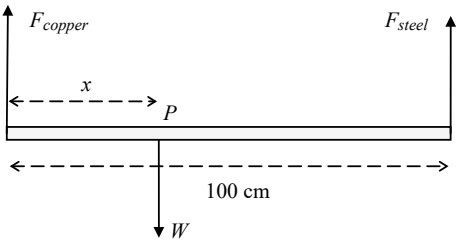

Fig. 13.7 Forces on the bar, Problem 13.8

The net force in the y direction is zero. This implies,

$$F_{copper} + F_{steel} - W = 0. \tag{13.1}$$

We want the stress in the copper and steel wires to be the same, this means,

$$\frac{F}{A} = \frac{F_{copper}}{1.0 \times 10^{-6}\ \text{m}^2} = \frac{F_{steel}}{2.0 \times 10^{-6}\ \text{m}^2}. \tag{13.2}$$

The net torque about P is zero. This means,

$$F_{steel}(1.0\ \text{m} - x) - F_{copper}\ x = 0. \tag{13.3}$$

From Eqs. (13.2) and (13.3), we get,

$$x = 0.67\ \text{m}.$$

The stress in each wire is the same if the weight W is placed at $x = 0.67$ m.

- wxMaxima codes:

```
(%i2)   fpprintprec:5; ratprint:false;
(fpprintprec)       5
(ratprint)     false
(%i4)       solve([Fcopper/1e-6=Fsteel/2e-6, Fsteel*(1-x)-
Fcopper*x=0], [x,Fcopper])$ float(%);
(%o4)   [[x=0.66667,Fcopper=0.5*Fsteel]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $\frac{F_{copper}}{1.0\times10^{-6}} = \frac{F_{steel}}{2.0\times10^{-6}}$ and $F_{steel}(1-x) - F_{copper}\ x = 0$ for x and F_{copper}.

(b) This means that we want the strains in both wires to be the same. This means that the quantities F/AY in both wires are the same,

$$\frac{F_{copper}}{A_{copper}\ Y_{copper}} = \frac{F_{steel}}{A_{steel}\ Y_{steel}},$$
$$\frac{F_{copper}}{(1.0 \times 10^{-6}\ \text{m}^2)(1.3 \times 10^{11}\ \text{N/m}^2)} = \frac{F_{steel}}{(2.0 \times 10^{-6}\ \text{m}^2)(2.3 \times 10^{11}\ \text{N/m}^2)}. \tag{13.4}$$

Solving Eqs. (13.3) and (13.4) gives,

$$x = 0.78\ \text{m}.$$

The strain in each wire is the same if $x = 0.78$ m.

- wxMaxima codes:

```
(%i6)          fpprintprec:5; ratprint:false; Acopper:1e-6;
Asteel:2e-6; Ycopper:1.3e11; Ysteel:2.3e11;
(fpprintprec)          5
(ratprint)       false
(Acopper)      10.0*10^-7
(Asteel)        2.0*10^-6
(Ycopper)      1.3*10^11
(Ysteel)        2.3*10^11
(%i8)              solve([Fcopper/(Acopper*Ycopper)=Fsteel/
(Asteel*Ysteel),Fsteel*(1-x)-Fcopper*x=0],    [x,Fcopper])$
float(%);
(%o8)    [[x=0.77966,Fcopper=0.28261*Fsteel]]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of A_{copper}, A_{steel}, Y_{copper}, and Y_{steel}.
(%i8) Solve $\frac{F_{copper}}{A_{copper}Y_{copper}} = \frac{F_{steel}}{A_{steel}Y_{steel}}$ and $F_{steel}(1-x) - F_{copper}\ x = 0$ for x and F_{copper}.

Problem 13.9 A 150 cm long wire is fastened between two walls. As shown in Fig. 13.8a, when weight W is suspended in the middle of the wire, the wire stretches and the weight moves downward by 6.0 cm. The wire has a cross-sectional area of 2.5 mm^2 and a Young's modulus of 2.0×10^{11} Pa. Determine (a) the extension of the wire, (b) the tension in the wire, and (c) the weight W.

Solution

(a) From Fig. 13.8b, the extension of the wire is,

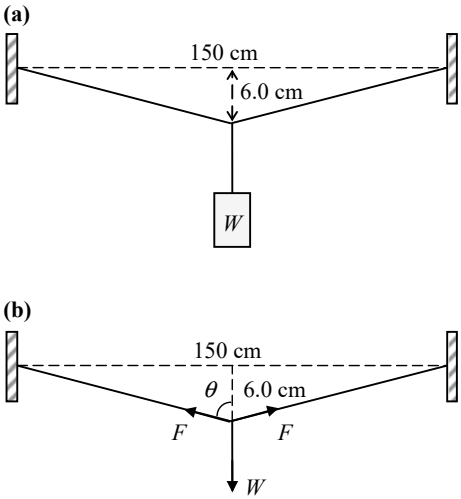

Fig. 13.8 **a** A wired fastened between two walls with a load at the middle, **b** forces on the wire, Problem 13.9

$$2 \times \left(\sqrt{75^2 + 6.0^2} - 75\right) \text{ cm} = 2 \times 0.24 \text{ cm} = 0.48 \text{ cm}.$$

(b) Using the definition of the Young's modulus (Eq. 13.4), the tension in the wire F can be calculated,

$$Y = \frac{Fl}{Ae},$$
$$F = \frac{YAe}{l} = \frac{(2.0 \times 10^{11} \text{ N/m}^2)(2.5 \times 10^{-6} \text{ m}^2)(0.24 \times 10^{-4} \text{ m})}{75 \times 10^{-2} \text{ m}}$$
$$= 1600 \text{ N}.$$

Figure 13.8b shows the tension F in the wire.

(c) The tension F acts along the wire, the vertical component of this force acting upward is (Fig. 13.8b),

$$\cos\theta \times F = \frac{6.0 \text{ cm}}{\sqrt{(75 \text{ cm})^2 + (6.0 \text{ cm})^2}} \times F.$$

The weight W is balanced by two of the above forces. The weight of the load is,

$$W = 2 \times \frac{6.0 \text{ cm}}{\sqrt{(75 \text{ cm})^2 + (6.0 \text{ cm})^2}} \times F$$

$$= 2 \times \frac{6.0 \text{ cm}}{\sqrt{(75 \text{ cm})^2 + (6.0 \text{ cm})^2}} (1600 \text{ N})$$
$$= 255 \text{ N}.$$

- wxMaxima codes:

```
(%i4)      fpprintprec:5; Y:2e11; A:2.5e-6; l:75e-2;
(fpprintprec)          5
(Y)        2.0*10^11
(A)        2.5*10^-6
(l)        0.75
(%i6)           extension:  2*(sqrt(75e-2^2+6e-2^2)-75e-2)$
float(%);
(%o6)    0.0047923
(%i7)     e: extension/2;
(e)       0.0023962
(%i8)    F: Y*A*e/l;
(F)       1597.4
(%i10) W: 2*F*6/sqrt(75^2+6^2)$ float(%);
(%o10) 254.78
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of Y, A, and l.
(%i6), (%i7), (%i8), (%i10) Calculate extension, e, F, and W.

Problem 13.10 The volume of a solid sphere of brass is 0.50 m^3 and the atmospheric pressure is 1.01×10^5 Pa at the surface of the earth. Calculate the change in volume of the sphere if it is placed at the sea floor where the pressure is 2.0×10^7 Pa. The bulk modulus of brass is 6.1×10^{10} Pa.

Solution

The change in volume of the sphere can be calculated based on the definition of the bulk modulus (Eq. 13.6),

$$B = -\frac{\Delta p}{\Delta V / V},$$
$$\Delta V = -\frac{V \Delta p}{B} = -\frac{(0.50 \text{ m}^3)(2.0 \times 10^7 \text{ Pa} - 1.01 \times 10^5 \text{ Pa})}{6.1 \times 10^{10} \text{ Pa}}$$
$$= -1.6 \times 10^{-4} \text{ m}^3.$$

The volume of the sphere decreases by 1.6×10^{-4} m^3.

- wxMaxima codes:

```
(%i3)     fpprintprec:5; V:0.5; B:6.1e10;
(fpprintprec)          5
(V)       0.5
(B)       6.1*10^10
(%i4)     dV: -V*(2e7-1.01e5)/B;
(dV)      -1.6311*10^-4
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of V and B.
(%i4) Calculate ΔV.

Problem 13.11 Prove that the energy stored per unit volume for a stretched wire is,

$$\frac{\tau^2}{2Y},$$

where τ represents the stress and Y represents the Young's modulus.

Solution

Let A, e, and l be cross-sectional area, extension, and original length of the wire, while F is the force applied. We have,

$$\begin{aligned} Y &= \frac{F/A}{e/l} = \frac{Fl}{Ae}, \\ F &= \frac{YAe}{l}, \\ \tau &= \frac{F}{A} = \frac{Ye}{l}, \\ F(x) &= \frac{YAx}{l}. \end{aligned}$$

The energy stored in the wire is,

$$\begin{aligned} W &= \int_0^e F(x)\,dx = \int_0^e \frac{YAx}{l}dx = \left[\frac{1}{2}\frac{YA}{l}x^2\right]_0^e \\ &= \frac{1}{2}\frac{YAe^2}{l} = \frac{1}{2}\frac{YAe^2}{l}\frac{Yl}{Yl} = \frac{1}{2}\left(\frac{Ye}{l}\right)^2\frac{Al}{Y} \\ &= \frac{1}{2}\frac{\tau^2 Al}{Y}. \end{aligned}$$

Thus, the energy stored per unit volume is,

$$\frac{W}{V} = \frac{W}{Al} = \frac{\tau^2}{2Y}.$$

Alternative solution: The energy stored in a wire that is stretched by e is,

$$W = \frac{1}{2}Fe.$$

Thus, the energy stored per unit volume is,

$$\begin{aligned}\frac{W}{V} &= \frac{W}{Al} = \frac{1}{2}\frac{Fe}{Al} \\ &= \frac{1}{2}\left(\frac{F}{A}\right)\left(\frac{F}{A}\right)\left(\frac{Ae}{Fl}\right) \\ &= \frac{\tau^2}{2Y}.\end{aligned}$$

Problem 13.12 A rubber cord with an initial length 0.20 m and a cross-sectional area of 2.0 mm^2 is stretched to 0.25 m. Rubber has a Young's modulus of 6.0×10^8 N m^{-2}.

(a) What is the stored potential energy in the stretched cord?
(b) A small object with a mass of 10 g is attached to and released from one end of the cord. What is the speed of the object?

Solution

(a) Figure 13.9 shows the stretched rubber cord and the small object. Here, the cross-sectional area is A, the initial length is l, and the extension is x.

The potential energy in the stretched cord is (Eq. 13.2),

$$\begin{aligned}W &= \frac{1}{2}Fx = \frac{1}{2}\frac{YAx}{l}x = \frac{1}{2}\frac{YAx^2}{l} \\ &= \frac{1}{2}\frac{(6.0 \times 10^8 \text{ N/m}^2)(2.0 \times 10^{-6} \text{ m}^2)(0.25 \text{ m} - 0.20 \text{ m})^2}{0.20 \text{ m}} \\ &= 7.5 \text{ J},\end{aligned}$$

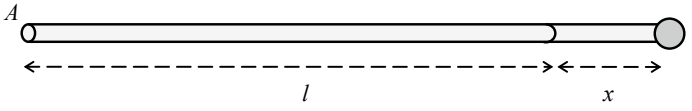

Fig. 13.9 A stretched rubber chord and a small object attached, Problem 13.12

where we invoke the fact that $Y = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{x/l} \Rightarrow F = \frac{YAx}{l}$.

(b) When the cord and object is released, the potential energy is transferred to the kinetic energy of the object, thus,

$$W = \frac{1}{2}mv^2,$$

$$v = \sqrt{\frac{2W}{m}} = \sqrt{\frac{2(7.5\text{ J})}{0.010\text{ kg}}} = 39\text{ m s}^{-1}.$$

- wxMaxima codes:

```
(%i7)   fpprintprec:5; ratprint:false; A:2e-6; l:0.2; x:0.25-
l; Y:6e8; m:0.01;
(fpprintprec)        5
(ratprint)       false
(A)       2.0*10^-6
(l)        0.2
(x)       0.05
(Y)       6.0*10^8
(m)       0.01
(%i8)     W: 0.5*Y*A*x^2/l;
(W)       7.5
(%i10)    solve(W=0.5*m*v^2, v)$ float(%);
(%o10)   [v=-38.73,v=38.73]
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of A, l, x, Y, and m.
(%i8) Calculate W, part (a).
(%i10) Solve $W = \frac{1}{2}mv^2$ for v, part (b).

13.3 Summary

- An object is elastic if it comes back to its original size when the stress vanishes.
- External forces on an object cause its deformation. The strength of the forces that cause the deformation is the stress. The degree of deformation is the strain.
- Extension, shear, and volume change are the three main deformations. The three elastic moduli corresponding to the three deformations are,

(a) Young's modulus, $Y = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{\Delta l/l}$,
(b) Shear modulus, $S = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F/A}{\Delta x/h}$,

(c) Bulk modulus, $B = \frac{\text{volume stress}}{\text{volume strain}} = -\frac{F/A}{\Delta V/V} = -\frac{\Delta p}{\Delta V/V}$.

13.4 Exercises

Exercise 13.1 A 5.0 kg mass is suspended to a 2.0 m long wire and of cross Sect. 2.5 $\times 10^{-5}$ m^2. Steel has Young's modulus of 2.0 $\times 10^{11}$ N m^{-2}. Calculate the extension of the wire.

(Answer: 2.0 $\times 10^{-5}$ m)

Exercise 13.2 A 100 N force is applied lengthwise to a 4.5 m long and 6.0 mm diameter nylon string tied to a nail. Calculate the elongation and decrease in diameter of the string. Nylon has Young's modulus of 4.8 $\times 10^{11}$ N m^{-2} and Poisson ratio of 0.2.

(Answer: 3.3 $\times 10^{-5}$ m, 8.8 $\times 10^{-9}$ m)

Exercise 13.3 A 20 N force is applied lengthwise to a 50 cm long and 3.0 mm^2 cross section wire tied at one end. The wire has Young's modulus of 1.9 $\times 10^{11}$ N m^{-2}. Calculate the elastic potential energy in the wire.

(Answer: 1.8 $\times 10^{-4}$ J)

Exercise 13.4 Figure 13.10 shows a uniform heavy rod of weight W, length l, cross-sectional area A, and Young's modulus Y, stands upright on the ground. Calculate the contraction of the rod due to its weight and elastic potential energy stored.

(Answer: $\frac{Wl}{2AY}$, $\frac{W^2l}{8AY}$)

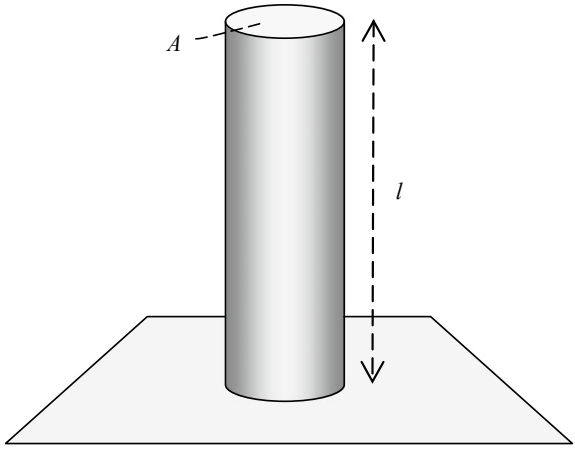

Fig. 13.10 A heavy rod standing upright on the ground, Exercise 13.4

Exercise 13.5 A uniform heavy iron rod that has a weight of 250 N, length of 2.0 m, cross-sectional area of 16×10^{-4} m^2, and Young's modulus of 2.1×10^{11} N m^{-2} is hanging from a fixed support. Calculate the elongation and the elastic potential energy of the rod due to its weight.

(Answer: 7.4×10^{-7} m, 4.6×10^{-5} J)

Chapter 14
Hydrostatics

14.1 Basic Concepts and Formulae

(1) The density of a material ρ is the amount of mass per unit volume,

$$\rho = \frac{m}{V}, \tag{14.1}$$

where m represents mass and V represents volume. The SI unit of density is kg m^{-3}.

(2) The fluid pressure p is the force per unit area in the fluid,

$$p = \frac{F}{A}, \tag{14.2}$$

where F represents the force and A represents the area. The SI unit of pressure is N m^{-2} or pascal (Pa),

$$1\,\mathrm{N\,m^{-2}} = 1\,\mathrm{Pa}.$$

The atmosperic pressure of 1 atm is defined as,

$$1\ \mathrm{atm} = 1.013 \times 10^5\ \mathrm{N\ m^{-2}}.$$

(3) The pressure in a fluid varies with depth h,

$$p = p_a + \rho g h, \tag{14.3}$$

where $p_a = 1.013 \times 10^5$ N m^{-2} is the atmospheric pressure and ρ is the density of the fluid.

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W. M. S. Wan Hassan, *Physics—Problems, Solutions, and Computer Calculations*,
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(4) According to Pascal's law, a pressure applied to a trapped fluid is transferred to every point of the fluid and the container's wall.
(5) When an object is completely or partially submerged in a fluid, the fluid exerts an upward force on the object. The force on the object is known as buoyant force or thrust. The Archimedes' principle states that the buoyant force equals the weight of the fluid an object displaces. Therefore, when an object floats in a liquid, its weight is equal to the weight of the liquid it displaces. This indicates that the mass of the object is equal to the mass of the liquid it displaces.
(6) The surface tension of a liquid denoted by γ, is the force per unit length acting on the liquid surface. The force is perpendicular to a line drawn on the liquid surface. The SI unit of surface tension is N m^{-1}. Thus,

$$\text{surface tension force} = \gamma \times \text{length of line on liquid surface.} \tag{14.4}$$

(7) For a bubble in a liquid, the difference in pressure is,

$$p_{in} - p_{out} = \frac{2\gamma}{r}, \tag{14.5}$$

where p_{in} represents the pressure in the bubble, p_{out} represents the pressure outside the bubble, and r represents the radius of the bubble.

14.2 Problems and Solutions

Problem 14.1 Ten percent of the volume of an ice chunk is above sea surface. Calculate the density of the seawater given the density of the ice chunk is 920 kg m^{-3}.

Solution

Figure 14.1 depicts the floating ice chunk and the forces acting upon it. The forces consist of the chunk weight and the bouyant force or the thrust on the chunk.

There is equilibrium when floating, meaning that the weight is equal in magnitude and opposite in direction to the thrust,

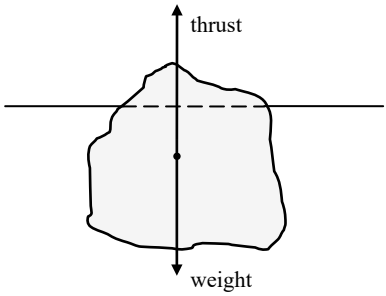

Fig. 14.1 A floating ice chunk, Problem 14.1

$$\text{weight of ice chunk} = \text{seawater thrust on ice chunk.}$$

Using Archimedes' principle, the thrust equals the weight of seawater that the chunk displaces,

$$\text{weight of ice chunk} = \text{weight of seawater the ice chunk displaces.}$$

This suggests,

$$V\rho_{ice}g = 0.9V\rho_{sea}g,$$

where V represents the volume of the ice chunk, ρ_{ice} represents its density, and ρ_{sea} represents the density of the seawater. Thus, the seawater's density is,

$$\rho_{sea} = \frac{\rho_{ice}}{0.9} = \frac{920 \text{ kg m}^{-3}}{0.9} = 1022 \text{ kg m}^{-3}.$$

- wxMaxima codes:

```
(%i2)   fpprintprec:5; rho_ice:920;
(fpprintprec)     5.
(rho_ice)     920.
(%i3)     rho_sea: rho_ice/0.9;
(rho_sea)     1022.2
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and assign ρ_{ice}.
(%i3) Calculate ρ_{sea}.

Problem 14.2 An alloy of 170 g mass is composed of metals X and Y. The weight of the alloy in a 1.5 g cm^{-3} density liquid is 0.93 N. Metals X and Y have relative densities of 4.0 and 3.0, respectively. What is the volume ratio of metal X to Y? Water has a density of 1.0 g cm^{-3}.

Solution

Figure 14.2 shows the weight of the alloy as measured in air and in the liquid.

From the measurements,

$$\text{weight of alloy in air} = 0.17 \text{ kg} \times 9.8 \text{ N/kg} = 1.67 \text{ N},$$
$$\text{weight of alloy in liquid} = 0.93 \text{ N}.$$

The thrust on the alloy block by the liquid is,

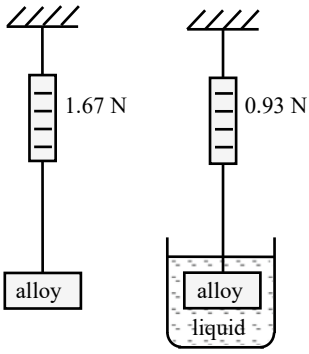

Fig. 14.2 Weight of an alloy as measured in air and in a liquid, Problem 14.2

$$1.67\,\text{N} - 0.93\,\text{N} = 0.74\,\text{N},$$

and this equals the weight of displaced liquid.

The mass of displaced liquid is,

$$\frac{0.74\,\text{N}}{9.8\,\text{N/kg}} = 0.075\,\text{kg}.$$

The volume of displaced liquid is,

$$\frac{m}{\rho} = \frac{75\text{ g}}{1.5\text{ g/cm}^3} = 50\text{ cm}^3 = \text{volume of the alloy block}.$$

Assume,

$$\text{volume of the alloy block} = \text{volume of metal } X + \text{volume of metal } Y,$$

$$50\text{ cm}^3 = V_X + V_Y. \tag{14.6}$$

Also, assume,

$$\text{mass of alloy block} = \text{mass of } X + \text{mass of } Y,$$

$$170\text{ g} = (4.0\text{ g/cm}^3)V_X + (3.0\text{ g/cm}^3)(50\text{ cm}^3 - V_X). \tag{14.7}$$

Solving Eqs. (14.6) and (14.7) gives,

$$V_X = 20\,\text{cm}^3 \text{ and } V_Y = 30\,\text{cm}^3.$$

The volume ratio of metal X to Y is,

$$V_X : V_Y = 20 : 30 = 2 : 3.$$

- wxMaxima codes:

```
(%i2)    fpprintprec:5; ratprint:false;
(fpprintprec)    5.
(ratprint)    false.
(%i3)    weight_in_air: 0.17*9.8;
(weight_in_air)    1.666.
(%i4)    thrust: weight_in_air - 0.93;
(thrust)    0.736.
(%i5)    mass_of_displaced_liquid: thrust/9.8;
(mass_of_displaced_liquid)    0.075102.
(%i6)    volume_of_block: mass_of_displaced_liquid/1.5;
(volume_of_block)    0.050068.
(%i7)    solve([50 = VX + VY, 170 = 4*VX + 3*(50-VX)], [VX,
VY]);
(%o7)    [[VX = 20,VY = 30]].
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3), (%i4) Calculate weight of alloy in air and thrust.
(%i5), (%i6) Calculate mass of displaced liquid and volume of the block.
(%i7) Solve Eqs. (14.6) and (14.7) for V_X and V_Y.

Problem 14.3 The density of aluminum is 2.3 g cm^{-3} and the density of copper is 6.5 g cm^{-3}. The density of an alloy made of both metals is 4.5 g cm^{-3}. Calculate the ratios of the metals in terms of their volumes and masses.

Solution

Assume the metal contents of the alloy are as in Fig. 14.3.

$$\text{Density of the alloy} = \frac{m_a + m_c}{V_a + V_c} = 4.5 \text{ g cm}^{-3}.$$

$$\text{Density of aluminum} = \frac{m_a}{V_a} = 2.3 \text{ g cm}^{-3}.$$

$$\text{Density of copper} = \frac{m_c}{V_c} = 6.5 \text{ g cm}^{-3}.$$

This means that,

$$4.5 = \frac{m_a + m_c}{V_a + V_c} = \frac{2.3V_a + 6.5V_c}{V_a + V_c},$$

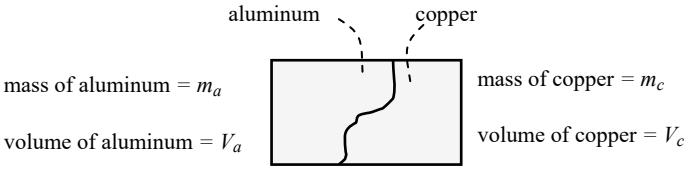

Fig. 14.3 An alloy of aluminum and copper, Problem 14.3

and this gives,

$$\frac{V_a}{V_c} = 0.91.$$

Also,

$$4.5 = \frac{m_a + m_c}{V_a + V_c} = \frac{m_a + m_c}{\frac{m_a}{2.3} + \frac{m_c}{6.5}},$$

and this gives,

$$\frac{m_a}{m_c} = 0.32.$$

In the alloy, the volume ratio of aluminum to copper is 0.91 and the mass ratio of aluminum to copper is 0.32.

- wxMaxima codes:

```
(%i2)    fpprintprec:5; ratprint:false;
(fpprintprec)    5.
(ratprint)    false.
(%i4)    solve([(ma + mc)/(Va + Vc) = 4.5, ma/Va = 2.3, mc/Vc
= 6.5], [Va, Vc, ma])$ float(%);
(%o4)    [[Va = 0.13986*mc,Vc = 0.15385*mc,ma = 0.32168*mc]].
(%i5)    volume_ratio: 0.13986/0.15385;
(volume_ratio)    0.90907.
(%i6)    mass_ratio: 0.32168;
(mass_ratio)    0.32168.
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $(m_a + m_c)/(V_a + V_c) = 4.5,\ m_a/V_a = 2.3,$ and $m_c/V_c = 6.5$ for V_a, V_c, and m_a.
(%i5) Calculate volume ratio.
(%i6) Calculate mass ratio.

Problem 14.4 The weight of a solid in air is 2.33 N. Its weight when completely submerged in liquid Y of relative density 0.90 is 0.12 N. Calculate,

(a) the relative density of the solid
(b) the relative density of liquid Z in which the solid floats with 1/5 of its volume is above the surface.

Solution

(a) Figure 14.4 shows the weights of the solid when measured in air and in liquid Y.

$$\begin{aligned}\text{Thrust on the solid by liquid } Y &= 2.33\text{ N} - 0.12\text{ N}\\ &= 2.21\text{ N}\\ &= \text{weight of displaced liquid } Y.\end{aligned}$$

$$\begin{aligned}\text{Volume of displaced liquid } Y &= \frac{(2.21/9.8)\text{ kg}}{900\text{ kg/m}^3}\\ &= 0.25\times 10^{-3}\text{ m}^3\\ &= \text{volume of the solid.}\end{aligned}$$

$$\begin{aligned}\text{Relative density of the solid} &= \frac{\text{density of the solid}}{\text{density of water}}\\ &= \frac{(2.21/9.8)\text{kg}/\left(0.25\times 10^{-3}\text{ m}^3\right)}{1000\text{ kg/m}^3}\\ &= 0.95.\end{aligned}$$

- wxMaxima codes:

```
(%i4)   fpprintprec:5; weight_solid:2.33; weight_in_Y:0.12;
density_Y:900;
(fpprintprec)    5.
(weight_solid)    2.33.
```

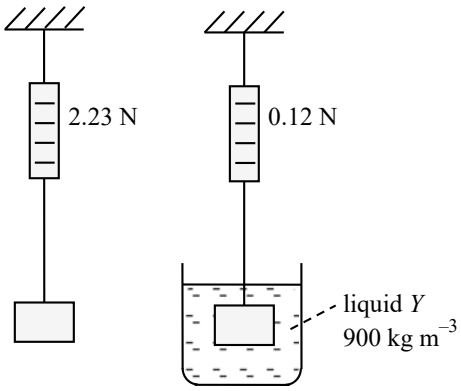

Fig. 14.4 Weight of a solid as measured in air and in liquid Y, Problem 14.4

```
(weight_in_Y)    0.12.
(density_Y)    900.
(%i5)    thrust: weight_solid - weight_in_Y;
(thrust)    2.21.
(%i6)    volume_displace_Y: thrust/(9.8*density_Y);
(volume_displace_Y)    2.5057*10^-4.
(%i7)     density_solid: weight_solid/(9.8*volume_displace_
Y);
(density_solid)    948.87.
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of weight of solid, weight of solid in liquid Y, and density of Y.
(%i5) Calculate thrust on the solid by liquid Y.
(%i6) Calculate volume of displaced liquid Y.
(%i7) Calculate density of solid.

(b) Figure 14.5 shows the solid floating in liquid Z.

Upon floating,

$$\text{weight of solid} = \text{weight of displaced liquid } Z,$$

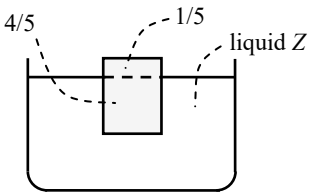

Fig. 14.5 The solid floats in liquid Z, Problem 14.4

$$2.33\ \text{N} = \frac{4}{5}\left(0.25 \times 10^{-3}\ \text{m}^3\right)(9.8\ \text{N/kg})\rho,$$

where ρ is the density of liquid Z. Thus, the density of liquid Z is,

$$\rho = 1190\ \text{kg m}^{-3}.$$

and the relative density of liquid Z is 1.19.

- wxMaxima codes:

```
(%i2)    fpprintprec:5; ratprint:false;
(fpprintprec)    5.
(ratprint)    false.
(%i4)    solve(2.33 = 4/5*0.25e-3*9.8*rho, rho)$ float(%);
(%o4)    [rho = 1188.8].
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $2.33 = \frac{4}{5} \times 0.25 \times 10^{-3} \times 9.8 \times \rho$ for ρ.

Problem 14.5 A hydrometer weighs 24 g. Its graduated stem reading is 1.00 when it floats in 1000 kg m^{-3} water and 0.99 when it floats in 990 kg m^{-3} liquid. The distance between the graduated marks 1.00 and 0.99 is 6.0 mm. Calculate the cross-sectional area of the hydrometer's graduated stem.

Solution

Figure 14.6 depicts a floating hydrometer in water and liquid. Let V denote the hydrometer's volume, a the cross-sectional area of the graduated stem, and m the hydrometer's mass.

Applying the Archimedes' principle, in water, Fig. 14.6a,
weight of the hydrometer = weight of the water displaced,

$$\begin{aligned} mg &= (V - ay_w)\rho_w g, \\ m &= (V - ay_w)\rho_w, \\ (24 \times 10^{-3}\ \text{kg}) &= (V - ay_w)(1000\ \text{kg/m}^3), \\ 24 \times 10^{-6}\ \text{m}^3 &= V - ay_w. \end{aligned} \tag{14.1}$$

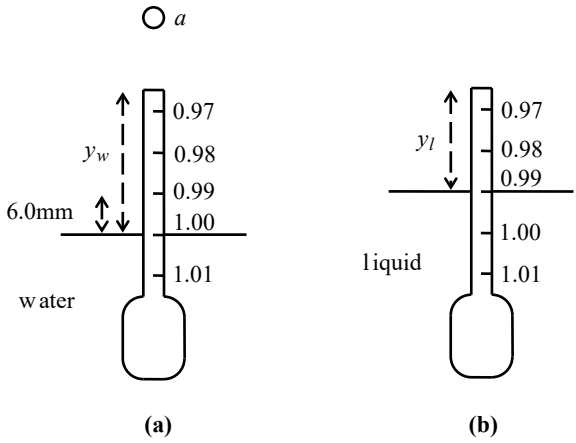

Fig. 14.6 A hydrometer floating in water (**a**), and in a liquid (**b**), Problem 14.5

Applying the Archimedes' principle, in liquid, Fig. 14.6b,

$$\text{weight of the hydrometer} = \text{weight of the liquid displaced},$$

$$\begin{aligned} mg &= (V - ay_l)\rho_l g, \\ m &= (V - ay_l)\rho_l, \\ (24 \times 10^{-3}\ \text{kg}) &= (V - ay_l)(990\ \text{kg/m}^3), \\ 2.424 \times 10^{-5}\ \text{m}^3 &= V - ay_l. \end{aligned} \tag{14.2}$$

The separation between the 1.00 and 0.99 graduated marks is 6.00 mm, that is,

$$y_w - y_l = 6.0 \times 10^{-3}\ \text{m}. \tag{14.3}$$

From Eqs. (14.1), (14.2), and (14.3), the cross-sectional area of the graduated stem is calculated as,

$$\begin{aligned} 2.424 \times 10^{-7}\ \text{m}^3 &= a(y_w - y_l) = a(6.0 \times 10^{-3}\ \text{m}), \\ a &= 4.0 \times 10^{-5}\ \text{m}^2. \end{aligned}$$

- wxMaxima codes:

```
(%i6)    fpprintprec:5; ratprint:false; m:24e-3; rho_w:1000;
rho_l:990; g:9.8;
(fpprintprec)    5.
(ratprint)    false.
```

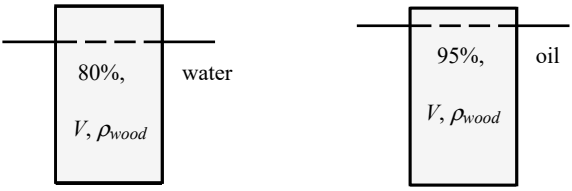

Fig. 14.7 A wooden block floating in water and oil, Problem 14.6

```
(m)   0.024.
(rho_w)   1000.
(rho_l)   990.
(g)   9.8
(%i8)     solve([m*g = (V-a*yw)*rho_w*g, m*g = (V-a*yl)*rho_
l*g, yw-yl = 6e-3],
[a, V, yw])$ float(%);
(%o8)    [[a = 4.0404*10^-5,V = 8.0808*10^6*(5.0*yl + 3.0),yw
= 0.002*(500.0*yl + 3.0)]].
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of m, ρ_w, ρ_l, and g.
(%i8) Solve $mg = (V - ay_w)\rho_w g$, $mg = (V - ay_l)\rho_l g$, and $y_w - y_l = 6 \times 10^{-3}$ for a, V, and y_w.

Problem 14.6 A block of wood floats in water with 80% of its volume submerged. In oil, the wooden block floats with 95% of its volume submerged. Calculate the density of the oil and wooden block.

Solution

Figure 14.7 shows the wooden block floating in water and oil.

Let the volume of the wood block be V and its density ρ_{wood}. Floating in water means,

$$\text{weight of wood} = \text{weight of water displaced},$$
$$V\rho_{wood}g = (0.8V)\rho_{water}g.$$

From the equation, by cancelling V and g, the density of the wooden block is,

$$\rho_{wood} = 0.8\rho_{water} = 0.8(1000 \text{ kg/m}^3) = 800 \text{ kg m}^{-3}.$$

Floating in oil means,

$$\text{weight of wood} = \text{weight of oil displaced},$$
$$V\rho_{wood}g = (0.95V)\rho_{oil}g.$$

This means that the density of the oil is,

$$\rho_{oil} = \frac{\rho_{wood}}{0.95} = \frac{800 \text{ kg/m}^3}{0.95} = 842 \text{ kg m}^{-3}.$$

- wxMaxima codes:

```
(%i3)     fpprintprec:5; ratprint:false; rho_water:1000;
(fpprintprec)     5.
(ratprint)     false.
(rho_water)     1000.
(%i5)     solve(V*rho_wood*g = 0.8*V*rho_water*g, rho_wood)$
float(%);
(%o5)     [rho_wood = 800.0].
(%i6)     rho_wood: 800.0;
(rho_wood)     800.0
(%i8)      solve(V*rho_wood*g = 0.95*V*rho_oil*g, rho_oil)$
float(%);
(%o8)     [rho_oil = 842.11].
```

Comments on the codes:

(%i3) Set floating point print precision to 5, internal rational number print to false, assign value of ρ_{water}.
(%i5) Solve $V\rho_{wood}g = (0.8V)\rho_{water}g$ for ρ_{wood}.
(%i6) Assign value of ρ_{wood}.
(%i8) Solve $V\rho_{wood}g = (0.95V)\rho_{oil}g$ for ρ_{oil}.

Problem 14.7 A hydrometer floats in water with 6.0 cm of its graduated stem above the water surface. In oil of density 0.80 g cm^{-3}, the hydrometer floats with 4.0 cm of its graduated stem above the oil surface. The hydrometer is floated in a liquid of density 0.90 g cm^{-3}. Calculate the length of the graduated stem that is above the surface of the liquid.

Solution

Figure 14.8 shows the hydrometer floating in water, oil, and the liquid. The weights of displaced water, oil, and liquid in these cases are the same and equal to the weight of the hydrometer.

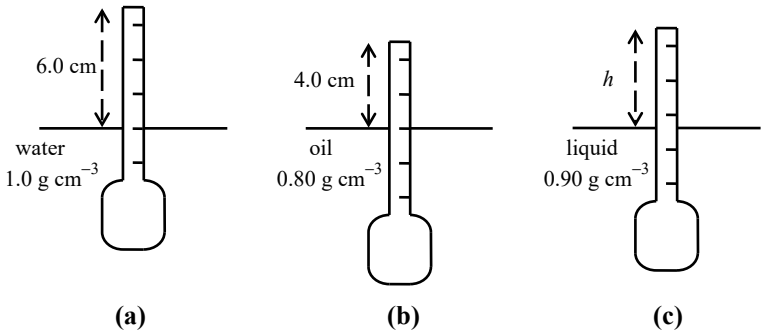

Fig. 14.8 A hydrometer floating in water (**a**), oil (**b**), and liquid (**c**), Problem 14.7

Let V be the volume of the hydrometer and a the cross-sectional area of the graduated stem. We write,

$$\begin{aligned} \text{weight of hydrometer } W &= (V-6a)(1)g && \text{(in water)} \\ &= (V-4a)(0.8)g && \text{(in oil)} \\ &= (V-ha)(0.9)g && \text{(in liquid)} \end{aligned}$$

These equations are solved and the length of the graduated stem that is above the surface of the liquid is,

$$h = 5.1\,\text{cm}.$$

- wxMaxima codes:

```
(%i3)     fpprintprec:5; ratprint:false; g:9.8;
(fpprintprec)     5.
(ratprint)     false.
(g)     9.8
(%i5)      solve([W = (V-6*a)*g, W = (V-4*a)*0.8*g, W = (V-
h*a)*0.9*g], [h,a,W])$ float(%);
(%o5)     [[h = 5.1111,a = 0.071429*V,W = 5.6*V]].
```

Comments on the codes:

(%i3) Set floating point print precision to 5, internal rational number print to false, and assign value of g.
(%i5) Solve $W = (V-6a)(1)g$, $W = (V-4a)(0.8)g$, and $W = (V-ha)(0.9)g$ for h, a, and W.

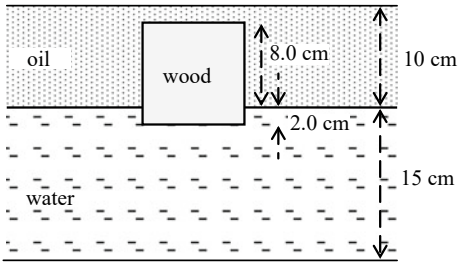

Fig. 14.9 A wooden cube floating between oil and water, Problem 14.8

Problem 14.8 As shown in Fig. 14.9, a wooden cube with sides of 10 cm floats between oil and water. Oil has a density of 0.90 g cm^{-3}. Calculate (a) the mass of the wooden cube, (b) the density of the wood, and (c) the gauge pressure at the cube's underside.

Solution

(a) Upon floating, the weight of the wooden cube is equal to the weight of displaced oil and water. This implies that the mass of the wood equals the mass of the displaced oil and water.

$$\begin{aligned}\text{mass of the wood} &= \text{mass of oil and water displaced}\\ &= (10\times 10\times 8.0)\text{cm}^3\times(0.90\,\text{g/cm}^3)\\ &\quad +(10\times 10\times 2.0)\text{cm}^3\times(1.0\,\text{g/cm}^3)\\ &= 920\,\text{g}.\end{aligned}$$

(b) The density of the wood is,

$$\begin{aligned}\text{density of wood} &= \frac{\text{mass}}{\text{volume}}\\ [0.1in] &= \frac{920\text{ g}}{(10\times 10\times 10)\text{ cm}^3}\\ &= 0.92\text{ g cm}^{-3}.\end{aligned}$$

(c) The gauge pressure at the underside of the wooden cube is,

$$\begin{aligned}\text{gauge pressure} &= \text{pressure due to oil and water}\\ &= \rho_{oil}gh_{oil}+\rho_{water}gh_{water}\\ &= (900\text{ kg/m}^3)(9.8\text{ N/kg})(0.10\text{ m})\\ &\quad +(1000\text{ kg/m}^3)(9.8\text{ N/kg})(0.02\text{ m})\\ &= 1078\text{ N m}^{-2}.\end{aligned}$$

- wxMaxima codes:

```
(%i1)    fpprintprec:5;
(fpprintprec)    5.
(%i2)    mass_of_wood: 10*10*8*0.9 + 10*10*2*1;
(mass_of_wood)    920.0
(%i3)    density_of_wood: float(920/(10*10*10));
(density_of_wood)    0.92.
(%i4)    gauge_pressure: 900*9.8*0.1 + 1000*9.8*0.02;
(gauge_pressure)    1078.0
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2), (%i3), (%i4) Calculate mass of wood, density of wood, and gauge pressure.

Problem 14.9 Half of a spherical shell with 10.0 cm external and 9.0 cm internal radii floats in liquid A with density 0.80 g cm^{-3}, as illustrated in Fig. 14.10a. Calculate (a) the density of the sphere material, and (b) the density of liquid B in which the spherical shell will get submerged as in Fig. 14.10b. Ignore the air within the shell.

Solution

(a) Upon floating and applying the Archimedes' principle, Fig. 14.10a, we have,

$$\text{weight of liquid } A \text{ displaced} = \text{weight of shell},$$

$$\text{mass of liquid } A \text{ displaced} = \text{mass of shell},$$

$$\frac{1}{2}\left(\frac{4}{3}\pi r_{ext}^3\right)\rho_A = \frac{4}{3}\pi\left(r_{ext}^3 - r_{int}^3\right)\rho_{material}.$$

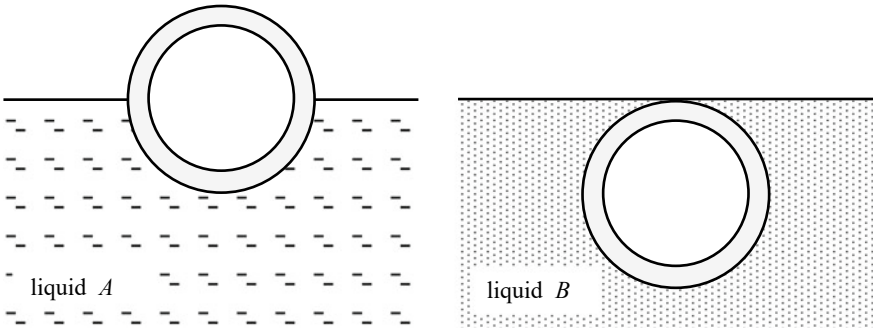

Fig. 14.10 A spherical shell floating in liquid A (**a**) and liquid B (**b**), Problem 14.9

The density of the shell material is,

$$\rho_{material} = \frac{r_{ext}^3 \rho_A}{2\left(r_{ext}^3 - r_{int}^3\right)} = \frac{(10.0\text{ cm})^3 (0.80\text{ g/cm}^3)}{2[(10.0\text{ cm})^3 - (9.0\text{ cm})^3]} = 1.5\text{ g cm}^{-3}.$$

(b) Upon floating and application of the Archimedes' principle, Fig. 14.10b, we have,

$$\text{weight of liquid } B \text{ displaced} = \text{weight of shell},$$

$$\text{mass of liquid } B \text{ displaced} = \text{mass of shell},$$

$$\frac{4}{3}\pi r_{ext}^3 \rho_B = \frac{4}{3}\pi \left(r_{ext}^3 - r_{int}^3\right)\rho_{material}.$$

From this equation and part (a), the density of liquid B is calculated as,

$$\frac{4}{3}\pi r_{ext}^3 \rho_B = \frac{1}{2}\left(\frac{4}{3}\pi r_{ext}^3\right)\rho_A,$$

$$\rho_B = \frac{\rho_A}{2} = \frac{0.80\text{ g/cm}^3}{2} = 0.40\text{ g cm}^{-3}.$$

- wxMaxima codes:

```
(%i5)       fpprintprec:5; ratprint:false; rext:10; rint:9;
rhoA:0.8;
(fpprintprec)     5.
(ratprint)     false.
(rext)     10.
(rint)     9.
(rhoA)     0.8
(%i7)     solve(1/2*(4/3*%pi*rext^3)*rhoA = 4/3*%pi*(rext^3-
rint^3)*rho_material, rho_material)$ float(%);
(%o7)     [rho_material = 1.476].
(%i8)     rho_material: rhs(%[1]);
(rho_material)     1.476.
(%i10)        solve(4/3*%pi*rext^3*rhoB = 4/3*%pi*(rext^3-
rint^3)*rho_material, rhoB)$ float(%);
(%o10)     [rhoB = 0.4].
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of r_{ext}, r_{int}, and ρ_A.

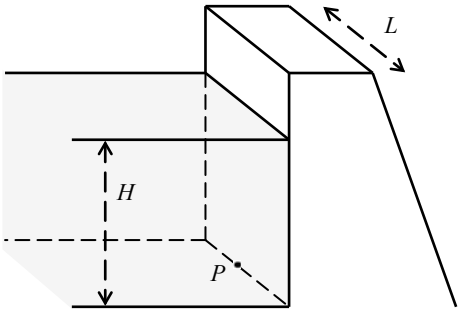

Fig. 14.11 A dam with water of depth H, Problem 14.10

(%i7) Solve $\frac{1}{2}\left(\frac{4}{3}\pi r_{ext}^3\right)\rho_A = \frac{4}{3}\pi\left(r_{ext}^3 - r_{int}^3\right)\rho_{material}$ for $\rho_{material}$, part (a).
(%i8) Assign value of $\rho_{material}$.
(%i10) Solve $\frac{4}{3}\pi r_{ext}^3\rho_B = \frac{4}{3}\pi(r_{ext}^3 - r_{int}^3)\rho_{material}$ for ρ_B, part (b).

Problem 14.10 A dam of width L has water of depth H, as in Fig. 14.11.

(a) Determine the net force on the dam due to water pressure
(b) What is the torque about point P?

Solution

(a) Figure 14.12 shows the face of the dam that holds the water as seen from left to right of Fig. 14.11. The pressure at point Q is ρgh. The force on the shaded strip is presure times area, $\rho ghL\, dh$.

The total force acting on the dam face is,

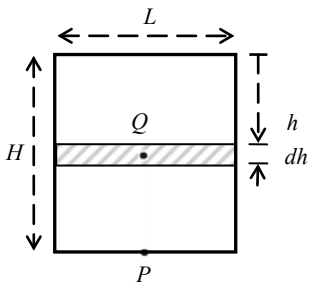

Fig. 14.12 Problem 14.10

$$F = \int_0^H \rho g L h \, dh = \left[\rho g L \left(\frac{h^2}{2} \right) \right]_0^H = \frac{\rho g L H^2}{2}.$$

- wxMaxima codes:

```
(%i1)     F: integrate(rho*g*L*h, h, 0, H);
(F)     (H^2*L*g*rho)/2.
```

Comments on the codes:

(%i1) Calculate $F = \int_0^H \rho g L h \, dh$.

(b) The torque on the shaded strip about point P is,

$$\begin{aligned} \text{force} \times \text{distance (from point } P) &= \rho g h L \, dh \times (H - h) \\ &= \rho g L \, (Hh - h^2) \, dh. \end{aligned}$$

The total torque is,

$$\tau = \int_0^H \rho g L \, (Hh - h^2) \, dh = \left[\rho g L \left(\frac{Hh^2}{2} - \frac{h^3}{3} \right) \right]_0^H = \frac{\rho g L H^3}{6}.$$

- wxMaxima codes:

```
(%i1)    torque: integrate(rho*g*L*(H*h-h^2), h, 0, H);
(torque)     (H^3*L*g*rho)/6.
```

Comments on the codes:

(%i1) Calculate $\tau = \int_0^H \rho g L \, (Hh - h^2) \, dh$.

Problem 14.11 Determine the internal and external pressure difference of a soap bubble of radius 2.0 cm. The surface tension of soap layer is 2.5×10^{-2} N m^{-1}.

Solution

The internal and external pressure difference of a soap bubble is (Eq. 14.5),

$$p_{int} - p_{ext} = \frac{4\gamma}{r} = \frac{4(2.5 \times 10^{-2}\ \text{N m}^{-1})}{2.0 \times 10^{-2}\ \text{m}} = 5.0\ \text{N m}^{-2}.$$

- wxMaxima codes:

```
(%i4)     fpprintprec:5; r:2e-2; gamma:2.5e-2; g:9.8;
(fpprintprec)     5.
(r)     0.02.
(gamma)     0.025.
(g)     9.8
(%i5)     pressure_difference: 4*gamma/r;
(pressure_difference)     5.0
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of r, γ, and g.
(%i5) Calculate pressure difference.

Problem 14.12 The length of mercury thread in a capillary tube is 12.2 cm and the mass of the mercury is 0.242 g, as illustrated in Fig. 14.13a. The capillary tube is then dipped in water, as shown Fig. 14.13b. Calculate,

(a) the radius of the capillary tube

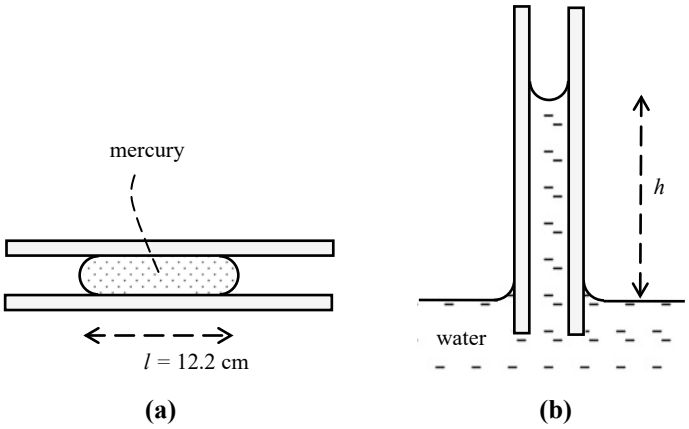

Fig. 14.13 A capillary tube with mercury (**a**), when dipped in water (**b**), Problem 14.12

(b) The height of the water h. The density of mercury is 13600 kg m^{-3}, the surface tension of water is 7.2×10^{-2} N m^{-1}.

Solution

(a) The mass of mercury in the capillary tube is, Fig. 14.13a,

$$m = \pi r^2 l \rho,$$

where r is the radius of the tube, l is the length of the mercury thread, and ρ is the density of mercury. The radius of the capillary tube is,

$$r = \sqrt{\frac{m}{\pi l \rho}} = \sqrt{\frac{0.242 \times 10^{-3} \text{ kg}}{\pi (12.2 \times 10^{-2} \text{ m})(13600 \text{ kg/m}^3)}} = 2.15 \times 10^{-4} \text{ m}.$$

(b) The surface tension force in the capillary tube supports the water column in the tube. Using Eq. (14.4) and Fig. 14.13b, we have,

$$\text{surface tension force} = \text{weight of the water column},$$

$$\gamma (2\pi r) = \pi r^2 h \rho_{water} g.$$

Thus, the height of the water is,

$$h = \frac{2\gamma}{r \rho_{water} g} = \frac{2(7.2 \times 10^{-2} \text{ N/m})}{(2.15 \times 10^{-4} \text{ m})(1000 \text{ kg/m}^3)(9.8 \text{ N/kg})} = 0.068 \text{ m}.$$

- wxMaxima codes:

```
(%i7)         fpprintprec:5; m:0.242e-3; l:12.2e-2; g:9.8;
rho:13600; rho_water:1000; gamma:7.2e-2;
(fpprintprec)     5.
(m)     2.42*10^-4.
(l)     0.122.
(g)     9.8
(rho)     13600.
(rho_water)     1000.
(gamma)     0.072.
(%i9)     r: sqrt(m/(%pi*l*rho)); float(%);
(r)     (3.8191*10^-4)/sqrt(%pi).
(%o9)     2.1547*10^-4.
(%i11)     h: 2*gamma/(r*rho_water*g); float(%);
```

```
(h)     0.038475*sqrt(%pi).
(%o11)     0.068195.
```

Comments on the codes:

(%i7) Set floating point print precision to 5, assign values of m, l, g, ρ, ρ_{water}, and γ.
(%i9) Calculate r, part (a).
(%i11) Calculate h, part (b).

Problem 14.13 Two capillary tubes A and B are vertically dipped in a liquid and the liquid raises to 7.6 and 4.4 cm, respectively. Calculate the ratio of radius of tube A to that of B.

Solution

Figure 14.14a shows the two capillary tubes dipped in the liquid.

Figure 14.14b shows the surface tension force F acting at an angle of contact θ. The vertical component of this force supports the weight of the liquid column in the capillary tube. Thus,

$$\gamma(2\pi r)\cos\theta = \pi r^2 h\rho g,$$
$$h = \frac{2\gamma\cos\theta}{r\rho g}.$$

Applying the formula, the heights of liquid in capillary tubes A and B are (Fig. 14.14a),

$$h_A = \frac{2\gamma\cos\theta}{r_A\rho g},$$

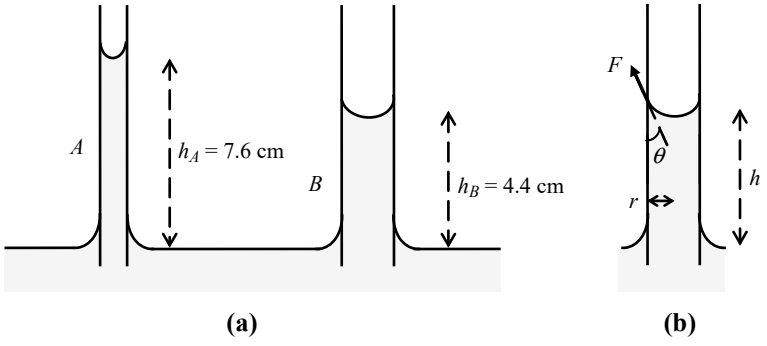

Fig. 14.14 **a** Two capillary tubes dipped in a liquid, **b** surface tension force and angle of contact, Problem 14.13

$$h_B = \frac{2\gamma \cos\theta}{r_B \rho g}.$$

The ratio of radius of tube A to that of B is,

$$\frac{r_A}{r_B} = \frac{h_B}{h_A} = \frac{4.4 \text{ cm}}{7.6 \text{ cm}} = 0.58.$$

- wxMaxima codes:

```
(%i3)     fpprintprec:5; hA:7.6; hB:4.4;
(fpprintprec)     5.
(hA)     4.4
(hB)     7.6
(%i4)     hB/hA;
(%o4)     0.57895.
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of h_A and h_B.
(%i4) Calculate h_B/h_A.

Problem 14.14 Figure 14.15 shows a capillary tube of radius 0.14 cm floating in water. The mass of the tube is 0.20 g. The surface tension of water is 7.2×10^{-2} N m^{-1} and the contact angle is zero. Calculate x.

Solution

Figure 14.16 depicts the forces acting on the tube. The weight of the tube and the surface tension force both exert a downward force, while the buoyant force exerts an upward force.

At equilibrium,

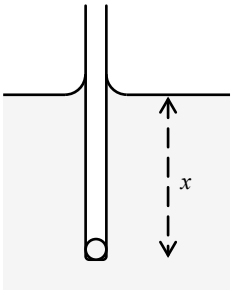

Fig. 14.15 A capillary tube floating in water, Problem 14.14

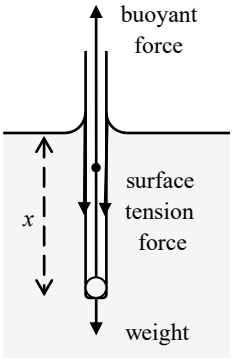

Fig. 14.16 Forces on the capillary tube, Problem 14.14

$$\begin{aligned}\text{surface tension force} + \text{weight of tube} &= \text{buoyant force on the tube}\\ &= \text{weight of water displaced},\end{aligned}$$

$$2\pi r\gamma + mg = \pi r^2 x\rho g.$$

Here, the surface tension force is $2\pi r\gamma$ as per Eq. (14.4) and the Archimedes' principle has been applied.

Substituting known values in the equation, gives x as,

$$\begin{aligned}2\pi(0.14\times10^{-2}\text{ m})(7.2\times10^{-2}\text{ N/m}) + (0.20\times10^{-3}\text{ kg})(9.8\text{ N/kg})\\ = \pi(0.14\times10^{-2}\text{ m})^2(1000\text{ kg/m}^3)(9.8\text{ N/kg})x,\\ x = 0.043\text{ m}.\end{aligned}$$

- wxMaxima codes:

```
(%i7)    fpprintprec:5; ratprint:false; r:0.14e-2; m:0.2e-3;
gamma:7.2e-2; rho:1000; g:9.8;
(fpprintprec)     5.
(ratprint)     false.
(r)     0.0014.
(m)     2.0*10^-4.
(gamma)     0.072.
(rho)     1000.
(g)     9.8
(%i9)      solve(gamma*2*%pi*r + m*g = %pi*r^2*x*rho*g, x)$
float(%);
(%o9)     [x = 0.042976].
```

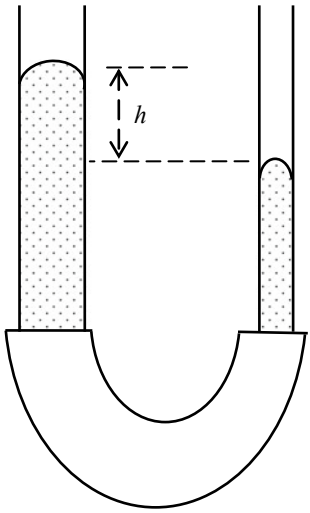

Fig. 14.17 A U tube filled with mercury, Problem 14.15

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of r, m, γ, and ρ.
(%i9) Solve $2\pi r\gamma + mg = \pi r^2 x\rho g$ for x.

Problem 14.15 Figure 14.17 depicts a U-tube constructed from tubes with radii of 4.0 mm and 2.0 mm connected by a rubber tube. The tube is filled with mercury. Calculate the mercury level difference h. Mercury's surface tension is 0.54 N m^{-1}, mercury-glass contact angle is 132°, and its density is 13,600 kg m^{-3}.

Solution

Figure 14.18a shows a curved surface of mercury in a capillary tube. The angle of contact is θ. The pressures below and above the surface are p_A and p_B, respectively, with $p_A > p_B$. The upward force due to p_A is equal in magnitude to the downward force due to p_B plus the downward component of the surface tension force. Thus,

$$p_A(\pi r^2) = p_B(\pi r^2) + \gamma(2\pi r)\cos(180^\circ - \theta),$$
$$p_A - p_B = \frac{2\gamma\cos(180^\circ - \theta)}{r}.$$

Here, the upward and downward forces due to p_A and p_B are $p_A(\pi r^2)$ and $p_B(\pi r^2)$, respectively. The surface tension force is $\gamma(2\pi r)$ as per Eq. (14.4) and its downwad component is $\gamma(2\pi r)\cos(180^\circ - \theta)$.

Applying the formula, in the big tube, the pressure difference is, Fig. 14.18b,

$$p_1 - p_{atm} = \frac{2\gamma\cos(180^\circ - \theta)}{r_1}.$$

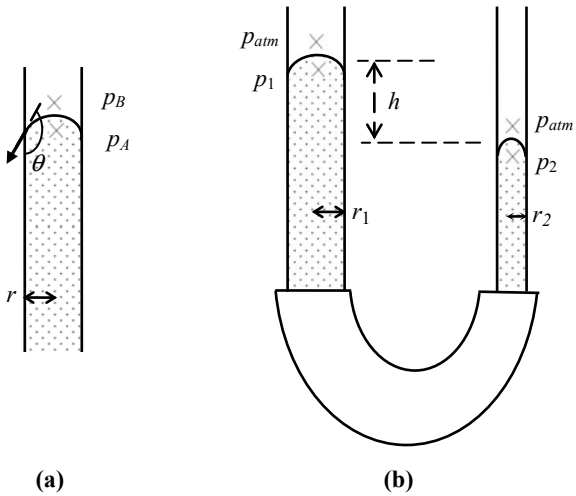

Fig. 14.18 **a** Contact angle and pressures at two points near the mercury surface, **b** pressures at different points of mercury filled U tube, Problem 14.15

In the small tube, the pressure difference is,

$$p_2 - p_{atm} = \frac{2\gamma \cos(180^\circ - \theta)}{r_2}.$$

The two equations give,

$$p_2 - p_1 = 2\gamma \cos(180^\circ - \theta)\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

But,

$$p_2 - p_1 = \rho g h.$$

Thus, the difference in mercury levels is,

$$\begin{aligned} h &= \frac{2\gamma \cos(180^\circ - \theta)\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}{\rho g} \\ &= \frac{2(0.54\ \text{N/m})\cos(180^\circ - 132^\circ)\left(\frac{1}{2.0\times10^{-3}\ \text{m}} - \frac{1}{4.0\times10^{-3}\ \text{m}}\right)}{(13600\ \text{kg/m}^3)(9.8\ \text{N/kg})} \\ &= 1.4 \times 10^{-3}\ \text{m}. \end{aligned}$$

- wxMaxima codes:

```
(%i7)          fpprintprec:5; r1:4e-3; r2:2e-3; gamma:0.54;
rho:13600; g:9.8; theta:(180-132)/180*%pi;
(fpprintprec)      5.
(r1)      0.004.
(r2)      0.002.
(gamma)      0.54.
(rho)      13600.
(g)      9.8
(theta)      (4*%pi)/15.
(%i9)          h: 2*gamma*cos(theta)*(1/r2 - 1/r1)/(rho*g)$
float(%);
(%o9)      0.0013555.
```

Comments on the codes:

(%i7) Set floating point print precision to 5, assign values of r_1, r_2, γ, ρ, g, and θ.
(%i9) Calculate h.

Problem 14.16 A bubble is created by combining 8.0 mg soap solution and hydrogen gas. The bubble floats through the air. What is the bubble's internal and external pressure difference? Soap solution has surface tension of 25×10^{-3} N m^{-1}, whereas hydrogen gas and air have densities of 0.09 kg m^{-3} and 1.29 kg m^{-3}, respectively.

Solution

Figure 14.19 shows the bubble floating in air.

Upon floating, there is equilibrium, implying,

$$\begin{aligned}\text{buoyant force on the bubble} = {} & \text{weight of soap solution} \\ & + \text{weight of hydrogen gas},\end{aligned}$$

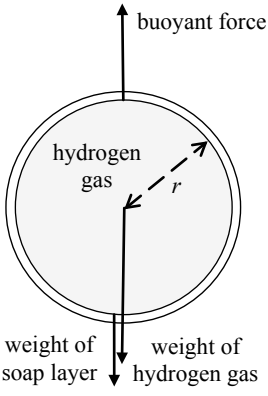

Fig. 14.19 A hydrogen filled soap bubble, Problem 14.16

$$\frac{4}{3}\pi r^3 \rho_{air} g = mg + \frac{4}{3}\pi r^3 \rho_{hydrogen} g.$$

Here, the buoyant force is due to the air displaced by the bubble. According to the Archimedes' principle the buoyant for is equal to the weight of the air displaced by the bubble. Thus, the radius of the bubble is calculated as,

$$\begin{aligned} r^3 &= \frac{3m}{4\pi(\rho_{air} - \rho_{hydrogen})} = \frac{3(8.0 \times 10^{-6}\ \text{kg})}{4\pi(1.29\ \text{kg/m}^3 - 0.09\ \text{kg/m}^3)} \\ &= 1.59 \times 10^{-6}\ \text{m}^3, \\ r &= 1.2 \times 10^{-2}\ \text{m}. \end{aligned}$$

The internal and external pressure difference of the bubble is,

$$p_{int} - p_{ext} = 2\left(\frac{2\gamma}{r}\right) = \frac{4\gamma}{r} = \frac{4(25 \times 10^{-3}\ \text{N/m})}{1.2 \times 10^{-2}\ \text{m}} = 8.6\ \text{N m}^{-2}.$$

Here, we mutiply $2\gamma/r$ by 2 because the soap buble has internal and external surfaces. Equation (14.5) is applicable to a bubble in a liquid that has only one surface.

- wxMaxima codes:

```
(%i5)            fpprintprec:5; m:8e-6; gamma:25e-3; rho_
hydrogen:0.09; rho_air:1.29;
(fpprintprec)     5.
(m)     8.0*10^-6.
(gamma)     0.025.
(rho_hydrogen)     0.09.
(rho_air)     1.29.
(%i7)        r: (3*m/(4*%pi*(rho_air-rho_hydrogen)))^(1/3)$
float(%);
(%o7)     0.011675.
(%i9)     pressure_difference: 4*gamma/r$ float(%);
(%o9)     8.565.
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of m, γ, $\rho_{hydrogen}$, and ρ_{air}.
(%i7) Calculate r.
(%i9) Calculate the pressure difference.

14.3 Summary

- Density is the mass per unit volume of a substance, $\rho = m/V$. The SI unit of density is kg m^{-3}.
- Pressure is the force per unit area over which the force is applied, $p = F/A$. The SI unit of pressure is pascal (Pa), 1 Pa = 1 N m^{-2}.
- Pressure due to weight of a liquid at depth h is $p = \rho gh$.
- Pascal's principle states that a change in pressure applied to an enclosed fluid is transmitted to all portions of the fluid and the wall of its container.
- Archimedes' principle states that the buoyant force on an object is equal to the weight of the fluid it displaces.

14.4 Exercises

Exercise 14.1 A beaker is filled with mercury to 10.0 cm high. What is the the absolute pressure at the bottom of the beaker? Density of mercury is 13,600 kg m^{-3}, atmospheric pressure is 101,300 Pa.

(Answer: 115,000 Pa)

Exercise 14.2 A metal ball weighs 0.096 N in air. When suspended in water, it has an apparent weight of 0.073 N, as shown in Fig. 14.20. Find the density of the metal.

(Answer: 4170 kg m^{-3})

Exercise 14.3 The average densities of an iceberg and seawater are 920 and 1020 kg m^{-3}, respectively. What fraction of the iceberg is beneath the seawater surface?

(Answer: 0.90)

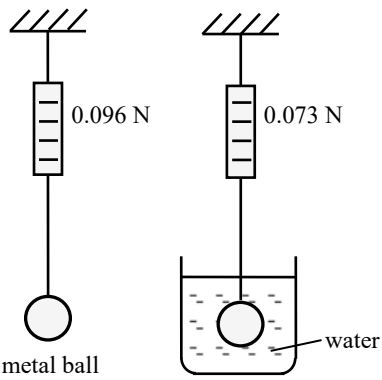

Fig. 14.20 Weight of a metal ball measured in air and in water, Exercise 14.2

Exercise 14.4 How high does methyl alcohol rise in a glass tube 0.04 cm in radius? The surface tension is 0.023 N m^{-1} and density of methyl alcohol is 0.80 g cm^{-3}.

(Answer: 1.5 cm)

Exercise 14.5 A soap bubble has a radius of 4.0 cm. What is the gauge pressure of the bubble? The surface tension of the soap solution is 3.0×10^{-2} N m^{-1}.

(Answer: 3.0 Pa)

Chapter 15
Hydrodynamics

15.1 Basic Concepts and Formulae

(1) The properties of non-turbulent flow of a non-viscous and incompressible fluid in a pipe are as follows:

(a) The volume flow rate in the pipe is constant. The product of the pipe's cross-sectional area A and the speed of the fluid v at any point in the pipe is a constant,

$$A_1 v_1 = A_2 v_2 = \text{volume flow rate} = \text{constant}. \tag{15.1}$$

The formula is known as the continuity equation.

(b) At any point in the pipe, the sum of pressure, kinetic energy per unit volume, and potential energy per unit volume is a constant,

$$p + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}. \tag{15.2}$$

This is known as Bernoulli's equation.

Theorem of Torricelli: The efflux speed of a non-viscous liquid from a tank hole at depth h equals the speed of a liquid falling freely from a height of h,

$$v = \sqrt{2gh}. \tag{15.3}$$

(2) The viscous force in laminar fluid flow is given by Newton as,

$$F = \eta A \frac{dv}{dr}, \tag{15.4}$$

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https://doi.org/10.1007/978-3-031-42678-0_15

where η is the coefficient of viscosity, A is the area between layers of fluid, and dv/dr is the velocity gradient. The SI units of the coefficient of viscosity, area between layers of fluid, and velocity gradient are N s m^{-2}, m^2, and s^{-1}, respectively.

The viscosity of a fluid is a measure of the resistance of fluid layers to shear motion. It is assumed that the fluid flow is laminar. The coefficient of viscosity is defined as shear stress divided by shear strain rate of change.

$$\eta = \frac{F/A}{v/l} = \frac{Fl}{Av}. \tag{15.5}$$

The SI unit of the coefficient of viscosity is N s m^{-2}, while the cgs unit is dyne s cm^{-2} or poise.

$$1 \text{ poise} = 1 \text{ dyne s cm}^{-2} = 10^{-1} \text{ N s m}^{-2}.$$

(3) In laminar flow of a viscous fluid in a horizontal pipe, the speed of fluid layer is,

$$v_r = \frac{(p_1 - p_2)}{4\eta L}(R^2 - r^2). \tag{15.6}$$

Here, R and L represent the pipe's radius and length, respectively, p_1 and p_2 are the fluid pressures at the pipe's ends, r is the fluid layer's distance from the pipe's axis, and η is the coefficient of viscosity of the fluid.

The volume rate of flow of the fluid is,

$$\frac{dV}{dt} = \frac{\pi R^4 (p_1 - p_2)}{8\eta L}. \tag{15.7}$$

This is the Poiseuille's equation.

(4) According to Stokes's law, the viscous force F on a sphere of radius r and velocity v moving in a fluid is,

$$F = 6\pi \eta r v, \tag{15.8}$$

where η is the coefficient of viscosity of the fluid. Using Stokes's law, the terminal velocity v_t of a falling sphere in a viscous fluid is calculated as,

$$v_t = \frac{2r^2 g}{9\eta}(\rho_{sphere} - \rho_{fluid}), \tag{15.9}$$

where r and ρ_{sphere} are the radius and density of the sphere, respectively, while η and ρ_{fluid} are the coefficient of viscosity and density of the fluid, respectively.

(5) Experiments demonstrate that the Reynolds number, N_R, can determine whether a fluid's flow in a tube is laminar or turbulent. The Reynolds number is,

$$N_R = \frac{\rho v D}{\eta}, \tag{15.10}$$

where ρ and η are the density and the coefficient of viscosity of the fluid, respectively, D is the diameter of the tube, and v is the speed of the fluid. The fluid flow is laminar if $N_R < 2000$.

15.2 Problems and Solutions

Problem 15.1 A hole of radius 0.50 cm is punched at the base of a cylindrical tank of radius 4.0 cm full with water, Fig. 15.1. The water pressure at $h = 1.2$ m from the base is 3.0×10^5 N m^{-2} and the water flows at 1.5 L per second through the hole. Calculate the pressure at the hole.

Solution
Let the water speeds at points 1 and 2 be v_1 and v_2. Using Eq. (15.1), the speeds of water are,

$$v_1 = \frac{\text{volume flow rate}}{\text{area}} = \frac{\text{volume/second}}{\text{area}} = \frac{1.5 \times 10^{-3}\ \text{m}^3/\text{s}}{\pi (4.0 \times 10^{-2}\ \text{m})^2} = 0.30\ \text{m s}^{-1},$$

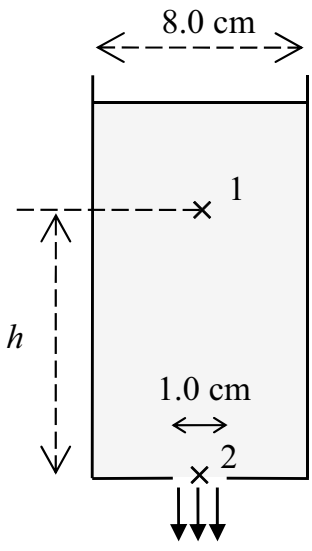


Fig. 15.1 A cylindrical tank with a hole at the bottom, Problem 15.1

$$v_2 = \frac{1.5 \times 10^{-3}\ \text{m}^3/\text{s}}{\pi(0.50 \times 10^{-2}\ \text{m})^2} = 19\ \text{m s}^{-1}.$$

Applying the Bernoulli's equation (Eq. 15.2) at points 1 and 2 and setting point 2 as reference level, we have,

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h = p_2 + \frac{1}{2}\rho v_2^2.$$

Substituting known numerical values into the equation, we can calculate the pressure at the hole,

$$\begin{aligned} &3.0 \times 10^5\ \text{N/m}^2 + \frac{1}{2}\left(1000\ \text{kg/m}^3\right)(0.30\ \text{m/s})^2 + \left(1000\ \text{kg/m}^3\right)(9.8\ \text{N/kg})(1.2\ \text{m}) \\ &\quad = p_2 + \frac{1}{2}\left(1000\ \text{kg/m}^3\right)(19\ \text{m/s})^2, \\ &p_2 = 1.3 \times 10^5\ \text{N m}^{-2}. \end{aligned}$$

- wxMaxima codes:

```
(%i9)  fpprintprec:5; ratprint:false; r1:4e-2; r2:0.5e-2;
flow_rate:1.5e-3; h:1.2; p1:3e5; g:9.8; rho:1000;
(fpprintprec)  5
(ratprint)  false
(r1)  0.04
(r2)  0.005
(flow_rate)  0.0015
(h)  1.2
(p1)  3.0*10^5
(g)  9.8
(rho)  1000
(%i10) v1: float(flow_rate/(%pi*r1^2));
(v1)  0.29842
(%i11) v2: float(flow_rate/(%pi*r2^2));
(v2)  19.099
(%i13) solve(p1+1/2*rho*v1^2 + rho*g*h = p2 + 1/2*rho*v2^2,
p2)$ float(%);
(%o13) [p2=1.2943*10^5]
```

Comments on the codes:

(%i9) Set floating point print precision to 5, internal rational number print to false, assign values of r_1, r_2, flow rate, h, p_1, g, and ρ.
(%i10), (%i11) Calculate v_1 and v_2.
(%i13) Solve $p_1 + \frac{1}{2}\rho v_1^2 + \rho g h = p_2 + \frac{1}{2}\rho v_2^2$ for p_2.

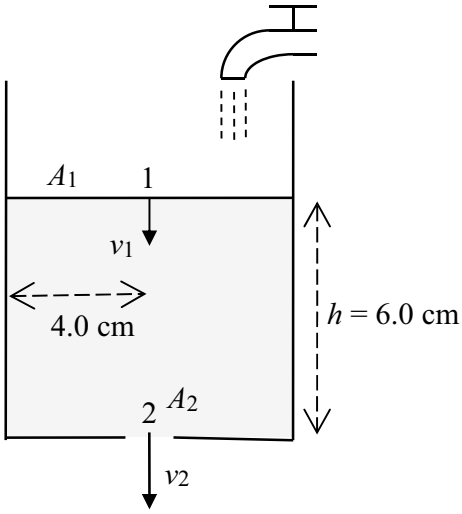


Fig. 15.2 Water stays at a constant level when water in and water out are at the same rate, Problem 15.2

Problem 15.2 A circular hole is punched at the base of a cylinder with a radius of 4.0 cm. The tank is filled with water at a rate of 150 cm^3 per second from the top, and some water escapes through the hole. The water level in the tank remains constant over time at 6.0 cm. Determine the diameter of the hole.

Solution

Figure 15.2 shows the cylindrical tank when the water level stays constant at 6.0 cm.

The continuity equation (Eq. 15.1) when the water level stays constant,

$$A_1 v_1 = A_2 v_2 = 150 \times 10^{-6}\ \text{m}^3/\text{s} = \text{volume flow rate}.$$

This gives the speeds of water at points 1 and 2 as,

$$v_1 = \frac{150 \times 10^{-6}\ \text{m}^3/\text{s}}{A_1} = \frac{150 \times 10^{-6}\ \text{m}^3/\text{s}}{\pi(4 \times 10^{-2}\ \text{m})^2} = 0.0298\ \text{m s}^{-1},$$
$$v_2 = \frac{150 \times 10^{-6}\ \text{m}^3/\text{s}}{A_2} = \frac{150 \times 10^{-6}\ \text{m}^3/\text{s}}{\pi r^2} = \frac{4.77 \times 10^{-5}\ \text{m}^3/\text{s}}{r^2},$$

where r is the radius of the hole. Applying the Bernoulli's equation (Eq. 15.2) at points 1 and 2 gives,

$$p_{atm} + \frac{1}{2}\rho v_1^2 + \rho g h = p_{atm} + \frac{1}{2}\rho v_2^2 + \rho g(0),$$
$$2gh = v_2^2 - v_1^2.$$

Substituting the known quantities into the equation gives the radius of the hole as,

$$2(9.8\ \text{m/s}^2)(6.0\ \times\ 10^{-2}\ \text{m}) = \left(\frac{4.77 \times 10^{-5}\ \text{m}^3/\text{s}}{r^2}\right)^2 - (0.0298\ \text{m/s})^2,$$
$$r = 6.6 \times 10^{-3}\ \text{m}.$$

- wxMaxima codes:

```
(%i6)      fpprintprec:5; ratprint:false; r1:4e-2; flow_
rate:150e-6; h:6e-2; g:9.8;
(fpprintprec)  5
(ratprint)  false
(r1)  0.04
(flow_rate)  1.5*10^-4
(h)  0.06
(g)  9.8
(%i7)  v1: float(flow_rate/(%pi*r1^2));
(v1)  0.029842
(%i8)  v2: float(flow_rate/(%pi*r^2));
(v2)  (4.7746*10^-5)/r^2
(%i10) solve(2*g*h=v2^2-v1^2,r)$ float(%);
(%o10)                    [r=0.0066342*%i,r=-0.0066342,r=-
0.0066342*%i,r=0.0066342]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of r_1, flow rate, h, and g.
(%i7), (%i8) Calculate v_1 and v_2.
(%i10) Solve $2gh = v_2^2 - v_1^2$ for r.

Problem 15.3 The top of a cylindrical tank is open. The tank's height is 20 cm and its diameter is 5.0 cm. The base of the tank is punched with a 1.0 cm^2 hole. The tank is filled with water at a rate of 140 $\text{cm}^3\ \text{s}^{-1}$. How much will the water level rise?

Solution
Figure 15.3 shows the tank and the flows when a steady state is acheived. At steady state, the rate of water coming in is equal to the rate of water getting out.

The continuity equation (Eq. 15.1) is,

$$A_1 v_1 = A_2 v_2 = 140\ \text{cm}^3/\text{s} = \text{volume flow rate}.$$

The water speeds at points 1 and 2 are,

$$v_1 = \frac{140\ \text{cm}^3/\text{s}}{\pi(5.0\ \text{cm})^2} = 1.78\ \text{cm s}^{-1},$$

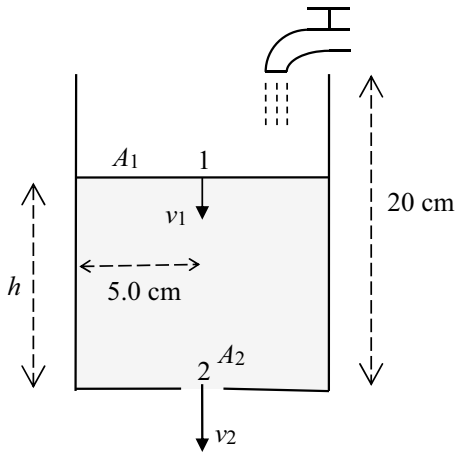


Fig. 15.3 At steady state, the rate of water coming in is equal to the rate of water getting out, Problem 15.3

$$v_2 = \frac{140\ \text{cm}^3/\text{s}}{1.0\ \text{cm}^2} = 140\ \text{cm s}^{-1}.$$

Applying the Bernoulli's equation (Eq. 15.2) at points 1 and 2, we have,

$$\frac{1}{2}\rho v_1^2 + \rho g h = \frac{1}{2}\rho v_2^2.$$

This gives the required water level height,

$$\begin{aligned} h &= \frac{1}{2g}(v_2^2 - v_1^2) = \frac{1}{2(980\ \text{cm/s}^2)}[(140\ \text{cm/s})^2 - (1.78\ \text{cm/s})^2] \\ &= 10\ \text{cm}. \end{aligned}$$

- wxMaxima codes:

```
(%i7)     fpprintprec:5; ratprint:false; r1:5; A2:1; flow_
rate:140; g:980; rho:1;
(fpprintprec)   5
(ratprint)   false
(r1)   5
(A2)   1
(flow_rate)   140
(g)   980
(rho)   1
(%i8)   v1: float(flow_rate/(%pi*r1^2));
(v1)   1.7825
(%i9)   v2: float(flow_rate/A2);
(v2)   140.0
```

```
(%i11) solve(1/2*rho*v1^2 + rho*g*h = 1/2*rho*v2^2, h)$
float(%);
(%o11) [h=9.9984]
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of r_1, A_2, flow rate, g, and ρ.
(%i8), (%i9) Calculate v_1 and v_2.
(%i11) Solve $\frac{1}{2}\rho v_1^2 + \rho g h = \frac{1}{2}\rho v_2^2$ for h.

Problem 15.4 A venturi meter, Fig. 15.4, is used to measure a liquid flow speed v_1. A liquid of density ρ_{liq} flows in the large tube of cross-sectional area A_1 then flows to a small tube of cross-sectional area A_2. The liquid in the manometer is mercury of density ρ_{mec} and the difference in mercury levels is h. Determine,

(a) the liquid flow speed v_1
(b) the volume flow per second.

Solution

(a) Figure 15.5 shows the venturi meter and the physical quantities related to the flow.

Applying the Bernoulli's equation (Eq. 15.2) at points 1 and 2, we have,

$$p_1 + \frac{1}{2}\rho_{liq} v_1^2 = p_2 + \frac{1}{2}\rho_{liq} v_2^2,$$

or,

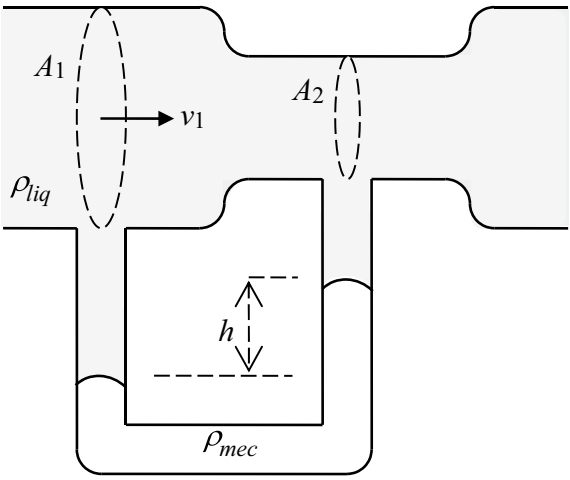


Fig. 15.4 A venturi meter, Problem 15.4

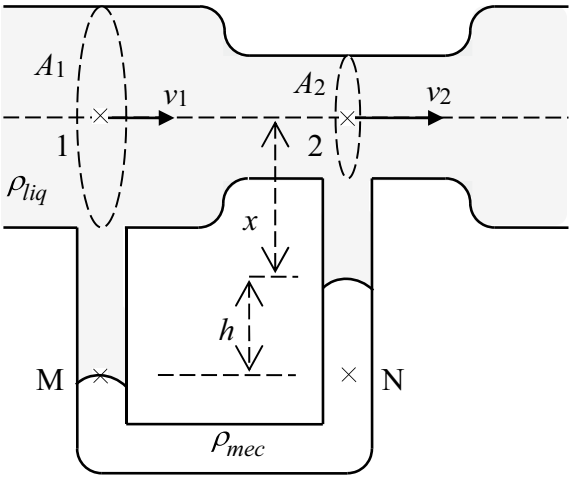


Fig. 15.5 Flow analysis of a venturi meter, Problem 15.4

$$p_1 - p_2 = \frac{1}{2}\rho_{liq}(v_2^2 - v_1^2).$$

The continuity equation (Eq. 15.1) at points 1 and 2 gives,

$$A_1 v_1 = A_2 v_2,$$

thus,

$$v_2 = \frac{A_1}{A_2} v_1.$$

The difference in pressure is written as,

$$p_1 - p_2 = \frac{1}{2}\rho_{liq}\left(\frac{A_1^2}{A_2^2}v_1^2 - v_1^2\right) = \frac{1}{2}\rho_{liq}v_1^2\left(\frac{A_1^2 - A_2^2}{A_2^2}\right). \tag{15.1}$$

Consider points M and N. Their pressures are,

$$p_M = p_1 + \rho_{liq}g(h + x),$$
$$p_N = p_2 + \rho_{mec}gh + \rho_{liq}gx.$$

Points M and N are at the same level of height, thus, $p_M = p_N$. So, from the two equations we have,

$$p_1 - p_2 = gh(\rho_{mec} - \rho_{liq}). \tag{15.2}$$

Solving Eqs. (15.1) and (15.2) for v_1, the liquid flow speed at point 1 is,

$$v_1 = A_2\sqrt{\frac{2gh(\rho_{mec}-\rho_{liq})}{(A_1^2-A_2^2)\rho_{liq}}}.$$

(b) The volume flow per second of the liquid is,

$$A_1 v_1 = A_1 A_2\sqrt{\frac{2gh(\rho_{mec}-\rho_{liq})}{(A_1^2-A_2^2)\rho_{liq}}}.$$

- wxMaxima codes:

```
(%i1)sol:                solve(1/2*rholiq*v1^2*(A1^2-A2^2)/
A2^2=g*h*(rhomec-rholiq), v1);
(sol)                    [v1=-sqrt(2)*A2*sqrt((g*h*rholiq)/
(A2^2*rholiq-A1^2*rholiq)-       (g*h*rhomec)/(A2^2*rholiq-
A1^2*rholiq)),v1=sqrt(2)*A2*
sqrt((g*h*rholiq)/(A2^2*rholiq-A1^2*rholiq)-(g*h*rhomec)/
(A2^2*rholiq-A1^2*rholiq))]
(%i2)   radcan(%);
(%o2)         [v1=-(sqrt(2)*A2*sqrt(g)*sqrt(h)*sqrt(rhomec-
rholiq))/(sqrt(A1-A2)*sqrt(A2+A1)*sqrt(rholiq)),
v1=(sqrt(2)*A2*sqrt(g)*sqrt(h)*sqrt(rhomec-rholiq))/
(sqrt(A1-A2)*sqrt(A2+A1)*sqrt(rholiq))]
(%i3)   v1: rhs(%o2[2]);
(v1)      (sqrt(2)*A2*sqrt(g)*sqrt(h)*sqrt(rhomec-rholiq))/
(sqrt(A1-A2)*sqrt(A2+A1)*sqrt(rholiq))
(%i4)   A1*v1;
(%o4)           (sqrt(2)*A1*A2*sqrt(g)*sqrt(h)*sqrt(rhomec-
rholiq))/(sqrt(A1-A2)*sqrt(A2+A1)*sqrt(rholiq))
```

Comments on the codes:

(%i1) Solve $\frac{1}{2}\rho_{liq}v_1^2\left(\frac{A_1^2-A_2^2}{A_2^2}\right) = gh(\rho_{mec}-\rho_{liq})$ for v_1.
(%i2) Simplify.
(%i3) Assign v_1 from the solution.
(%i4) Calculate $A_1 v_1$.

Problem 15.5 Water is used as the manometer liquid in a Pitot tube to measure air flow speed, as shown in Fig. 15.6. If the difference in water level is 0.10 m, what is the speed of the air flow? The densities of water and air are 1000 and 1.3 kg m^{-3} respectively.

Solution
In Fig. 15.6 the air enters at point 1. Point 2 is the stagnation point, a point where the air is not moving. The pressure difference at point 2 and 1 is,

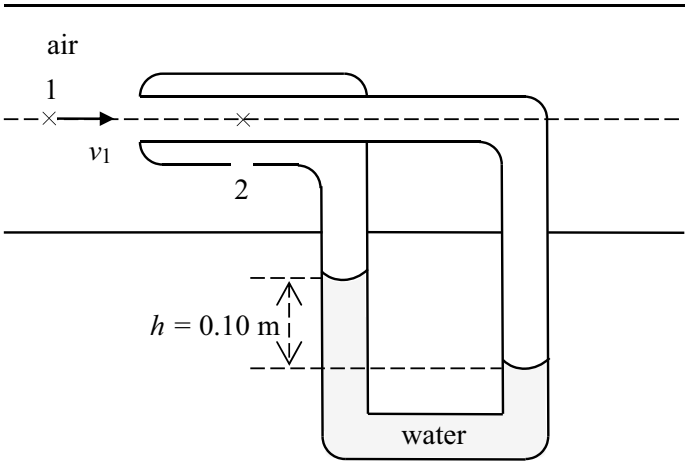


Fig. 15.6 Pitot tube, Problem 15.5

$$p_2 - p_1 = \rho_{water} g h. \tag{15.1}$$

Applying the Bernoulli's equation (Eq. 15.2) at points 1 and 2, we have,

$$p_1 + \frac{1}{2}\rho_{air} v_1^2 = p_2,$$

or,

$$p_2 - p_1 = \frac{1}{2}\rho_{air} v_1^2. \tag{15.2}$$

From Eqs. (15.1) and (15.2), we write,

$$\rho_{water} g h = \frac{1}{2}\rho_{air} v_1^2.$$

The speed of air flow is,

$$v_1 = \sqrt{2gh\frac{\rho_{water}}{\rho_{air}}} = \sqrt{2(9.8\ \text{m/s}^2)(0.10\ \text{m})\frac{1000\ \text{kg/m}^3}{1.3\ \text{kg/m}^3}}$$
$$= 39\ \text{m s}^{-1}.$$

- wxMaxima codes:

```
(%i6)     fpprintprec:5; ratprint:false; rho_air:1.3; rho_
water:1000; g:9.8; h:0.1;
(fpprintprec)   5
(ratprint)   false
(rho_air)   1.3
(rho_water)   1000
(g)   9.8
(h)   0.1
(%i8)  solve([p2-p1=rho_water*g*h, p2-p1=1/2*rho_air*v1^2],
[v1, p1])$ float(%);
(%o8)   [[v1=-38.829,p1=p2-980.0],[v1=38.829,p1=p2-980.0]]
```

Comments on the codes:

(%i6) Set floting point print precision to 5, internal rational number print to false, assign values of ρ_{air}, ρ_{water}, g, and h.
(%i8) Solve $p_2 - p_1 = \rho_{water} gh$ and $p_2 - p_1 = \frac{1}{2}\rho_{air} v_1^2$ for v_1 and p_1.

Problem 15.6 A big tank is filled with water to 10 m depth and a hole of 1.0 cm radius is punched at the base of the tank. Calculate,

(a) the speed of water coming out of the hole
(b) the volume per second of the water coming out of the hole.

Solution

(a) Applying the Torricelli's theorem (Eq. 15.3), the speed of efflux is the speed of the liquid in a free fall, Fig. 15.7,

$$v = \sqrt{2gh} = \sqrt{2(9.8\ \text{m/s}^2)(10\ \text{m})} = 14\ \text{m s}^{-1}.$$

(b) The rate of flow of water is,

$$\begin{aligned}\text{volume per second} &= \text{cross sectional area of hole} \times v \\ &= \pi(1.0 \times 10^{-2}\ \text{m})^2 \times 14\ \text{m/s}\end{aligned}$$

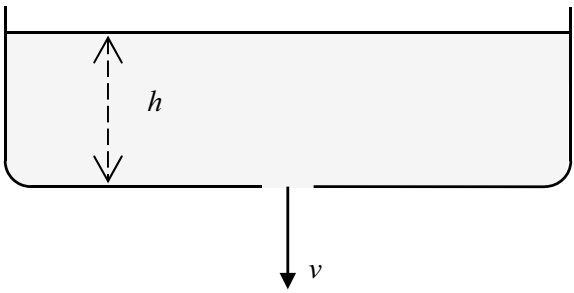


Fig. 15.7 A big tank filles with water, Problem 15.6

$$= 4.4 \times 10^{-3}\ \mathrm{m}^3\ \mathrm{s}^{-1}.$$

- wxMaxima codes:

```
(%i4)   fpprintprec:5; h:10; r:1e-2; g:9.8;
(fpprintprec)   5
(h)   10
(r)   0.01
(g)   9.8
(%i5)   v: sqrt(2*g*h);
(v)   14.0
(%i6)   volume_per_second: float(%pi)*r^2*v;
(volume_per_second) 0.0043982
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of h, r, and g.
(%i5) Calculate v, part (a).
(%i6) Calculate volume per second, part (b).

Problem 15.7 A large tank is filled with water to a depth of y_0. A hole is punched on the wall a distance y below the water surface, Fig. 15.8. The water comes out of the hole and hits the floor a distance x from the tank.

(a) Show that $x = 2\sqrt{(y_0 - y)y}$
(b) Determine the value of y such that x is a maximum.

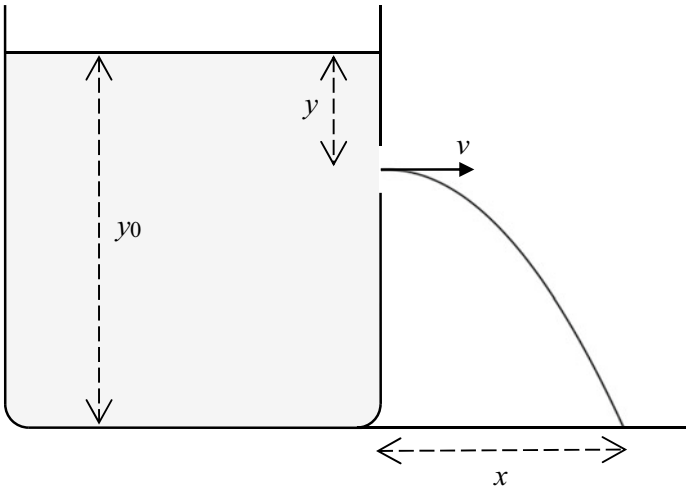


Fig. 15.8 A tank with a hole punched at its wall, Problem 15.7

Solution

(a) Applying the Torricelli's theorem (Eq. 15.3), the speed of efflux of the water from the hole is,

$$v = \sqrt{2gy}.$$

In the vertical direction, the distance s travelled by the water in time t is,

$$s = \frac{1}{2}gt^2.$$

Thus, the time for the water to hit the floor is,

$$t = \sqrt{\frac{2s}{g}} = \sqrt{\frac{2(y_0 - y)}{g}}.$$

The distance from the wall of the tank to the point the water hits the floor is,

$$x = vt = \sqrt{2gy}\sqrt{\frac{2(y_0 - y)}{g}} = 2\sqrt{(y_0 - y)y}.$$

(b) To get a maximum, differentiate x with respect to y, and equate it with zero,

$$\begin{aligned} x &= 2[(y_0 - y)y]^{1/2} \\ &= 2(y_0 y - y^2)^{1/2}, \\ \frac{dx}{dy} &= 2\left(\frac{1}{2}\right)(y_0 y - y^2)^{-1/2}(y_0 - 2y) \\ &= 0, \\ y &= \frac{1}{2}y_0. \end{aligned}$$

The maximum of x at $y = y_0/2$ is,

$$x = 2\sqrt{(y_0 - y)y} = 2\sqrt{\left(y_0 - \frac{1}{2}y_0\right)\left(\frac{1}{2}y_0\right)} = y_0.$$

This means that to get a maximum x of y_0, the hole is at $y = y_0/2$.

Problem 15.8 A solid block of weight 1.5 N is glued to a wall by a grease, as illustrated in Fig. 15.9. The area in contact with the wall is 8.0×10^{-4} m^2 and the grease is 2.0×10^{-6} m thick. In two weeks, the block moves down by 5.0 mm. Calculate the viscosity coefficient of the grease.

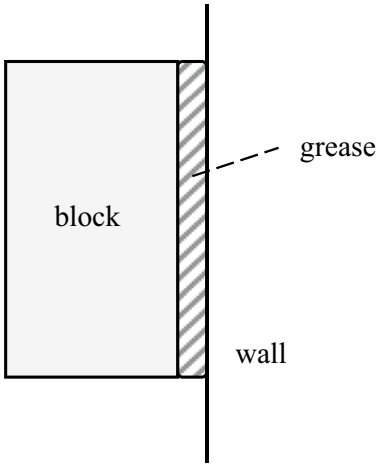


Fig. 15.9 A solid block glued to the wall by a grease, Problem 15.8

Solution

The viscous force is given by Newton as (Eq. 15.4),

$$F = \eta A \frac{dv}{dr},$$

where η is the coefficient of viscosity, A is the area, and dv/dr is the velocity gradient. Substituting known values into the equation gives,

$$1.5\ \text{N} = \eta(8.0 \times 10^{-4}\ \text{m}^2)\left(\frac{5.0 \times 10^{-3}\ \text{m}}{2 \times 7 \times 24 \times 60 \times 60\ \text{s}}\right)\left(\frac{1}{2.0 \times 10^{-6}\ \text{m}}\right).$$

The coefficient of viscosity of the grease is,

$$\eta = 9.1 \times 10^5\ \text{N m}^{-2}\ \text{s}.$$

- wxMaxima codes:

```
(%i2)  fprintprec:5; ratprint:false;
(fprintprec)  5
(ratprint)  false
(%i3)  solve(1.5=eta*8e-4*(5e-3/(2*7*24*60*60*2e-6)),eta);
(%o3)  [eta=907200]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3) Solve $1.5 = \eta(8.0 \times 10^{-4}) \left(\frac{5.0\times10^{-3}}{2\times7\times24\times60\times60}\right) \left(\frac{1}{2.0\times10^{-6}}\right)$ for η.

Problem 15.9 The experiment depicted in Fig. 15.10 measures the viscosity of water by observing its flow in a horizontal tube. Thirty cm water separates the ends of the horizontal tube in terms of pressure. The tube's diameter and length are 0.40 mm and 28 cm, respectively. In 10 min, 3.6 cm^3 of water was collected in the measuring cylinder. Determine the coefficient of water's viscosity.

Solution

Applying the Poiseuille's formula (Eq. 15.7),

$$\text{water volume per second} = \text{flow rate} = \frac{\pi r^4 \Delta p}{8\eta L}.$$

Substituting known values into the equation gives,

$$\frac{3.6 \times 10^{-6}\ \text{m}^3}{10 \times 60\ \text{s}} = \frac{\pi\left(\frac{0.40}{2} \times 10^{-3}\ \text{m}\right)^4 (0.30\ \text{m})(1000\ \text{kg/m}^3)(9.8\ \text{m/s}^2)}{8\eta(0.28\ \text{m})}.$$

The coefficient of viscosity of water is,

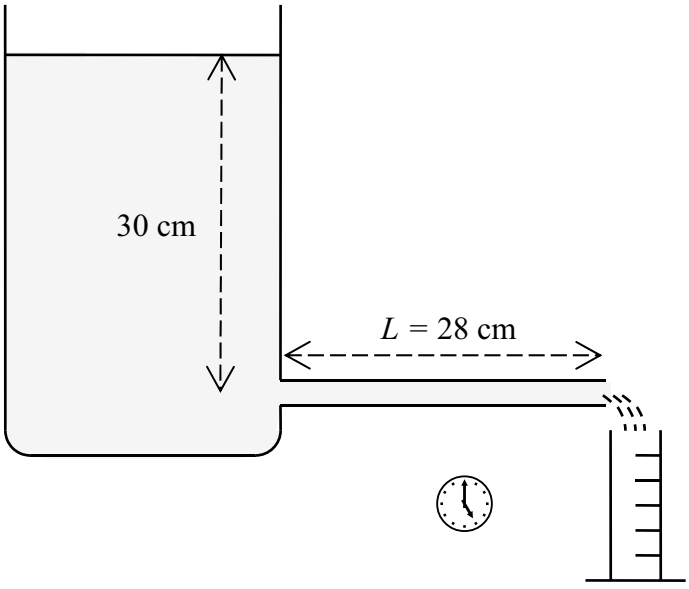


Fig. 15.10 Experiment to measure viscosity of water, Problem 15.9

$$\eta = 1.1 \times 10^{-3} \text{ N s m}^{-2}.$$

- wxMaxima codes:

```
(%i6)  fpprintprec:5; ratprint:false; delta_p:1000*9.8*0.3;
r:0.2e-3; L:0.28; flow_rate:3.6e-6/(10*60);
(fpprintprec)  5
(ratprint)  false
(delta_p)  2940.0
(r)  2.0*10^-4
(L)  0.28
(flow_rate)  6.0*10^-9
(%i8)  solve(flow_rate = %pi*r^4*delta_p/(8*eta*L), eta)$
float(%);
(%o8)  [eta=0.0010996]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of Δp, r, L, and flow rate.
(%i8) Solve, flow rate $= \frac{\pi r^4 \Delta p}{8\eta L}$ for η.

Problem 15.10 An oil droplet of density 900 kg m^{-3} attains a terminal speed of 0.20 m s^{-1} as it falls in a gas whose coefficient of viscosity is 15×10^{-6} N m^{-2} s. Calculate the radius of the oil droplet.

Solution
Assume the oil droplet is spherical in shape and neglect the density of the gas. Applying the Stokes's law, the terminal speed is (Eq. 15.9),

$$v_t = \frac{2r^2 g}{9\eta}(\rho_{oil} - \rho_{gas}).$$

Substituting the knows quantities into the equation gives,

$$0.20 \text{ m/s} = \frac{2r^2\left(9.8 \text{ m/s}^2\right)}{9\left(15 \times 10^{-6} \text{ N s/m}^2\right)}(900 \text{ kg/m}^3 - 0).$$

The radius of the oil droplet is,

$$r = 3.9 \times 10^{-5} \text{ m}.$$

- wxMaxima codes:

```
(%i7)  fpprintprec:5; ratprint:false; vt:0.2; g:9.8; eta:15e-
6; rho_oil:900; rho_gas:0;
(fpprintprec)   5
(ratprint)   false
(vt)   0.2
(g)   9.8
(eta)   1.5*10^-5
(rho_oil)   900
(rho_gas)   0
(%i9)      solve(vt=2*r^2*g*(rho_oil-rho_gas)/(9*eta),  r)$
float(%);
(%o9)   [r=-3.9123*10^-5,r=3.9123*10^-5]
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of v_t, g, η, ρ_{oil}, and ρ_{gas}.
(%i9) Solve $v_t = \frac{2r^2 g}{9\eta}(\rho_{oil} - \rho_{gas})$ for r.

Problem 15.11 When particles with a radius of 5.0×10^{-7} m and a density of 4.0×10^3 kg m^{-3} are mixed with water and stirred, a suspension is produced. The suspension is 5.0 cm tall and the particles are evenly dispersed. Determine the proportion of particles that have not settled after an hour. Water has a viscosity coefficient of 1.0×10^{-3} N m^{-2} s.

Solution
Assume the particles attain the terminal speed very quickly. The terminal speed of the particle is (Eq. 15.9),

$$\begin{aligned} v_t &= \frac{2gr^2}{9\eta}(\rho_{particle} - \rho_{water}) \\ &= \frac{2(9.8 \text{ N/kg})(5.0 \times 10^{-7} \text{ m})^2}{9(10^{-3} \text{ N s/m}^2)}(4000 \text{ kg/m}^3 - 1000 \text{ kg/m}^3) \\ &= 1.6 \times 10^{-6} \text{ m s}^{-1}. \end{aligned}$$

Figure 15.11 shows the situation after one hour. The top part of the container is clear water as the particles have fallen down in the water. The bottom part is the particles yet to fall and settle down.

The distance d is,

$$d = v_t t = (1.6 \times 10^{-6} \text{ m/s})(60 \times 60 \text{ s}) = 5.9 \times 10^{-3} \text{ m}.$$

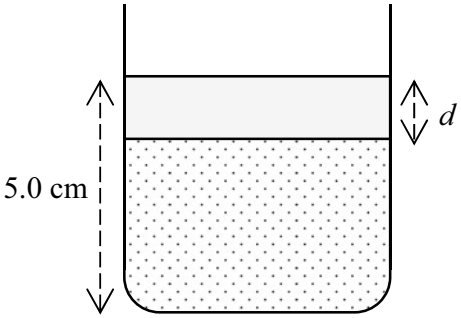


Fig. 15.11 Partilces in water, Problem 15.11

The percentage of the suspension that has not settled down is,

$$\frac{5.0 \times 10^{-2}\ \text{m} - 5.9 \times 10^{-3}\ \text{m}}{5.0 \times 10^{-2}\ \text{m}} \times 100\% = 88\%.$$

- wxMaxima codes:

```
(%i6)       fpprintprec:5;  g:9.8;  r:5e-7;  eta:1e-3;  rho_
particle:4000; rho_water:1000;
(fpprintprec)   5
(g)   9.8
(r)   5.0*10^-7
(eta)   0.001
(rho_particle)   4000
(rho_water)   1000
(%i7)   vt: 2*g*r^2*(rho_particle - rho_water)/(9*eta);
(vt)   1.6333*10^-6
(%i8)   d: vt*60*60;
(d)   0.00588
(%i9)   percentage: (5e-2 - d)/5e-2*100;
(percentage)   88.24
```

Comments on the codes:

(%i6) Set floating point print precision to 5, assign values of g, r, η, $\rho_{particle}$, and ρ_{water}.
(%i7), (%i8) Calculate v_t and d.
(%i9) Calculate percentage.

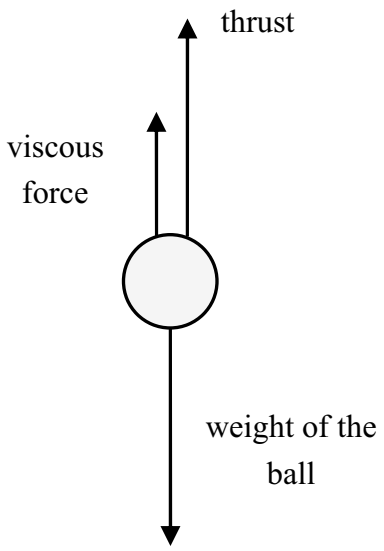


Fig. 15.12 Forces on the ball bearing, Problem 15.12

Problem 15.12 A ball bearing of radius 1.0 mm is dropped in a viscous liquid. The terminal speed attained is 4.0 mm s^{-1}. Calculate the coefficient of viscosity of the liquid. The density of the ball bearing is 8000 kg m^{-3} and the density of the liquid is 2000 kg m^{-3}.

Solution

Figure 15.12 shows the forces acting on the ball bearing in the liquid.

When the terminal speed is attained,

$$\begin{matrix}\text{thrust on}\\ \text{the ball by}\\ \text{the liquid}\end{matrix} + \begin{matrix}\text{viscous force}\\ \text{between the moving}\\ \text{ball and the liquid}\end{matrix} = \begin{matrix}\text{weight of}\\ \text{the ball,}\end{matrix}$$

that is,

$$\frac{4}{3}\pi r^3 \rho_{liq} g + 6\pi \eta r v_t = \frac{4}{3}\pi r^3 \rho_{ball} g.$$

The coefficient of viscosity of the liquid is,

$$\begin{aligned}\eta &= \frac{2gr^2}{9v_t}(\rho_{ball} - \rho_{liq})\\ &= \frac{2(9.8\ \text{N/kg})(1.0 \times 10^{-3}\ \text{m})^2}{9(4.0 \times 10^{-3}\ \text{m/s})}(8000\ \text{kg/m}^3 - 2000\ \text{kg/m}^3)\end{aligned}$$

$$= 3.3 \text{ N s m}^{-2}.$$

- wxMaxima codes:

```
(%i7)  fpprintprec:5; ratprint:false; g:9.8; r:1e-3; vt:4e-3;
rho_ball:8000; rho_liq:2000;
(fpprintprec)   5
(ratprint)   false
(g)   9.8
(r)   0.001
(vt)   0.004
(rho_ball)   8000
(rho_liq)   2000
(%i9)    solve(4/3*%pi*r^3*rho_liq*g + 6*%pi*eta*r*vt = 4/
3*%pi*r^3*rho_ball*g, eta)$ float(%);
(%o9)   [eta=3.2667]
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of g, r, v_t, ρ_{ball}, and ρ_{liq}.
(%i9) Solve $\frac{4}{3}\pi r^3 \rho_{liq} g + 6\pi\eta r v_t = \frac{4}{3}\pi r^3 \rho_{ball} g$ for η.

Problem 15.13 A small sphere of radius r and density σ, is released from the bottom of a liquid column, the density of the liquid is ρ. If $\rho > \sigma$, the coefficient of viscosity is η, and the acceleration of gravity is g. Determine,

(a) the intial acceleration of the sphere
(b) the terminal speed of the sphere.

Solution

(a) Before the sphere is released, there are two forces acting on the sphere, that is, the thrust on the sphere by the liquid in the upward direction and the weight of the sphere in the downward direction, Fig. 15.13a.

Net force in the upward direction is,

$$\begin{aligned}\text{thrust} - \text{weight of the sphere} &= \frac{4}{3}\pi r^3 \rho g - \frac{4}{3}\pi r^3 \sigma g \\ &= \frac{4}{3}\pi r^3 g(\rho - \sigma).\end{aligned}$$

Applying the Newton's second law, $\Sigma F = ma$,

$$\frac{4}{3}\pi r^3 g(\rho - \sigma) = ma$$

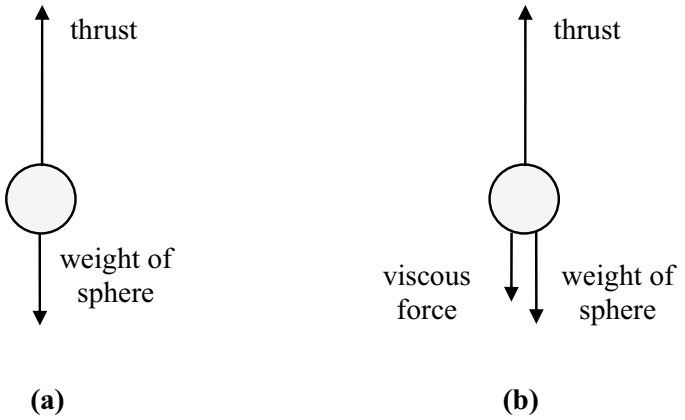


Fig. 15.13 Forces on the sphere **a** before and **b** after it is released in the liquid, Problem 15.13

$$= \frac{4}{3}\pi r^3 \sigma a,$$

where m is the mass of the sphere and and a is the initial acceleration. Thus, the initial acceleration of the sphere is,

$$a = g\left(\frac{\rho}{\sigma} - 1\right).$$

(b) When the teminal speed is attained, Fig. 15.13b,

thrust on the sphere = viscous force + weight of sphere,

$$\frac{4}{3}\pi r^3 \rho g = 6\pi \eta r v_t + \frac{4}{3}\pi r^3 \sigma g.$$

The terminal speed of the sphere is,

$$v_t = \frac{2r^2 g}{9\eta}(\rho - \sigma).$$

- wxMaxima codes:

```
(%i1)          solve((4/3)*%pi*r^3*g*(rho-sigma)   =   (4/
3)*%pi*r^3*sigma*a, a);
(%o1)   [a=-(g*sigma-g*rho)/sigma]
(%i2)         solve((4/3)*%pi*r^3*rho*g=6*%pi*eta*r*vt+(4/
3)*%pi*r^3*sigma*g, vt);
```

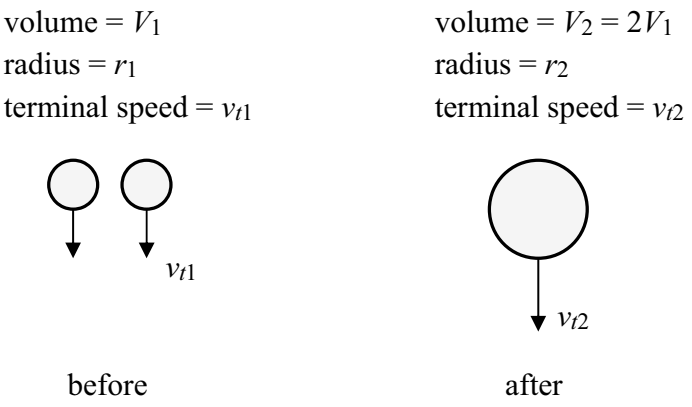


Fig. 15.14 Rain droplets before and after combination, Problem 15.14

```
(%o2)   [vt=-(2*g*r^2*sigma-2*g*r^2*rho)/(9*eta)]
```

Comments on the codes:

(%i1) Solve $\frac{4}{3}\pi r^3 g(\rho - \sigma) = \frac{4}{3}\pi r^3 \sigma a$ for a, part (a).
(%i2) Solve $\frac{4}{3}\pi r^3 \rho g = 6\pi\eta r v_t + \frac{4}{3}\pi r^3 \sigma g$ for v_t, part (b).

Problem 15.14 Two spherical rain droplets of the same size fall in air with terminal speed of 0.15 m s^{-1}. What is the new terminal speed of a larger droplet formed from the combination of the two droplets? Assume the Stoke's law is applicable.

Solution
Assume all rain droplets are spherical in shape. Figure 15.14 shows the rain droplets before and after combination.

The formula of terminal speed is (Eq. 15.9),

$$v_t = \frac{2r^2 g}{9\eta}(\rho_{water} - \rho_{air}).$$

The terminal speed of the rain droplet is proportional to the square of its radius. We write,

$$v_{t1} \propto r_1^2, \quad v_{t2} \propto r_2^2,$$
$$v_{t2} = \frac{r_2^2}{r_1^2} v_{t1}.$$

The volume of a rain droplet is proportinal to the cube of its radius. This implies,

$$V_1 \propto r_1^3, \quad V_2 \propto r_2^3,$$
$$\frac{r_2^3}{r_1^3} = \frac{V_2}{V_1} = \frac{2V_1}{V_1} = 2,$$
$$\left(\frac{r_2}{r_1}\right)^2 = 2^{2/3}.$$

Here, $V_2 = 2V_1$ because two droplets of volume V_1 combime to to form droplet of volume V_2. The terminal speed of the combined rain droplet is,

$$\begin{aligned} v_{t2} &= \left(\frac{r_2}{r_1}\right)^2 v_{t1} \\ &= 2^{2/3}(0.15 \text{ m s}^{-1}) \\ &= 0.24 \text{ m s}^{-1}. \end{aligned}$$

- wxMaxima codes:

```
(%i3)   fpprintprec:5; 2^(2/3)*0.15; float(%);
(fpprintprec)   5
(%o2)   0.15*2^(2/3)
(%o3)   0.23811
```

Comments on the codes:

(%i3) Set floating point print precision to 5 and calculate value of $2^{2/3} \times 0.15$.

Problem 15.15 Water flows in a horizontal tube of radius 1.0 mm and length 1.5 m. The pressure difference between ends of tube is 5.3×10^3 Pa. The coefficient of viscosity of water is 1.0×10^{-3} N m^{-2} s. Calculate,

(a) the water flow rate
(b) the speed of water flow.

Solution

(a) The water flow rate is given by Poiseuille's formula (Eq. 15.7), Fig. 15.15,

$$\begin{aligned} Q &= \frac{\pi r^4 \Delta p}{8 l \eta} \\ &= \frac{\pi(1.0 \times 10^{-3} \text{ m})^4 (5.3 \times 10^3 \text{ Pa})}{8(1.5 \text{ m})(1.0 \times 10^{-3} \text{ N m}^{-2} \text{ s})} \\ &= 1.4 \times 10^{-6} \text{ m}^3 \text{ s}^{-1}. \end{aligned}$$

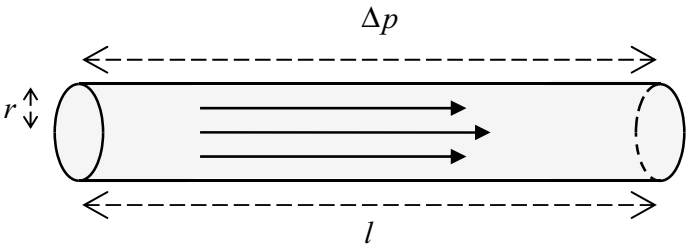


Fig. 15.15 Viscous fluid flow, Problem 15.15

(b) The speed of water flow is,

$$\begin{aligned} v &= \frac{Q}{A} = \frac{Q}{\pi r^2} \\ &= \frac{1.4 \times 10^{-6}\ \text{m}^3/\text{s}}{\pi (1.0 \times 10^{-3}\ \text{m})^2} \\ &= 0.44\ \text{m s}^{-1}. \end{aligned}$$

- wxMaxima codes:

```
(%i1)   fpprintprec:5;
(fpprintprec)   5
(%i3)   Q: %pi*(1e-3)^4*5.3e3/(8*1.5*1e-3)$ float(%);
(%o3)   1.3875*10^-6
(%i5)   v: Q/(%pi*(1e-3)^2)$ float(%);
(%o5)   0.44167
```

Comments on the codes:

(%i3) Calculation of flow rate, Q, part (a)
(%i5) Calculation of speed of water flow, v, part (b)

Problem 15.16 The pressure difference Δp between ends of a horizontal tube causes a constant laminar liquid flow in the tube, Fig. 15.16a. A new tube is made in which the diameter of one half length of the tube is reduced to a half of the original diameter, Fig. 15.16b. What is the pressure difference between the ends of the new tube so that the liquid flow rate is the same as in the original tube?

Solution

Applying the Poiseuille's formula (Eq. 15.7) to the original tube,

$$\text{volume flow rate} = \frac{\text{volume}}{\text{time}} = \frac{\pi (d/2)^4 \Delta p}{8 l \eta}. \tag{15.1}$$

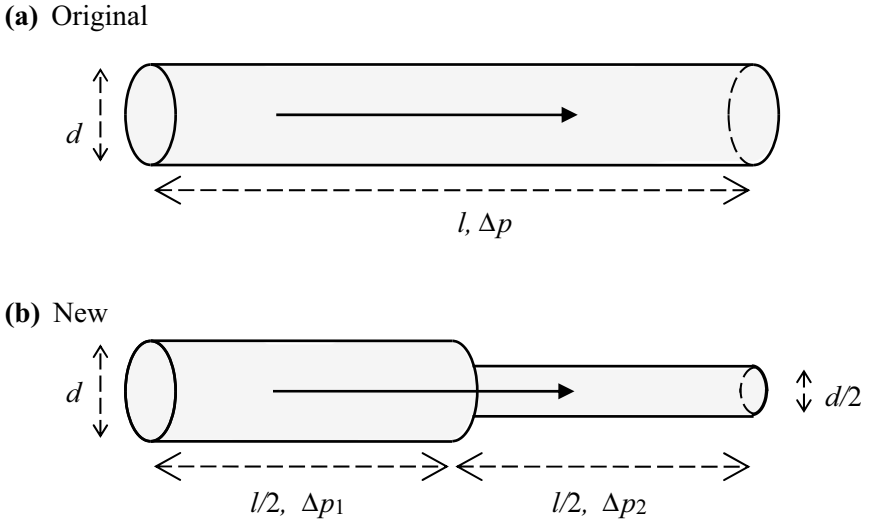


Fig. 15.16 Viscous fluid flow in **a** original and **b** new tubes, Problem 15.16

Similarly, for the new tube,

$$\text{volume flow rate} = \frac{\pi(d/2)^4 \Delta p_1}{8(l/2)\eta} = \frac{\pi(d/4)^4 \Delta p_2}{8(l/2)\eta}. \tag{15.2}$$

We want volume flow rates in (15.1) and (15.2) to be the same. From Eqs. (15.1) and (15.2), we have,

$$\Delta p_1 = \frac{1}{2}\Delta p \quad \text{and} \quad \Delta p_2 = 8\ \Delta p.$$

The pressure difference between the ends of the new tube is,

$$\begin{aligned} \Delta p_1 + \Delta p_2 &= \frac{1}{2}\Delta p + 8\ \Delta p \\ &= 8.5\ \Delta p. \end{aligned}$$

- wxMaxima codes:

```
(%i1)  equation1: %pi*(d/2)^4*dp/(8*l*eta)=%pi*(d/2)^4*dp1/
(8*(l/2)*eta);
(equation1)         (%pi*d^4*dp)/(128*eta*l)=(%pi*d^4*dp1)/
(64*eta*l)
```

```
(%i2)      equation2: %pi*(d/2)^4*dp/(8*l*eta) = %pi*(d/
4)^4*dp2/(8*(l/2)*eta);
(equation2)        (%pi*d^4*dp)/(128*eta*l)=(%pi*d^4*dp2)/
(1024*eta*l)
(%i3)  solve ([equation1, equation2], [dp1, dp2]);
(%o3)  [[dp1=dp/2,dp2=8*dp]]
```

Comments on the codes:

(%i1) Assign Eq. 15.1 as $\frac{\pi(d/2)^4\Delta p}{8l\eta} = \frac{\pi(d/2)^4\Delta p_1}{8(l/2)\eta}$.
(%i2) Assign Eq. 15.2 as $\frac{\pi(d/2)^4\Delta p}{8l\eta} = \frac{\pi(d/4)^4\Delta p_2}{8(l/2)\eta}$.
(%i3) Solve Eqs. 15.1 and 15.2 for Δp_1 and Δp_2.

15.3 Summary

- For flow of incompressible fluid, the equation of continuity states that the mass flowing into a pipe must equal the mass flowing out of the pipe.
- For flow of incompressible frictionless fluid, Bernoulli's equation states that the sum of pressure, kinetic energy per unit volume, and potential energy per unit volume at any point in the pipe is a constant,

$$p + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}.$$

- Fluid viscosity η is due to friction between layers in fluid. The volume rate of flow of the fluid is,

$$\frac{dV}{dt} = \frac{\pi R^4(p_1 - p_2)}{8\eta L}.$$

- The Stokes's law states that the viscous force F on a sphere of radius r and velocity v moving in a fluid is,

$$F = 6\pi\eta r v.$$

15.4 Exercises

Exercise 15.1 Figure 15.17 shows a venturi meter that is used to measure a liquid flow speed v_1. The liquid of density ρ_{liq} that flows in the large tube of cross-sectional area A_1 with speed v_1 flows into a small tube of cross-sectional area A_2 with speed v_2. The pressure at points 1 and 2 are p_1 and p_2, respectively. The liquid in the manometer

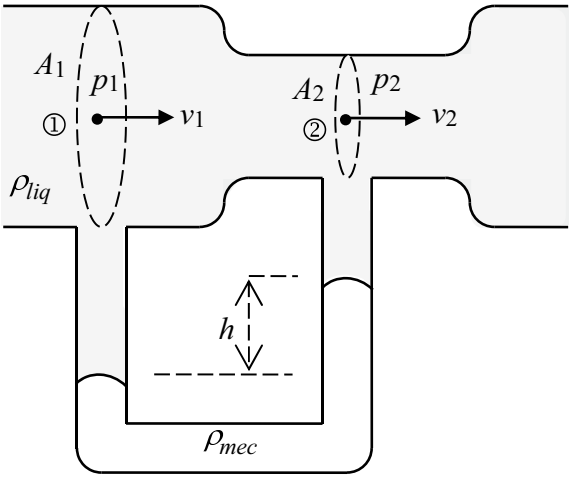


Fig. 15.17 A venture tube, Exercise 15.1

is mercury of density ρ_{mec} and the difference in mercury levels is h. In terms of A_1, A_2, v_1, v_2, and ρ_{liq}, what is,

(a) the ratio of the speeds v_1/v_2?
(b) the pressure difference between points 1 and 2.

(Answer: (a) A_2/A_1 (b) $\rho_{liq}(v_2^2 - v_1^2)/2$)

Exercise 15.2 A water container is at a height of h above the floor. A small hole is punched in the side of the container at its base, and the resultant stream of water strikes the floor at a horizontal distance of R from the container, as in Fig. 15.18. Determine the depth d of the water in the container.

(Answer: $d = R^2/(4\,h)$)

Exercise 15.3 Figure 15.19 shows a marble of radius r and density ρ in water near the surface. The density of water is ρ_w and its coefficient of viscosity is η. The marble is release from rest, falls down and attained a terminal speed. Show that,

(a) the net force on the marble is $\frac{4\pi r^3 g}{3}(\rho - \rho_w)$,
(b) the terminal speed of the marble is $\frac{2r^2 g}{9\eta}(\rho - \rho_w)$.

Exercise 15.4 A silt particle has a radius of 20 μm and density of 2.0×10^3 kg m^{-3}. The particle is in water whose coefficient of viscosity is 8.9×10^{-4} N s m^{-2} and density of 1.0×10^3 kg m^{-3}. Calculate the terminal speed of the particle in water.

(Answer: 9.8×10^{-4} m s^{-1})

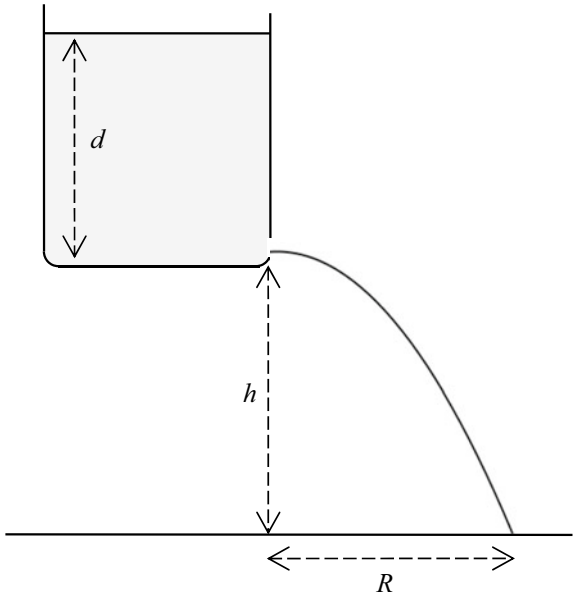
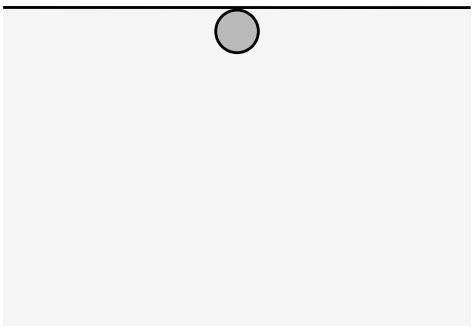


Fig. 15.18 A container with a hole at the bottom, Exercise 15.2



Marble:
radius r,
density ρ.

Water:
density ρ_w,
coefficient of
viscosity η.

Fig. 15.19 A marble in water, Exercise 15.3

Exercise 15.5 A motor oil with a coefficient of viscosity 0.25 N s m^{-2} flows through a horizontal pipe of radius 3.0 mm and of length 1.0 m. The drop in pressure is 2.3 $\times$ 10^5 Pa. Calculate the average speed of the flow.

(Answer: 1.0 m s^{-1})

Chapter 16
Temperature and Thermal Expansion

16.1 Basic Concepts and Formulae

(1) When two bodies in contact have the same temperature, they are in thermal equilibrium.
(2) According to the zeroth law of thermodynamics, if bodies A and B are each in thermal equilibrium with body C, then bodies A and B are also in thermal equilibrium.
(3) The SI unit of temperature is Kelvin (K). One kelvin is 1/273.16 of the temperature of the triple point of water. The temperature of the triple point of water is 273.16 K. The triple point of water is the temperature and pressure at which the three phases (gas, liquid, and solid) of water coexist in thermodynamic equilibrium.
(4) The temperature scale can be defined as,

$$\frac{T(X)}{T(X_3)} = \frac{X}{X_3}, \tag{16.1}$$

where X and X_3 are measures of thermometry property at the temperatures to be defined and at the triple point of water, respectively, while $T(X_3) =$ temperature of triple point of water $= 273.16$ K.

The temperature scale can also be defined as,

$$T = \left(\frac{X - X_{lower\ point}}{X_{upper\ point} - X_{lower\ point}}\right) N + T_{lower\ point}, \tag{16.2}$$

where X, $X_{lower\ point}$, and $X_{upper\ point}$ are measures of thermometry property at the temperatures to be defined, at lower point, and at upper point, respectively, N is the number of temperature divisions between lower and upper points, and $T_{lower\ point}$ is the temperature defined for the lower point.

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(5)

(a) Temperature T in K and temperature θ_C in Celsius, °C, are related by,

$$\theta_C = T - 273.15. \tag{16.3}$$

Temperature difference of 1 K is equal to temperature difference of 1°C, but 1 K is not equal to 1°C.

(b) Celsius temperature θ_C in °C and Fahrenheit temperature θ_F in °F are related by,

$$\theta_F = \frac{9}{5}\theta_C + 32. \tag{16.4}$$

(c) Temperature T in K and rankine temperature T_R in °R are related by,

$$T_R = \frac{9}{5}T. \tag{16.5}$$

(6) The thermal coefficient of volume expansion γ is the fractional volume change $\Delta V/V$ divided by the temperature change $\Delta\theta$,

$$\gamma = \frac{\Delta V}{V\,\Delta\theta}. \tag{16.6}$$

The SI unit of γ is K^{-1}.

The thermal coefficient of area expansion β is the fractional area change $\Delta A/A$ divided by the temperature change $\Delta\theta$,

$$\beta = \frac{\Delta A}{A\,\Delta\theta}. \tag{16.7}$$

The SI unit of β is K^{-1}.

The thermal coefficient of linear expansion α is the fractional change in length $\Delta l/l$ divided by the temperature change $\Delta\theta$,

$$\alpha = \frac{\Delta l}{l\,\Delta\theta}. \tag{16.8}$$

The SI unit of α is K^{-1}.

(7) A rod of length l_0 made of material with thermal linear expansion coefficient of α is subjected to a temperature increase of $\Delta\theta$. The increase in length Δl and the new length l_{new} of the rod are,

$$\Delta l = \alpha\ \Delta\theta\ l_0, \tag{16.9}$$

$$l_{new} = l_0 + \Delta l = (1 + \alpha\ \Delta\theta) l_0. \tag{16.10}$$

A plate of area A_0 made of material with thermal area expansion coefficient of β is subjected to a temperature increase of $\Delta\theta$. The increase in area ΔA and the new area A_{new} of the plate are,

$$\Delta A = \beta\ \Delta\theta\ A_0, \tag{16.11}$$

$$A_{new} = A_0 + \Delta A = (1 + \beta\ \Delta\theta) A_0. \tag{16.12}$$

A material of volume V_0 with thermal volume expansion coefficient of γ is subjected to a temperature increase of $\Delta\theta$. The increase in volume ΔV and the new volume V_{new} are,

$$\Delta V = \gamma\ \Delta\theta\ V_0, \tag{16.13}$$

$$V_{new} = V_0 + \Delta V = (1 + \gamma\ \Delta\theta) V_0. \tag{16.14}$$

(8) For a material with thermal coefficient of linear expansion α, its thermal coefficient of area expansion β and thermal coefficient of volume expansion γ are related to α as,

$$\beta = 2\alpha, \tag{16.15}$$

$$\gamma = 3\alpha. \tag{16.16}$$

(9) A material with thermal coefficient of volume expansion γ has a volume V_0 and a density of ρ_0. An increase in temperature of $\Delta\theta$, gives new volume and density of,

$$V_{new} = V_0(1 + \gamma\ \Delta\theta), \tag{16.17}$$

$$\rho_{new} = \frac{\rho_0}{(1 + \gamma\ \Delta\theta)}. \tag{16.18}$$

16.2 Problems and Solutions

Problem 16.1 The length of mercury column of a mercury thermometer at freezing point of water (273.15 K) is 30 mm and at steam point (373.15 K) is 282 mm. What is the temperature measured by the thermometer if the length of the mercury column is 180 mm?

Solution

The temperature is calculated as follows (Eq. 16.2),

$$\begin{aligned} T &= \left(\frac{X - X_{lower\ point}}{X_{upper\ point} - X_{lower\ point}}\right) N + T_{lower\ point} \\ &= \left(\frac{180\ \text{mm} - 30\ \text{mm}}{282\ \text{mm} - 30\ \text{mm}}\right) \times 100\ \text{K} + 273.15\ \text{K} \\ &= 333\ \text{K}. \end{aligned}$$

Here, X is length of mercury column, we set there are 100 K between the lower and upper points, and defined the temperature of the lower point (freezing point of water) as 273.15 K.

- wxMaxima codes:

```
(%i1)  fpprintprec:5;
(fpprintprec)      5
(%i3)  T: (180-30)/(282-30)*100 + 273.15$ float(%);
(%o3)  332.67
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i3) Calculate T.

Alternative solution: The temperature can also be determined by the two lines of scales as in Fig. 16.1.

Each line represents the temperature scale and a linear relationship between X and T is assumed. The temperature is calculated as follows,

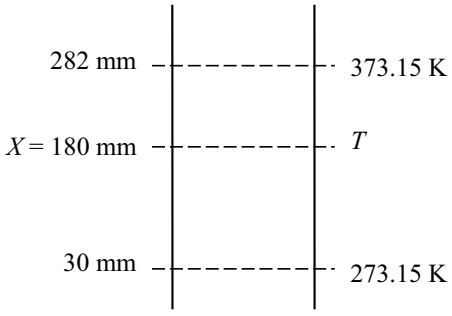

Fig. 16.1 Conversion of length of mercury column to temperature, Problem 16.1

$$\frac{T - 273.15\ \text{K}}{373.15\ \text{K} - 273.15\ \text{K}} = \frac{180\ \text{mm} - 30\ \text{mm}}{282\ \text{mm} - 30\ \text{mm}},$$
$$T = 333\ \text{K}.$$

- wxMaxima codes:

```
(%i2)  fpprintprec:5; ratprint:false;
(fpprintprec)      5
(ratprint)  false
(%i4)  solve((T-273.15)/(373.15-273.15)=(180-30)/(282-30),
T)$ float(%);
(%o4)  [T=332.67]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i4) Solve $\frac{T-273.15}{373.15-273.15} = \frac{180-30}{282-30}$ for T.

A similar question: The length of mercury column of a mercury thermometer at freezing point of water (32°F) is 30 mm and at steam point (212°F) is 282 mm. What is the temperature in Fahrenheit measured by the thermometer if the length of the mercury column is 180 mm?

Answer: The temperature in Fahrenheit is (Eq. 16.2),

$$\begin{aligned}\theta_F &= \left(\frac{X - X_{lower\ point}}{X_{upper\ point} - X_{lower\ point}}\right)N + \theta_{lower\ point} \\ &= \left(\frac{180\ \text{mm} - 30\ \text{mm}}{282\ \text{mm} - 30\ \text{mm}}\right) \times 180°\text{F} + 32°\text{F} \\ &= 139°\text{F}.\end{aligned}$$

Here, there are 180°F between the lower and upper points, and the temperature of the lower point (freezing point of water) is defined as 32°F.

- wxMaxima codes:

```
(%i1)  fpprintprec:5;
(fpprintprec)      5
(%i3)  thetaF: (180-30)/(282-30)*180 +32$ float(%);
(%o3)  139.14
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i3) Calculate θ_F.

Similarly, defining there are 100°C between lower and upper points, and the temperature of the lower point (freezing point of water) as 0°C (Eq. 16.2), the temperature in Celsius is,

$$\begin{aligned}\theta_C &= \left(\frac{X - X_{lower\ point}}{X_{upper\ point} - X_{lower\ point}}\right) N + \theta_{lower\ point} \\ &= \left(\frac{180\ \text{mm} - 30\ \text{mm}}{282\ \text{mm} - 30\ \text{mm}}\right) \times 100°\text{C} + 0°\text{C} \\ &= 59.5°\text{C}.\end{aligned}$$

- wxMaxima codes:

```
(%i1)  fpprintprec:5;
(fpprintprec)      5
(%i3)  thetaC: (180-30)/(282-30)*100 +0$ float(%);
(%o3)  59.524
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i3) Calculate θ_C.

Problem 16.2 At the triple point of water, a constant volume gas thermometer records a pressure of 15.0×10^4 Pa. When measuring temperature, the thermometer records a pressure of 20.5×10^4 Pa. What is the temperature at this pressure?

Solution

The measured temperature is (Eq. 16.1),

$$\begin{aligned}T &= \frac{p}{p_{triple}} \times 273.16\ \text{K} \\ &= \frac{20.5 \times 10^4\ \text{Pa}}{15 \times 10^4\ \text{Pa}} \times 273.16\ \text{K} \\ &= 373\ \text{K}.\end{aligned}$$

- wxMaxima codes:

```
(%i1)  fpprintprec:5;
(fpprintprec)      5
(%i2)  T: 20.5e4/15e4*273.16;
(T)    373.32
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i2) Calculate T.

Problem 16.3 The resistance R_θ (in ohm) at temperature θ (in °C) of a resistance thermometer is given by,

$$R_\theta = R_0(1 + a\theta + b\theta^2)$$

where R_0 is the resistance at 0°C, $b = -1.54 \times 10^{-4} a\ ^\circ\text{C}^{-1}$, and a is a constant.

(a) Write expressions for resistance at ice and steam points.
(b) What is the temperature recorded by the resistance thermometer at $\theta = 40.0^\circ$C?

Solution

(a) The resistance is given by,

$$R_\theta = R_0(1 + a\theta + b\theta^2).$$

At ice point, $\theta = 0^\circ$C, the resistance is,

$$R_0 = R_0(1 + 0 + 0) = R_0.$$

At steam point, $\theta = 100^\circ$C,

$$\begin{aligned} R_{100} &= R_0\left[1 + a(100) - 1.54 \times 10^{-4} a(100)^2\right] \\ &= R_0(1 + 98.46a). \end{aligned}$$

(b) At $\theta = 40.0^\circ$C, the resistance is,

$$\begin{aligned} R_{40} &= R_0\left[1 + a(40) - 1.54 \times 10^{-4} a(40)^2\right] \\ &= R_0(1 + 39.75a). \end{aligned}$$

The temperature recorded by the resistance thermometer is,

$$\begin{aligned}\theta_C &= \frac{R_{40} - R_0}{R_{100} - R_0} \times 100 \\ &= \frac{R_0(1 + 39.75a) - R_0}{R_0(1 + 98.46a) - R_0} \times 100 \\ &= 40.4^\circ\text{C}.\end{aligned}$$

- wxMaxima codes:

```
(%i2)   fpprintprec:5; ratprint:false;
(fpprintprec)         5
(ratprint)   false
(%i3)   Rtheta: R0*(1+a*theta-1.54e-4*a*theta^2);
(Rtheta)   R0*(-1.54*10^-4*a*theta^2+a*theta+1)
(%i5)   theta: 0; R0: R0*(1+a*theta-1.54e-4*a*theta^2);
(theta)   0
(R0)    R0
(%i7)   theta: 100; R100: R0*(1+a*theta-1.54e-4*a*theta^2);
(theta)   100
(R100)   R0*(98.46*a+1)
(%i9)   theta: 40; R40: R0*(1+a*theta-1.54e-4*a*theta^2);
(theta)   40
(R40)   R0*(39.754*a+1)
(%i12)   thetaC: (R40-R0)/(R100-R0)*100$ radcan(%)$ float(%);
(%o12)   40.375
```

Comments on the codes:

(%i2) Set floating point precision to 5 and internal rational number print to false.

(%i3) Assign R_θ.

(%i5), (%i7), (%i9) Calculate R_0, R_{100}, and R_{40}, part (a).

(%i12) Calculate θ_C, part (b).

Problem 16.4

(a) Convert 528 K to Celsius and Fahrenheit.
(b) Convert 440°F to Kelvin.
(c) Convert 360°C to Fahrenheit.

Solution

(a) A simple way for temperature conversion is as in Fig. 16.2.
From the figure, we write,

$$\frac{\theta_C - 0^\circ\text{C}}{100^\circ\text{C} - 0^\circ\text{C}} = \frac{T - 273.15\ \text{K}}{373.15\ \text{K} - 273.15\ \text{K}} = \frac{528\ \text{K} - 273.15\ \text{K}}{373.15\ \text{K} - 273.15\ \text{K}}. \quad (16.1)$$

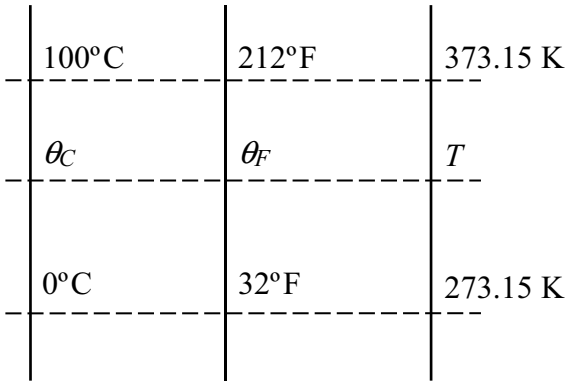

Fig. 16.2 Temperature conversions, Problem 16.4

Temperature $T = 528$ K in Celsius is,

$$\theta_C = 255°\text{C}.$$

Similarly, we write,

$$\frac{\theta_F - 32°\text{F}}{212°\text{F} - 32°\text{F}} = \frac{T - 273.15\ \text{K}}{373.15\ \text{K} - 273.15\ \text{K}} = \frac{528\ \text{K} - 273.15\ \text{K}}{373.15\ \text{K} - 273.15\ \text{K}}. \quad (16.2)$$

Temperature $T = 528$ K in Fahrenheit is,

$$\theta_F = 491°\text{F}.$$

(b) Similarly,

$$\frac{T - 273.15\ \text{K}}{373.15\ \text{K} - 273.15\ \text{K}} = \frac{\theta_F - 32°\text{F}}{212°\text{F} - 32°\text{F}} = \frac{440°\text{F} - 32°\text{F}}{212°\text{F} - 32°\text{F}}. \quad (16.3)$$

Temperature $\theta_F = 440°$F in Kelvin is,

$$T = 500\,\text{K}.$$

(c) Similarly,

$$\frac{\theta_F - 32°\text{F}}{212°\text{F} - 32°\text{F}} = \frac{\theta_C - 0°\text{C}}{100°\text{C} - 0°\text{C}} = \frac{360°\text{C} - 0°\text{C}}{100°\text{C} - 0°\text{C}}. \quad (16.4)$$

Temperature $\theta_C = 360°$C in Fahrenheit is,

$$\theta_F = 680°\text{F}.$$

- wxMaxima codes:

```
(%i2)  fpprintprec:5; ratprint:false;
(fpprintprec)       5
(ratprint)  false
(%i4)     solve(thetaC/100 = (528-273.15)/(373.15-273.15),
thetaC)$ float(%);
(%o4)  [thetaC=254.85]
(%i6)    solve((thetaF-32)/(212-32) = (528-273.15)/(373.15-
273.15), thetaF)$ float(%);
(%o6)  [thetaF=490.73]
(%i8)    solve((T-273.15)/(373.15-273.15) = (440-32)/(212-
32), T)$ float(%);
(%o8)  [T=499.82]
(%i10)      solve((thetaF-32)/(212-32)  =  (360-0)/(100-0),
thetaF)$ float(%);
(%o10)  [thetaF=680.0]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i4) Solve Eq. (16.1) for θ_C. Part (a).

(%i6) Solve Eq. (16.2) for θ_F. Part (a).

(%i8) Solve Eq. (16.3) for T. Part (b).

(%i10) Solve Eq. (16.4) for θ_F. Part (c).

Problem 16.5 The induced emf in a thermocouple is,

$$\mathcal{E} = a\Delta T + b(\Delta T)^2,$$

where ΔT is the temperature difference between hot and cold junctions in kelvin while $a = 6.93 \times 10^{-6}$ V K^{-1} and $b = -2.10 \times 10^{-9}$ V K^{-2} are constants. The cold junction is maintained at 273 K.

(a) What is the emf $\mathcal{E}$ when the hot junction is at 373 K?
(b) If $\mathcal{E} = 3.25$ mV, what is the temperature of the hot junction?

Solution

(a) At 373 K, the temperature difference of the junctions is,

$$T = 373\,\text{K} - 273\,\text{K} = 100\,\text{K}.$$

The induced emf in the thermocouple is,

$$\begin{aligned}\mathcal{E} &= a\Delta T + b(\Delta T)^2 \\ &= 6.93 \times 10^{-6}\,\mathrm{V\,K^{-1}}(100\,\mathrm{K}) - 2.10 \times 10^{-9}\,\mathrm{V\,K^{-2}}(100\,\mathrm{K})^2 \\ &= 6.72 \times 10^{-4}\,\mathrm{V}.\end{aligned}$$

- wxMaxima codes:

```
(%i4)  fpprintprec:5; a:6.93e-6; b:-2.1e-9; dT:373-273;
(fpprintprec)      5
(a)    6.93*10^-6
(b)    -2.1*10^-9
(dT)   100
(%i5)  emf: a*dT + b*dT^2;
(emf)  6.72*10^-4
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of a, b, and ΔT.

(%i5) Calculate the emf.

(b) Substituting known values in the equation gives,

$$\begin{aligned}\mathcal{E} &= a\Delta T + b(\Delta T)^2, \\ 3.25 \times 10^{-3} &= 6.93 \times 10^{-6}\Delta T - 2.10 \times 10^{-9}(\Delta T)^2.\end{aligned}$$

Solving the quadratic equation gives,

$$\Delta T = 566\,\mathrm{K}.$$

The temperature of the hot junction of the thermocouple is,

$$T = 273\,\mathrm{K} + 566\,\mathrm{K} = 839\,\mathrm{K}.$$

- wxMaxima codes:

```
(%i4)  fpprintprec:5; ratprint:false; a:6.93e-6; b:-2.1e-9;
(fpprintprec)      5
(ratprint)  false
```

```
(a)    6.93*10^-6
(b)    -2.1*10^-9
(%i6)  solve(3.25e-3 = a*dT + b*dT^2, dT)$ float(%);
(%o6)  [dT=566.08,dT=2733.9]
(%i7)  T: 273 + 566.08;
(T)    839.08
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of a and b.

(%i6) Solve $3.25 \times 10^{-3} = a\Delta T + b(\Delta T)^2$ for ΔT.

(%i7) Calculate T.

Problem 16.6 The difference in lengths of two rods of different metals is 5.0 cm. The thermal coefficient of linear expansion of one rod is 1.2×10^{-5} K^{-1} and the other is 1.8×10^{-5} K^{-1}. As the temperature increases, the difference in lengths of the rods stays the same. What are the original lengths of the rods?

Solution

Let the thermal coefficient of linear expansion of the first and second rods be $\alpha_1 = 1.2 \times 10^{-5}$ K^{-1} and $\alpha_2 = 1.8 \times 10^{-5}$ K^{-1} respectively. When the temperature increases by $\Delta\theta$, the lengths of rod 1 and 2 become (Eq. 16.10),

$$l_1 = l_{1original}(1 + \alpha_1 \Delta\theta),$$

$$l_2 = l_{2original}(1 + \alpha_2 \Delta\theta).$$

We want the difference in lengths to be the same at any temperature, that is, 5.0 cm. This implies,

$$l_{1original} - l_{2original} = 5.0 \text{ cm}, \tag{16.1}$$

$$l_1 - l_2 = 5.0 \text{ cm}.$$

The equation implies,

$$\begin{aligned} l_1 - l_2 &= l_{1original}(1 + \alpha_1 \Delta\theta) - l_{2original}(1 + \alpha_2 \Delta\theta) \\ &= l_{1original} - l_{2original} + \Delta\theta\left(\alpha_1 l_{1original} - \alpha_2 l_{2original}\right), \end{aligned}$$

$$5.0 \text{ cm} = 5.0 \text{ cm} + \Delta\theta\left(\alpha_1 l_{1original} - \alpha_2 l_{2original}\right),$$

$$\alpha_1 l_{1original} = \alpha_2 l_{2original}. \tag{16.2}$$

Solving Eqs. (16.1) and (16.2) gives the original lengths of the rods as,

$$l_{2original} = \frac{5\alpha_1}{\alpha_2 - \alpha_1} = \frac{5\left(1.2 \times 10^{-5}\ \mathrm{K}^{-1}\right)}{1.8 \times 10^{-5}\ \mathrm{K}^{-1} - 1.2 \times 10^{-5}\ \mathrm{K}^{-1}} = 10\ \mathrm{cm},$$
$$l_{1original} = 5\ \mathrm{cm} + l_{2original} = 5\ \mathrm{cm} + 10\ \mathrm{cm} = 15\ \mathrm{cm}.$$

- wxMaxima codes:

```
(%i4)       fpprintprec:5;  ratprint:false;  alpha1:1.2e-5;
alpha2:1.8e-5;
(fpprintprec)       5
(ratprint)   false
(alpha1)  1.2*10^-5
(alpha2)  1.8*10^-5
(%i6)      solve([l_1original  -  l_2original=5,  alpha1*l_
1original = alpha2*l_2original], [l_1original, l_2original])
$ float(%);
(%o6)  [[l_1original=15.0,l_2original=10.0]]
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of α_1 and α_2.

(%i6) Solve $l_{1original} - l_{2original} = 5$ and $\alpha_1 l_{1original} = \alpha_2 l_{2original}$ for $l_{1original}$ and $l_{2original}$.

Problem 16.7 A solid cube of copper has 10 cm sides at 20°C. As shown in Fig. 16.3, a cylindrical hole with a radius of 3.0 cm is drilled through the cube. The thermal coefficient of linear expansion of copper is $1.7 \times 10^{-5}\ \mathrm{K}^{-1}$. The temperature of the cube is raised to 70°C. Calculate,

(a) the change in area of one of its faces
(b) the change in the volume of the hole.

Solution

(a) The original area of a face of the cube at 20°C is,

$$A_0 = l_0 \times l_0 = 10\,\mathrm{cm} \times 10\,\mathrm{cm} = 100\,\mathrm{cm}^2.$$

When the temperature is raised to 70°C, the sides of the cube become (Eq. 16.10),

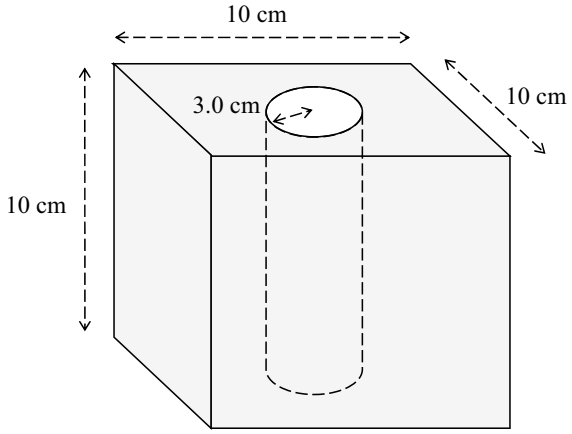

Fig. 16.3 A cylindrical hole in a solid cube, Problem 16.7

$$\begin{aligned} l_{new} &= (1 + \alpha\ \Delta\theta) l_0 \\ &= \left[1 + 1.7 \times 10^{-5}\ \mathrm{K}^{-1}(70^\circ\mathrm{C} - 20^\circ\mathrm{C})\right] \times 10\,\mathrm{cm} \\ &= 10.0085\,\mathrm{cm}. \end{aligned}$$

The new area is,

$$A_{new} = l_{new} \times l_{new} = 10.0085\,\mathrm{cm} \times 10.0085\,\mathrm{cm} = 100.17\,\mathrm{cm}^2.$$

The increase in area is,

$$\Delta A = A_{new} - A_0 = 100.17\,\mathrm{cm}^2 - 100\,\mathrm{cm}^2 = 0.17\,\mathrm{cm}^2.$$

- wxMaxima codes:

```
(%i4)  fpprintprec:5; l0:10; alpha:1.7e-5; deltatheta:70-20;
(fpprintprec)       5
(l0)   10
(alpha)   1.7*10^-5
(deltatheta)   50
(%i5)   A0: l0*l0;
(A0)    100
(%i6)   lnew: (1+alpha*deltatheta)*l0;
(lnew)   10.008
(%i7)   Anew: lnew*lnew;
(Anew)   100.17
(%i8)   deltaA: Anew-A0;
(deltaA)   0.17007
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of l_0, α, and $\Delta\theta$.

(%i5) Calculate area A_0.

(%i6) Calculate new length l_{new}.

(%i7) Calculate new area A_{new}.

(%i8) Calculate ΔA.

Alternative solution: The thermal coefficient of area expansion is $\beta = \frac{\Delta A}{A\,\Delta\theta}$ and its relation with the thermal coefficient of linear expansion α is $\beta = 2\alpha$ (Eq. 16.15). Thus, the increase in area is,

$$\begin{aligned}\Delta A &= \beta A_0\,\Delta\theta = 2\alpha A_0\,\Delta\theta \\ &= 2\big(1.7\times 10^{-5}\ \mathrm{K}^{-1}\big)\big(100\ \mathrm{cm}^2\big)(70-20)\,\mathrm{K} \\ &= 0.17\ \mathrm{cm}^2.\end{aligned}$$

- wxMaxima codes:

```
(%i5)    fpprintprec:5; l0:10; alpha:1.7e-5; beta:2*alpha;
deltatheta:70-20;
(fpprintprec)       5
(l0)   10
(alpha)   1.7*10^-5
(beta)   3.4*10^-5
(deltatheta)   50
(%i6)   A0: l0*l0;
(A0)   100
(%i7)   deltaA: beta*A0*deltatheta;
(deltaA)   0.17
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of l_0, α, β, and $\Delta\theta$.

(%i6) Calculate area A_0.

(%i7) Calculate ΔA.

(b) The original volume of the cylindrical hole at 20°C is,

$$V_0 = \pi r_0^2 l_0 = \pi(3.0\,\mathrm{cm})^2(10\,\mathrm{cm}) = 282.74\,\mathrm{cm}^3.$$

When the temperature is 70°C, the radius and length are,

$$r_{new} = (1 + \alpha\ \Delta\theta) r_0 = \left(1 + 1.7 \times 10^{-5}\ \mathrm{K}^{-1} \times 50\ \mathrm{K}\right) \times 3.0\ \mathrm{cm}$$
$$= 3.0026\ \mathrm{cm},$$

$$l_{new} = (1 + \alpha\ \Delta\theta) l_0 = \left(1 + 1.7 \times 10^{-5}\ \mathrm{K}^{-1} \times 50\ \mathrm{K}\right) \times 10\ \mathrm{cm}$$
$$= 10.0085\ \mathrm{cm}.$$

The new volume of the hole is,

$$V_{new} = \pi r_{new}^2 l_{new} = \pi (3.0026\ \mathrm{cm})^2 (10.0085\ \mathrm{cm}) = 283.46\ \mathrm{cm}^3.$$

The increase in the volume of the hole is,

$$\Delta V = V_{new} - V_0 = 283.46\,\mathrm{cm}^3 - 282.74\,\mathrm{cm}^3 = 0.72\,\mathrm{cm}^3.$$

- wxMaxima codes:

```
(%i5)        fpprintprec:5;   l0:10;   r0:3;   alpha:1.7e-5;
deltatheta:70-20;
(fpprintprec)      5
(l0)   10
(r0)   3
(alpha)  1.7*10^-5
(deltatheta)  50
(%i6)  V0: float(%pi)*r0^2*l0;
(V0)   282.74
(%i7)  rnew: (1+alpha*deltatheta)*r0;
(rnew)  3.0026
(%i8)  lnew: (1+alpha*deltatheta)*l0;
(lnew)  10.008
(%i9)  Vnew: float(%pi)*rnew^2*lnew;
(Vnew)  283.46
(%i10)  deltaV: Vnew-V0;
(deltaV)  0.72161
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of l_0, r_0, α, and $\Delta\theta$.

(%i6), (%i7), (%i8), (%i9), (%i10) Calculate V_0, r_{new}, l_{new}, V_{new}, and ΔV.

Alternative solution: From the definition of thermal coefficient of volume expansion $\gamma = \frac{\Delta V}{V \Delta\theta}$ and its relation with thermal coefficient of linear expansion α, namely $\gamma = 3\alpha$ (Eq. 16.16), the increase in volume is,

$$\begin{aligned}\Delta V &= \gamma V_0\ \Delta\theta = 3\alpha V_0\ \Delta\theta \\ &= 3\big(1.7\times 10^{-5}\ \mathrm{K}^{-1}\big)\big(282.74\ \mathrm{cm}^3\big)(70-20)\ \mathrm{K} \\ &= 0.72\ \mathrm{cm}^3.\end{aligned}$$

- wxMaxima codes:

```
(%i5)        fpprintprec:5;   l0:10;   r0:3;   alpha:1.7e-5;
deltatheta:70-20;
(fpprintprec)      5
(l0)   10
(r0)   3
(alpha)   1.7*10^-5
(deltatheta)   50
(%i6)  V0: float(%pi)*r0^2*l0;
(V0)   282.74
(%i7)  deltaV: 3*alpha*V0*deltatheta;
(deltaV)   0.721
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of l_0, r_0, α, and $\Delta\theta$.

(%i6), (%i7) Calculate V_0 and ΔV.

Problem 16.8

(a) Show that for a material that has a thermal coefficient of linear expansion α, its thermal coefficients of area and volume expansions are $\beta = 2\alpha$ and $\gamma = 3\alpha$, respectively.

(b) The radius of a metal disk increases by 0.15% when the temperature increases by 80°C. Calculate the following

(i) the percentage increase in the disk's surface area, thickness, and volume,
(ii) the percentage decrease in the disk's density,
(iii) the thermal coefficient of volume expansion of the metal, and
(iv) the percentage increase in the disk's moment of inertia about its symmetry axis.

Solution

(a) Consider a square plate of the material with side of length l_0. An increase in temperature of $\Delta\theta$, will give a new side of length $l_0(1 + \alpha\ \Delta\theta)$. Thus, the original area A_0, new area A_{new}, increase in area $\Delta A = A_{new} - A_0$, and thermal coefficient of area expansion β are,

$$
\begin{aligned}
&A_0 = l_0^2,\\
&A_{new} = l_0^2(1+\alpha\ \Delta\theta)^2 = l_0^2\left[1+2\alpha\ \Delta\theta+\alpha^2\ (\Delta\theta)^2\right],\\
&\Delta A = A_{new} - A_0 = 2\alpha\ \Delta\theta\ l_0^2 + \alpha^2\ (\Delta\theta)^2 l_0^2 \approx 2\alpha\ \Delta\theta\ l_0^2,\\
&\beta = \frac{\Delta A}{A_0 \Delta\theta} = \frac{2\alpha\ \Delta\theta\ l_0^2}{l_0^2 \Delta\theta} = 2\alpha.
\end{aligned}
$$

Next, consider a cube of the material with side of length l_0. An increase in temperature of $\Delta\theta$, will give a new side of length $l_0(1+\alpha\ \Delta\theta)$. Thus, the original volume V_0, new volume V_{new}, increase in volume $\Delta V = V_{new} - V_0$, and thermal coefficient of volume expansion γ are,

$$
\begin{aligned}
&V_0 = l_0^3,\\
&V_{new} = l_0^3(1+\alpha\ \Delta\theta)^3 = l_0^3\left[1+3\alpha\ \Delta\theta+3(\alpha\ \Delta\theta)^2+(\alpha\ \Delta\theta)^3\right],\\
&\Delta V = V_{new} - V_0 \approx 3\alpha\ \Delta\theta\ l_0^3,\\
&\gamma = \frac{\Delta V}{V_0 \Delta\theta} = \frac{3\alpha\ \Delta\theta\ l_0^3}{l_0^3 \Delta\theta} = 3\alpha.
\end{aligned}
$$

(b)

(i) From the definitions of thermal coefficient of linear, area, and volume expansions (α, β, and γ, Eqs. 16.8, 16.7, and 16.6) and their relationships (namely $\beta = 2\alpha$, $\gamma = 3\alpha$, Eqs. 16.15 and 16.16), we have,

$$
\begin{aligned}
&\frac{\Delta r}{r} = \frac{\Delta t}{t} = \frac{\Delta l}{l} = \alpha\ \Delta\theta = 0.15\%,\\
&\frac{\Delta A}{A} = \beta\ \Delta\theta = 2\alpha\ \Delta\theta = 2(0.15\%) = 0.30\%,\\
&\frac{\Delta V}{V} = \gamma\ \Delta\theta = 3\alpha\ \Delta\theta = 3(0.15\%) = 0.45\%,
\end{aligned}
$$

where r, t, and l are radius, thickness, and length, respectively, while A and V are area and volume, respectively. The percentage of increase of surface area, thickness, and volume are 0.30%, 0.15%, and 0.45%, respectively. The percentage of increase in radius and thickness are the same, as both are linear expansions.

(ii) Density is mass divided by volume,

$$\rho = \frac{m}{V} = mV^{-1}.$$

Using the calculus, this means,

$$\Delta\rho = m(-1)V^{-2}\Delta V = \frac{-m\Delta V}{V^2},$$

$$\frac{\Delta\rho}{\rho} = \frac{\frac{-m\Delta V}{V^2}}{\frac{m}{V}} = -\frac{\Delta V}{V}.$$

The percentage decrease in density is,

$$\frac{\Delta\rho}{\rho} = -\frac{\Delta V}{V} = -0.45\%.$$

(iii) The thermal coefficient of volume expansion is,

$$\gamma = \frac{\Delta V}{V\ \Delta\theta} = \frac{0.45}{100\ \times\ 80\ \text{K}} = 5.6 \times 10^{-5}\,\text{K}^{-1}.$$

(iv) The moment of inertia of the disk about its symmetry axis is,

$$I = \frac{1}{2}mr^2.$$

This means that,

$$\Delta I = mr\Delta r.$$

The percentage change in the moment of inertia of the disk is,

$$\frac{\Delta I}{I} = \frac{mr\Delta r}{\frac{1}{2}mr^2} = 2\frac{\Delta r}{r} = 2(0.15\%) = 0.30\%.$$

Problem 16.9

(a) The string of a simple pendulum is made of a metal wire with thermal coefficient of linear expansion α. If the temperature increases by $\Delta\theta$, what is the fractional change in the period of the oscillation?

(b) A simple pendulum with a brass string ($\alpha = 1.9 \times 10^{-5}\ \text{K}^{-1}$) is accurate at 288 K. If the temperature is 293 K, what is the time error in one day?

Solution

(a) The period of a simple pendulum is,

$$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi g^{-1/2} l^{1/2}.$$

This means that,

$$\Delta T = 2\pi g^{-1/2}\left(\frac{1}{2}\right) l^{-1/2}\Delta l = \pi g^{-1/2} l^{-1/2}\Delta l,$$

$$\frac{\Delta T}{T} = \frac{1}{2}\frac{\Delta l}{l}.$$

But,

$$\frac{\Delta l}{l} = \alpha \Delta\theta.$$

The fractional change in period is,

$$\frac{\Delta T}{T} = \frac{1}{2}\alpha \Delta\theta.$$

The fractional change of the period is proportional to the change in temperature and the thermal coefficient of linear expansion of the string.

(b) Let every oscillation makes the time indicator changes by T second. In a day the accurate pendulum makes,

$$\frac{1 \times 24 \times 60 \times 60}{T} \text{ oscillations},$$

and the time shown by the accurate pendulum is,

$$\frac{1 \times 24 \times 60 \times 60}{T} \times T = 86{,}400 \text{ s}. \tag{16.1}$$

On the other hand, the inaccurate pendulum makes,

$$\frac{1 \times 24 \times 60 \times 60}{T + \Delta T} \text{ oscillations},$$

in a day and the time shown by the inaccurate pendulum is,

$$\frac{1 \times 24 \times 60 \times 60}{T + \Delta T} \times T \text{ seconds}. \tag{16.2}$$

Thus, in one day, the error or lag in time is Eq. (16.1) minus Eq. (16.2),

$$
\begin{aligned}
86{,}400 - 86{,}400\left(\frac{T}{T+\Delta T}\right) &= 86{,}400 - \frac{86{,}400}{1+\frac{\Delta T}{T}} \\
&= 86{,}400 - \frac{86{,}400}{1+\frac{1}{2}\alpha\Delta\theta} \\
&= 86{,}400 - \frac{86{,}400}{1+\frac{1}{2}(1.9\times10^{-5}\ \mathrm{K}^{-1})(293-288)\ \mathrm{K}} \\
&= 4.1\ \mathrm{s}.
\end{aligned}
$$

Here, we have used result of part (a), $\Delta T/T = \alpha\ \Delta\theta/2$, to arrive at the answer.

- wxMaxima codes:

```
(%i3)  fpprintprec:5; alpha:1.9e-5; deltatheta:293-288;
(fpprintprec)      5
(alpha)  1.9*10^-5
(deltatheta)  5
(%i4)  86400 - 86400/(1 + 1/2*alpha*deltatheta);
(%o4)  4.1038
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of α and $\Delta\theta$.

(%i4) Calculate lag in time in one day.

Problem 16.10 The thermal coefficient of mercury's volume expansion is $182 \times 10^{-6}\ \mathrm{K}^{-1}$. Mercury's density at 20.0°C is 13,550 kg m^{-3}. What temperature does the density of 13,420 kg m^{-3} occur?

Solution

The relation between density and temperature is (Eq. 16.18),

$$\frac{\rho}{\rho_0} = \frac{1}{1+\gamma(\theta-\theta_0)}.$$

Substituting the known quantities, we have,

$$\frac{13{,}420\ \mathrm{kg/m^3}}{13{,}550\ \mathrm{kg/m^3}} = \frac{1}{1+182\times10^{-6}\ \mathrm{K}^{-1}(\theta-20.0^{\circ}\mathrm{C})}.$$

Solving this equation gives the temperature as,

$$\theta = 73.2^{\circ}\text{C}.$$

- wxMaxima codes:

```
(%i5)       fpprintprec:5;  ratprint:false;  gamma:182e-6;
rho0:13550; rho:13420;
(fpprintprec)      5
(ratprint)  false
(gamma)  1.82*10^-4
(rho0)  13550
(rho)  13420
(%i7)   solve(rho/rho0 = 1/(1 + gamma*(theta-20)), theta)$
float(%);
(%o7)  [theta=73.225]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of γ, ρ_0, and ρ.

(%i7) Solve $\frac{\rho}{\rho_0} = \frac{1}{1+\gamma(\theta-20)}$ for θ.

Problem 16.11 Given that the period of oscillation of a compound pendulum is,

$$T = 2\pi\sqrt{\frac{I}{Mgd}},$$

where I is the moment of inertia of the pendulum about the pivot, M is the mass of the pendulum, and d is the distance from the pivot to the center of mass.

(a) By pivoting one of the ends of a metal bar of length L_0 and mass M_0, a compound pendulum is formed. What is the compound pendulum's period?
(b) The ambient temperature increases by $\Delta\theta$. The thermal coefficient of linear expansion of the metal is α. Determine the change in the period of the pendulum's oscillation.

Solution

(a) For the oscillating metal bar, $I = M_0{L_0}^2/3$, $d = L_0/2$. The period of oscillation of the compound pendulum is,

$$T_0 = 2\pi\sqrt{\frac{I}{Mgd}} = 2\pi\sqrt{\frac{M_0L_0^2/3}{M_0gL_0/2}} = 2\pi\sqrt{\frac{2L_0}{3g}}.$$

(b) When temperature is raised by $\Delta\theta$, the length of the pendulum becomes,

$$L_0(1+\alpha\ \Delta\theta).$$

The period of oscillation becomes,

$$\begin{aligned} T &= 2\pi\sqrt{\frac{2L_0(1+\alpha\ \Delta\theta)}{3g}} = 2\pi\sqrt{\frac{2L_0}{3g}}(1+\alpha\ \Delta\theta)^{1/2} \\ &= 2\pi\sqrt{\frac{2L_0}{3g}}\left(1+\frac{\alpha\Delta\theta}{2}+\cdots\right). \end{aligned}$$

The increase in the period of oscillation is,

$$\begin{aligned} T - T_0 &= 2\pi\sqrt{\frac{2L_0}{3g}}\left(1+\frac{\alpha\ \Delta\theta}{2}+\cdots\right) - 2\pi\sqrt{\frac{2L_0}{3g}} \\ &= 2\pi\sqrt{\frac{2L_0}{3g}}\frac{\alpha\ \Delta\theta}{2}+\cdots \\ &\approx T_0\frac{\alpha\ \Delta\theta}{2}. \end{aligned}$$

16.3 Summary

- Temperature is degree of hotness. Temperature is proportional to the average kinetic energy of atoms and molecules in a system.
- The zeroth law of thermodynamics states that if body A and body B separately are in thermal equilibrium with body C, then bodies A and B are in thermal equilibrium as well.
- The three main temperature scales are Celsius, Fahrenheit, and Kelvin.
- Thermal expansion is the increase in length, area, or volume of a body due to in increase in temperature.

16.4 Exercises

Exercise 16.1 Convert 68°F and 5°F to Celsius and Kelvin scales.

(Answer: 20°C, 293 K and −15°C, 258 K)

Exercise 16.2 Compute the increase in length of 50 m copper wire when its temperature changes from 20 to 40°C. The thermal coefficient of linear expansion of copper is 1.7×10^{-5} K^{-1}.

(Answer: 0.017 m)

Exercise 16.3 A steel wire of 2.0 mm^2 cross section is attached to two firm tie points a distance of 1.5 m apart at 25°C. If the temperature drops to −5°C, what is the tension in the wire? The coefficient of linear expansion of steel is 1.5×10^{-5} K^{-1} and the Young's modulus of steel is 2.0×10^{11} Pa.

(Answer: 180 N)

Exercise 16.4 A glass bottle holds 50,000 mm^3 at 25°C. What is its capacity at 35°C? The coefficient of linear expansion of glass is 8.3×10^{-6} K^{-1}.

(Answer: 50,012 mm^3)

Exercise 16.5 The density of gold at 20°C is 19.30 g cm^{-3}. Compute the density of gold at 70°C. The coefficient of linear expansion of gold is 1.4×10^{-5} K^{-1}.

(Answer: 19.26 g cm^{-3})

Chapter 17
Heat and Calorimetry

17.1 Basic Concepts and Formulae

(1) Heat flow is a type of heat transfer caused by temperature differences. The internal energy of a body, which represents its state function, increases as its temperature rises.

(2) A calorie (cal) is the amount of heat required to raise the temperature of 1 g of water from 14.5 to 15.5°C. The equivalence of heat and mechanical energy is,

$$1\,\text{cal} = 4.186\,\text{J}. \tag{17.1}$$

(3) The heat capacity C of a material is the amount of heat required to raise the material's temperature by one degree Celsius. The specific heat capacity or specific heat c is the material's heat capacity per unit of mass.

$$c = \frac{C}{m}. \tag{17.2}$$

The SI units of C and c are J K^{-1} and J kg^{-1} K^{-1}, respectively.
Heat Q needed to raise the temperature of a material by $\Delta\theta$ is,

$$Q = mc\Delta\theta. \tag{17.3}$$

(4) Heat Q needed to change the phase of a material of mass m at a constant temperature is,

$$Q = mL, \tag{17.4}$$

where L is the specific latent heat of the material. The SI unit of L is J kg^{-1}.

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17.2 Problems and Solutions

Problem 17.1 Calculate the heat calories needed to raise the temperature of 5.0 kg aluminum from 25 to 50°C. The specific heat capacity of aluminum is 0.215 cal g^{-1} K^{-1}.

Solution

The heat needed is (Eq. 17.3),

$$\begin{aligned} Q &= mc\Delta\theta \\ &= \left(5.0 \times 10^3\ \text{g}\right)\left(0.215\ \text{cal g}^{-1}\ \text{K}^{-1}\right)(50-25)\ \text{K} \\ &= 26{,}875\ \text{cal}. \end{aligned}$$

- wxMaxima codes:

```
(%i2)m:5e3; c:0.215;
(m)5000.0
(c)0.215
(%i3)Q: m*c*(50-25);
(Q) 26875.0
```

Comments on the codes:

(%i2) Assign values of m and c.
(%i3) Calculate Q.

Problem 17.2 Twenty grams of water at 10°C is mixed with 150 g of water at 90°C. What is the equilibrium temperature of the mixture?

Solution

Let the equilibrium temperature of the mixture be θ. Applying the conservation of energy, and using the formula $Q = mc\Delta\theta$ (Eq. 17.3), we write,

$$\begin{gathered} \text{heat received by 20 g of water with temperature rise from 10°C to } \theta \\ = \\ \text{heat released by 150 g of water with temperature fall from 90°C to } \theta, \end{gathered}$$

$$(20\ \text{g})(1\ \text{cal g}^{-1}\ ^\circ\text{C}^{-1})(\theta - 10)^\circ\text{C} = (150\ \text{g})(1\ \text{cal g}^{-1}\ ^\circ\text{C}^{-1})(90-\theta)^\circ\text{C}.$$

Solving the equation gives the temperature. The equilibrium temperature of the water mixture is,

$$\theta = 80.6^\circ\text{C}.$$

- wxMaxima codes:

```
(%i2)fpprintprec:5; ratprint:false;
(fpprintprec)5
(ratprint)false
(%i4)solve(20*1*(theta-10) = 150*1*(90-theta), theta)$
float(%);
(%o4)[theta=80.588]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $20(1)(\theta - 10) = 150(1)(90 - \theta)$ for θ.

Problem 17.3 An aluminum container of 0.30 kg mass has 0.10 kg of water at 20°C in it. Water of mass 0.20 kg at 100°C is added into the container. What is the final temperature of the mixture? Specific heat capacity of aluminum is 900 J kg^{-1} K^{-1}, specific heat capacity of water is 4190 J kg^{-1} K^{-1}.

Solution

Let the final temperature be θ. Applying the conservation of energy and using the formula $Q = mc\Delta\theta$ (Eq. 17.3), we have,

$$\begin{aligned} &\text{heat absorbed by the aluminum container and the 0.10 kg of water} \\ &\quad = \text{heat released by the 0.20 kg of water,} \\ &\left[(0.30\,\text{kg})\left(900\,\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}\right)(\theta - 20)\,\text{K}\right. \\ &\left.+(0.10\,\text{kg})\left(4190\,\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}\right)(\theta - 20)\,\text{K}\right] \\ &\quad = (0.20\,\text{kg})\left(4190\,\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}\right)(100 - \theta). \end{aligned}$$

Solving the equation gives the final temperature of the mixture.

$$\theta = 64^{\circ}\text{C}.$$

- wxMaxima codes:

```
(%i2)fpprintprec:5; ratprint:false;
(fpprintprec)5
(ratprint)false
(%i4)solve(0.3*900*(theta-20) + 0.1*4190*(theta-20) =
0.2*4190*
(100-theta), theta)$ float(%);
(%o4)[theta = 63.903]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $0.3(900)(\theta - 20) + 0.1(4190)(\theta - 20) = 0.2(4190)(100 - \theta)$ for θ.

Problem 17.4 A bullet of 30 g mass at a speed of 0.25 km s^{-1} hits a target and gets embedded in the target.

(a) What are the changes in energy?
(b) What is the increase in bullet and target internal energies?
(c) Calculate the increase in bullet temperature if 75% of the bullet's internal energy is absorbed. The specific heat capacity of the bullet is 0.13 kJ kg^{-1} K^{-1}.

Solution

(a) A bullet in motion has kinetic energy. Upon hitting the target, the majority of kinetic energy is transformed into heat energy.
(b) Assuming that all of the bullet's kinetic energy is converted to the internal energy of the bullet and the target, the increase in the internal energy of the bullet and the target is,

$$\frac{1}{2}mv^2 = \frac{1}{2}(30 \times 10^{-3}\,\text{kg})(0.25 \times 10^3\,\text{m/s})^2 = 938\,\text{J}.$$

(c) The heat absorbed by the bullet heats the bullet, and we write,

$$\text{heat absorbed by the bullet} = mc\Delta\theta,$$
$$75\% \times 938\,\text{J} = \left(30 \times 10^{-3}\,\text{kg}\right)\left(0.13 \times 10^3\,\text{J kg}^{-1}\,\text{K}^{-1}\right)\Delta\theta.$$

The increase in temperature of the bullet is,

$$\Delta\theta = 180\,\text{K}.$$

- wxMaxima codes:

```
(%i5)fpprintprec:5;  ratprint:false;  m:30e-3;  v:0.25e3;
c:0.13e3;
(fpprintprec)5
(ratprint)false
(m)0.03
(v)250.0
(c)130.0
(%i6)kinetic_energy: 0.5*m*v^2;
(kinetic_energy)937.5
```

```
(%i8) solve(0.75*kinetic_energy = m*c*deltatheta, delta-
theta)$ float(%);
(%o8) [deltatheta = 180.29]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of m, v, and c.
(%i6) Calculate kinetic energy.
(%i8) Solve 0.75 × kinetic energy $= m \times c \times \Delta\theta$ for $\Delta\theta$.

Problem 17.5 A 0.30 kg mass copper container contains 0.45 kg of water at temperature 20°C. A 1.0 kg mass hot metal block at 100°C is submerged into the container. The final temperature of the system is 40°C. Calculate specific heat capacity of the metal. The specific heat capacities of water and copper are 4200 J kg^{-1} K^{-1} and 390 J kg^{-1} K^{-1}, respectively.

Solution

Figure 17.1 shows the situations described by the problem.

Applying the conservation of energy and using the formula $Q = mc\Delta\theta$ (Eq. 17.3), we write,

$$\begin{aligned}
\text{heat released by the metal block} &= \text{heat absorbed by water and the copper container,}\\
m_{metal}c_{metal}(100-40)\,\text{K} &= (m_{water}c_{water} + m_{copper}c_{copper})(40-20)\,\text{K},\\
(1.0\,\text{kg})c_{metal}(100-40)\,\text{K} &= (0.45\,\text{kg} \times 4200\,\text{J kg}^{-1}\,\text{K}^{-1}\\
&\quad + 0.30\,\text{kg} \times 390\,\text{J kg}^{-1}\,\text{K}^{-1})(40-20)\,\text{K}.
\end{aligned}$$

The specific heat capacity of the metal is obtained by solving this equation. The specific heat capacity is,

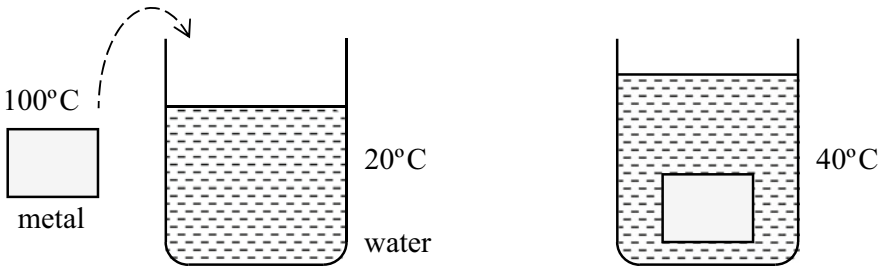

Fig. 17.1 A hot metal block is submerged into water in a copper container, Problem 17.5

$$c_{metal} = 670\,\mathrm{J\,kg^{-1}\,K^{-1}}.$$

- wxMaxima codes:

```
(%i7)fpprintprec:5;     ratprint:false;     mcopper:0.3;
mwater:0.45; mmetal:1; cwater:4200; ccopper:390;
(fpprintprec)5
(ratprint)false
(mcopper)0.3
(mwater)0.45
(mmetal)1
(cwater)4200
(ccopper)390
(%i9)solve(mmetal*cmetal*(100-40)  =  (mwater*cwater  +
mcopper*ccopper)*
(40-20), cmetal)$ float(%);
(%o9)[cmetal = 669.0]
```

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of m_{copper}, m_{water}, m_{metal}, c_{water} and c_{copper}.
(%i9) Solve $m_{metal}c_{metal}(100-40) = (m_{water}c_{water} + m_{copper}c_{copper})(40-20)$ for c_{metal}.

Problem 17.6 A Nernst calorimeter used to determine the specific heat capacity of a 0.30 kg metal block is depicted in Fig. 17.2. The experimental results are tabulated in Table 17.1. Determine (a) the specific heat capacity of the metal, and (b) the heat power loss.

Solution

(a) For the Nernst calorimeter, conservation of energy implies that,

$$\begin{gathered}\text{electric energy supplied} = \text{heat absorbed by the metal block} + \text{heat loss},\\ VIt = mc\Delta\theta + h.\end{gathered}$$

Here, V is the voltage across the calorimeter, I is the electric current through the calorimeter, and t is the time. Thus, VIt is the electric energy supplied.
In Experiment 1,

$$(2.0\,\mathrm{V})(3.0\,\mathrm{A})(22\,\mathrm{s}) = (0.30\,\mathrm{kg})c(1.20\,\mathrm{K}) + h.$$

In Experiment 2,

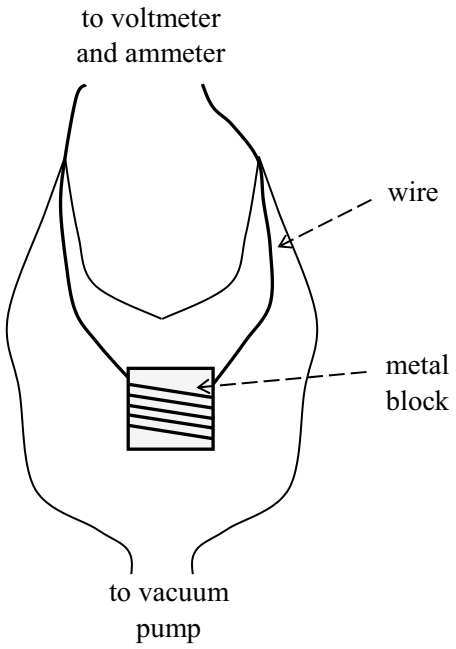

Fig. 17.2 A Nernst calorimeter, Problem 17.6

Table 17.1 Data of Nernst calorimeter experiments

	Voltage, V (V)	Current, I (A)	Temperature rise, $\Delta\theta$ (K)	Time, t (s)
Experiment 1	2.0	3.0	1.20	22
Experiment 2	1.8	2.7	0.97	22

$$(1.8\,\mathrm{V})(2.7\,\mathrm{A})(22\,\mathrm{s}) = (0.30\,\mathrm{kg})c(0.97\,\mathrm{K}) + h.$$

We assume the heat losses h in both experiments are the same. By solving the two equations, the specific heat capacity of the metal block and the heat loss are,

$$c = 364\,\mathrm{J\,kg^{-1}\,K^{-1}},$$

$$h = 1.15\,\mathrm{J}.$$

(b) The power loss of heat is,

$$\text{heat power loss} = \frac{\text{heat loss}}{\text{time}} = \frac{h}{\text{time}} = \frac{1.15\,\text{J}}{22\,\text{s}} = 0.05\,\text{W}.$$

- wxMaxima codes:

```
(%i2)fpprintprec:5; ratprint:false;
(fpprintprec)5
(ratprint)false
(%i4)solve([2*3*22 = 0.3*c*1.2 + h, 1.8*2.7*22 = 0.3*c*0.97
+ h], [c,h])$ float(%);
(%o4)[[c = 363.48,h = 1.1478]]
(%i5)h: rhs(%[1][2]);
(h) 1.1478
(%i6)power_loss: h/22;
(power_loss) 0.052174
```

Comments on the codes:

(%i4) Solve $(2)(3)(22) = (0.3)c(1.2) + h$ and $(1.8)(2.7)(22) = (0.3)c(0.97) + h$ for c and h.
(%o4) The solutions.
(%i5) Assign value of h.
(%i6) Calculate power loss.

Problem 17.7 A Callendar and Barnes continuous flow method is used to determine the specific heat capacity of a liquid. Data of such measurements are tabulated in Table 17.2. Calculate the specific heat capacity of the flowing liquid.

Solution

In the Callendar and Barnes' apparatus, a fluid flowing at a constant rate is heated electrically by a spiral resistance carrying a steady electric current. The heat supplied by the resistance is carried away by the flowing liquid. We have,

Here, V is the voltage across the resistance in volt, I is the current through it in ampere, and VI is the rate of electric energy converted to heat. The heat is absorbed by the fluid at a rate of $mc\ \Delta\theta/t$, where m/t is the rate of fluid flow in kg s^{-1}, $\Delta\theta$ is the temperature rise in K, c is the specific heat of the fluid, and H is the rate of heat loss from the apparatus.For the two measurements we write,

Table 17.2 Data of Callendar and Barnes continuous flow measurements

	Voltage, V (V)	Current, I (A)	Flow rate of liquid (g s^{-1})	Increase in temperature, $\Delta\theta$ (K)
Measurement 1	2.4	2.6	2.8	2.0
Measurement 2	1.2	1.9	1.0	2.0

In Measurement 1,

$$(2.4\,\mathrm{V})(2.6\,\mathrm{A}) = \left(2.8 \times 10^{-3}\,\mathrm{kg/s}\right)c(2.0\,\mathrm{K}) + H. \tag{17.1}$$

In Measurement 2,

$$(1.2\,\mathrm{V})(1.9\,\mathrm{A}) = \left(1.0 \times 10^{-3}\,\mathrm{kg/s}\right)c(2.0\,\mathrm{K}) + H. \tag{17.2}$$

Here, we assumed the rates of heat loss in the two measurements are the same. The specific heat capacity of the flowing liquid is obtained by solving the two equations,

$$c = 1100\,\mathrm{J\,kg^{-1}\,K^{-1}}.$$

- wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec) 5
(ratprint) false
(%i4) solve([2.4*2.6 = 2.8e-3*c*2 + H, 1.2*1.9 = 1e-3*c*2 +
H], [c,H])$ float(%);
(%o4) [[c = 1100.0,H = 0.08]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve Eqs. (17.1) and (17.2) for c and H.

Problem 17.8 A liter of water at 25°C is to be cooled to 10°C. How much ice at 0°C needs to be mixed with the water to achieve this? Ice has a specific latent heat of fusion of 334,000 J $\mathrm{kg^{-1}}$ and water has a specific heat capacity of 4190 J $\mathrm{kg^{-1}\,K^{-1}}$.

Solution

Apply the conservation of energy. Let the mass of ice be m. We write,

$$\begin{aligned} &\text{heat absorbed by the ice} \;=\; \text{heat released from 1 L of water,} \\ &mL + mc_{\mathrm{water}}\,(10-0)\,\mathrm{K} = m_{\mathrm{water}}\,c_{\mathrm{water}}\,(25-10)\,\mathrm{K}. \end{aligned}$$

The mass of the ice is,

$$m = \frac{m_{water} c_{water} (25 - 10)\,\mathrm{K}}{L + c_{water}(10 - 0)\,\mathrm{K}} = \frac{(1.0\,\mathrm{kg})(4190\,\mathrm{J\,kg^{-1}\,K^{-1}})(15\,\mathrm{K})}{334{,}000\,\mathrm{J/kg} + (4190\,\mathrm{J\,kg^{-1}\,K^{-1}})(10\,\mathrm{K})}$$
$$= 0.17\,\mathrm{kg}.$$

- wxMaxima codes:

```
(%i5)fpprintprec:5; ratprint:false; mwater:1; L:334000;
cwater:4190;
(fpprintprec)5
(ratprint)false
(mwater) 1
(l) 334000
(cwater) 4190
(%i7)solve(m*L + m*cwater*(10-0) = mwater*cwater*(25-10),
m)$ float(%);
(%o7)[m = 0.1672]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of m_{water}, L, and c_{water}.
(%i7) Solve $mL + mc_{water}(10 - 0) = m_{water} c_{water}(25 - 10)$ for m.

Problem 17.9 In an experiment, 6.0 g of steam at 100°C is mixed with 80 g of water at 14°C in a copper calorimeter of 150 g mass. What is the final temperature? Specific latent heat of vaporization of water is $2.3 \times 10^6\,\mathrm{J\,kg^{-1}}$, specific heat capacity of water is 4200 J $\mathrm{kg^{-1}\,K^{-1}}$, and specific heat capacity of copper is 400 J $\mathrm{kg^{-1}\,K^{-1}}$.

Solution

We apply the conservation of energy. Let the final temperature be θ. We have,

$$\begin{aligned}
\text{heat released by the steam} &= \text{heat absorbed by the calorimeter and water,}\\
m_{\text{steam}}\, L + m_{\text{steam}}\, c_{\text{water}}\,(100 - \theta) &= \left(m_{\text{copper}}\, c_{\text{copper}} + m_{\text{water}}\, c_{\text{water}}\right)(\theta - 14),\\
\left(6.0 \times 10^{-3}\,\mathrm{kg}\right)\left(2.3 \times 10^{6}\,\mathrm{J/kg}\right) & \\
+\left(6.0 \times 10^{-3}\,\mathrm{kg}\right)\left(4200\,\mathrm{J\,kg^{-1}\,K^{-1}}\right)(100 - \theta) &= \begin{array}{c}\left[\left(150 \times 10^{-3}\,\mathrm{kg}\right)\left(400\,\mathrm{J\,kg^{-1}\,K^{-1}}\right)\right.\\ \left.+\left(80 \times 10^{-3}\,\mathrm{kg}\right)\left(4200\,\mathrm{J\,kg^{-1}\,K^{-1}}\right)\right](\theta - 14).\end{array}
\end{aligned}$$

By solving the equation for θ, the final temperature of the water is,

$$\theta = 52^{\circ}\mathrm{C}.$$

- wxMaxima codes:

```
(%i8)fpprintprec:5;     ratprint:false;     msteam:6e-3;
mwater:80e-3;   mcopper:150e-3;   L:2.3e6;   cwater:4200;
ccopper:400;
(fpprintprec)5
(ratprint)false
(msteam)0.006
(mwater)0.08
(mcopper)0.15
(L)2.3*10^6
(cwater)4200
(ccopper)400
(%i10)  solve(msteam*L  +  msteam*cwater*(100-theta)  =
(mcopper*ccopper  +  mwater*cwater)*(theta-14),  theta)$
float(%);
(%o10) [theta = 51.909]
```

Comments on the codes:

(%i8) Set floating point print precision to 5, internal rational number print to false, assign values of m_{steam}, m_{water}, m_{copper}, L, c_{water}, and c_{copper}.
(%i10) Solve $m_{steam}L + m_{steam}c_{water}(100 - \theta) = (m_{copper}c_{copper} + m_{water}c_{water})(\theta - 14)$ for θ.

Problem 17.10 A 400 g mass liquid at 32°C is poured into a 1.0 kg mass copper container. A heater supplies a constant rate of heat to the container. It takes 5.0 min to bring the liquid to a boil at its boiling temperature of 172°C. Calculate the time required to evaporate the liquid completely.

Specific latent heat of evaporation of the liquid is 289.8 J g^{-1}, specific heat capacity of the liquid is 1.97 J g^{-1} K^{-1}, and specific heat capacity of copper is 0.42 J g^{-1} K^{-1}.

Solution

Heat required to raise the temperature of the liquid and copper container from 32 to 172°C is,

$$\begin{aligned} & m_{\text{liquid}}\, c_{\text{liquid}}\, \Delta\theta + m_{\text{copper}}\, c_{\text{copper}}\, \Delta\theta \\ &= (400\,\text{g})\left(1.97\,\frac{\text{J}}{\text{g K}}\right)(172-32)\,\text{K} + (1000\,\text{g})\left(0.42\,\frac{\text{J}}{\text{g K}}\right)(172-32)\,\text{K} \\ &= 169{,}120\,\text{J}. \end{aligned}$$

The rate of heat supplied is,

$$\frac{\text{heat}}{\text{time}} = \frac{169{,}120\,\text{J}}{5 \times 60\,\text{s}} = 563.73\,\text{J s}^{-1}.$$

Heat required to evaporate the liquid is,

$$m_{liquid} L_{evaporation} = (400\,\text{g})\left(289.8\,\frac{\text{J}}{\text{g}}\right) = 115{,}920\,\text{J}.$$

The time required to evaporate the liquid is,

$$\frac{\text{heat}}{\text{rate of heat supplied}} = \frac{115{,}920\,\text{J}}{563.73\,\text{J/s}} = 206\,\text{s} = 3.4\,\text{min}.$$

- wxMaxima codes:

```
(%i1)fpprintprec:5;
(fpprintprec)5
(%i2)heat_to_raise_temperature:   400*1.97*(172-32)   +
1000*0.42*(172-32);
(heat_to_raise_temperature) 1.6912*10^5
(%i3)heating_rate: %/(5*60);
(heating_rate) 563.73
(%i4)heat_to_evaporate: 400*289.8;
(heat_to_evaporate) 1.1592*10^5
(%i5)time: %/heating_rate;
(time) 205.63
(%i6)%/60;
(%o6)3.4272
```

Comments on the codes:

(%i2) Calculate heat needed to raise the temperature.
(%i3) Calculate the heating rate.
(%i4) Calculate heat needed to evaporate liquid.
(%i5), (%i6) Calculate time needed to evaporate liquid.

Problem 17.11 An iron fragment from a rocket at $-100°$C enters the earth's atmosphere. It melts as it enters the atmosphere. Assuming that the fragment's kinetic energy is converted into heat, determine the minimum speed at which the fragment enters the atmosphere. Specific heat capacity of iron is 0.11 kcal kg^{-1} K^{-1}, specific latent heat of fusion of iron is 30 kcal kg^{-1}, boiling point of iron is 1535°C, and 1 kcal = 4200 J.

Solution

The kinetic energy of the iron fragment is converted to heat. This implies,

$$\frac{1}{2}mv^2 = ml + mc\Delta\theta,$$

$$\frac{1}{2}mv^2 = m(30 \times 4200\,\mathrm{J/kg}) + m(0.11 \times 4200\,\mathrm{J\,kg^{-1}\,K^{-1}})[1535 - (-100)]\,\mathrm{K},$$
$$v = 1.3 \times 10^3\,\mathrm{m\,s^{-1}},$$

where m is the mass of the fragment, v is its speed, l is specific latent heat of fusion of iron, and c is its specific heat capacity. Multiplication by 4200 is to convert kcal to joule. The minimum speed of the iron fragment is 1.3 km s^{-1}.

- wxMaxima codes:

```
(%i2)fpprintprec:5; ratprint:false;
(fpprintprec)5
(ratprint)false
(%i4)solve(0.5*m*v^2 = m*30*4200 + m*0.11*4200*(1535 +
100), v)$ float(%);
(%o4)[v = -1327.7,v = 1327.7]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $\frac{1}{2}mv^2 = m(30 \times 4200) + m(0.11 \times 4200)(1535 + 100)$ for v.

17.3 Summary

- The heat capacity C of a substance is the ratio of the heat ΔQ supplied to its temperature increase $\Delta\theta$,

$$C = \frac{\Delta Q}{\Delta\theta}.$$

- The specific heat capacity c of a substance of mass m is the heat capacity per unit mass,

$$c = \frac{C}{m} = \frac{\Delta Q}{m\,\Delta\theta}.$$

- Heat Q needed to raise the temperature of a material by $\Delta\theta$ is,

$$Q = mc\,\Delta\theta.$$

- The heat liberated or absorbed by a substance of mass m undergoing a phase change is given by,

$$Q = mL,$$

where L is the specific latent heat.

17.4 Exercises

Exercise 17.1 An electric heater supplies 1.6 kW of power in the form of heat to a tank containing 200 kg of water. What is the time taken to heat the water from 20 to 70°C? Specific heat capacity of water is 4200 J kg^{-1} K^{-1}.
(Answer: 7.3 h)

Exercise 17.2 Water of mass 50 g at 10°C is added to 250 g of water at 80°C. What is the final temperature? Specific heat capacity of water is 4200 J kg^{-1} K^{-1}.
(Answer: 68°C)

Exercise 17.3 How many gram of ice at 0°C must one add to 200 g water at 90°C to cool it to 10°C? Specific heat capacity of water is 4200 J kg^{-1} K^{-1} and latent heat of fusion of ice is 3.34×10^5 J kg^{-1}.
(Answer: 179 g)

Exercise 17.4 Ice of mass 50.0 g of is in an aluminum calorimeter of mass 2.0 g at 0°C. Then, 75.0 g of water at 80°C is poured into the calorimeter. What is the final temperature? Specific heat capacity of water and aluminum are 4200 J kg^{-1} K^{-1} and 900 J kg^{-1} K^{-1}, respectively, and latent heat of fusion of ice is 3.34×10^5 J kg^{-1}.
(Answer: 16°C)

Exercise 17.5 Aluminum of mass 0.3 kg at 20°C is heated to its melting point at 660°C and heated further on until all the aluminum is converted to liquid. How much heat is required? Latent heat of fusion and specific heat capacity of aluminum are 3.96×10^5 J kg^{-1} and 900 J kg^{-1} K^{-1}, respectively.
(Answer: 2.9×10^5 J)

Chapter 18
Heat Transfer

18.1 Basic Concepts and Formulae

(1) Three mechanisms transfer heat from one point to another: conduction, convection, and radiation.

(2) Conduction is the transfer of heat between particles through the exchange of kinetic energies. The rate of heat flow $H = dQ/dt$ by conduction across an area A is,

$$H = -kA\frac{d\theta}{dx}, \tag{18.1}$$

where k is the coefficient of thermal conduction of the material (or the thermal conductivity of the material) and $d\theta/dx$ is the temperature gradient. The SI units of H, k, and $d\theta/dx$ are W, W m^{-1} K^{-1}, and K m^{-1}, respectively.

(3) The steady state rates of heat conduction for three simple geometries are depicted in Table 18.1.

(4) Convection is the transfer of heat through the motion of a hot body from one point to another. Heat loss or absorbed by an area per unit time is,

$$H = hA\Delta\theta, \tag{18.6}$$

where h is the coefficient of convection (unit W m^{-2} K^{-1}), A is the area of the surface, and $\Delta\theta$ is the temperature difference between the surface and the surrounding (such as air or water).

There are two types of convection: free convection and forced convection.

Newton's law of cooling states that in forced convection, the rate of heat loss dQ/dt from a body is proportional to the temperature difference between the body and its surrounding,

$$\frac{dQ}{dt} = -k'(\theta - \theta_S), \tag{18.7}$$

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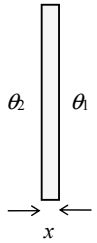
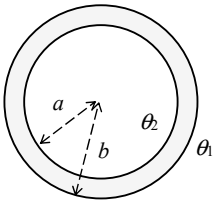
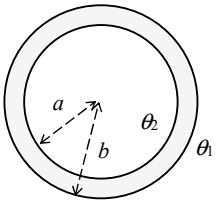
W. M. S. Wan Hassan, *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-42678-0_18

Table 18.1 Rates of heat conduction of three simple configurations

Configuration	Rate of heat conduction
	Plate of thickness x, area A $H = \frac{kA(\theta_2-\theta_1)}{x}, \quad \theta_2 > \theta_1$ A few plates sandwiched together $H = \frac{A(\theta_2-\theta_1)}{\sum_i x_i/k_i}, \quad \theta_2 > \theta_1$
	Cylindrical shell, length L, internal radius a, external radius b $H = \frac{2\pi k(\theta_2-\theta_1)L}{\ln(b/a)}, \quad \theta_2 > \theta_1$
	Spherical shell, internal radius a, external radius b $H = \frac{4\pi k(\theta_2-\theta_1)ab}{b-a}, \quad \theta_2 > \theta_1$

where θ and θ_S are temperatures of the body and surrounding, respectively, and k' is a proportional constant. For a body of mass m, $Q = mc\,\Delta\theta$, and the equation becomes,

$$\frac{d\theta}{dt} = -k''(\theta - \theta_S), \tag{18.8}$$

where $d\theta/dt$ is the rate of temperature change or the rate of cooling of the body, and k'' is a proportional constant.

(5) All bodies radiate and absorb energy of electromagnetic forms. The rate of radiation emission or radiation power P (in W) is given by the Stefan law,

$$P = eA\sigma T^4, \tag{18.9}$$

where A, T, e, and σ are the area, temperature (in K), emissivity of the body, and Stefan-Boltzmann constant, respectively. The Stefan-Boltzmann constant has a value of,

$$\sigma = 5.6696 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}. \tag{18.10}$$

Emissivity of a body has a value between 0 and 1, and that of a black body is $e = 1$.

(6) For a black body, the Wien displacement law states that the wavelength λ_{max} corresponding to the radiation with maximum intensity at temperature T satisfies,

$$\lambda_{\max} T = 2.898 \times 10^{-3}\,\mathrm{m\,K} = \text{constant}. \tag{18.11}$$

18.2 Problems and Solutions

Problem 18.1 A rod is made up of 100 cm of copper section joined to l length of steel section. The rod has a cross-sectional area of 5.0 $\mathrm{cm^2}$. To reduce heat loss, the rod is insulated. The copper end is in contact with boiling water at 100°C, while the steel end is in contact with melting ice at 0°C. The temperature of the copper-steel joint is 60°C in the steady state.

(a) Calculate the rate of heat flow in the rod.
(b) What is the length of the steel section l?

The thermal conductivity of copper is 385 $\mathrm{W\ m^{-1}\ K^{-1}}$ and that of steel is 50.2 $\mathrm{W\ m^{-1}\ K^{-1}}$.

Solution

(a) Figure 18.1 shows the rod made of copper and steel. Assume no heat is lost through the curved surface of the rod.

In steady state, the rate of heat flow in the rod is the same as the rate of heat flow in the copper section of the rod. The rate of heat flow in the rod is (Eq. 18.2),

$$\begin{aligned} H &= k_{\text{copper}}\, A\left(\frac{\theta_2 - \theta_1}{x}\right) \\ &= \left(385\,\mathrm{W\,m^{-1}\,K^{-1}}\right)\left(5.0 \times 10^{-4}\,\mathrm{m^2}\right)\frac{(100-60)\,\mathrm{K}}{100 \times 10^{-2}\,\mathrm{m}} \end{aligned}$$

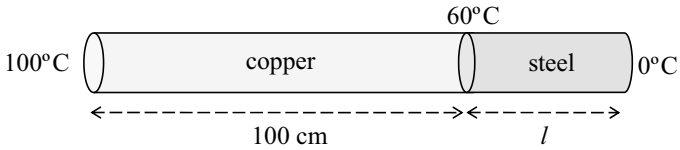


Fig. 18.1 A rod made of copper and steel, Problem 18.1

$$= 7.7\,\text{W}.$$

(b) The rate of heat flow in the steel section is the same as in part (a). We write,

$$\begin{aligned} H &= k_{\text{steel}}\, A\left(\frac{\theta_2' - \theta_1'}{x}\right) \\ &= \left(50.2\,\text{W m}^{-1}\,\text{K}^{-1}\right)\left(5.0 \times 10^{-4}\,\text{m}^2\right)\frac{(60-0)\,\text{K}}{l} \\ &= 7.7\,\text{W}, \end{aligned}$$

where l is the length of the steel section. Solving this equation gives the length of the steel section as,

$$l = 0.20\,\text{m}.$$

- wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec) 5
(ratprint) false
(%i4) solve(50.2*5e-4*60/l = 7.7, l)$ float(%);
(%o4) [l = 0.19558]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $(50.2)(5 \times 10^{-4})\frac{(60-0)}{l} = 7.7$ for l.

Problem 18.2 A boiler is a steel tank mounted on a heater. The steel base measures 1.5 cm thick and has a cross-sectional area of 1500 cm^2. The boiler's water temperature is 100°C, and 750 g of water vaporizes every 5.0 min. Calculate the temperature of the boiler's outer surface that is in contact with the heater. Thermal conductivity of steel is 50.2 W m^{-1} K^{-1} and specific latent heat of vaporization of water is 2.268 $\times$ 10^6 J kg^{-1}.

Solution

Figure 18.2 shows the tank, heater, and the water.

Assume the heat from the heater flows through the base of the boiler. The heat then vaporizes the boiling water. So we write (Eqs. 18.2 and 17.4),

$$\begin{pmatrix}\text{rate of heat flow through}\\ \text{the base of the boiler}\end{pmatrix} = \begin{pmatrix}\text{rate of heat received by the boiling}\\ \text{water to change it into steam,}\end{pmatrix}$$

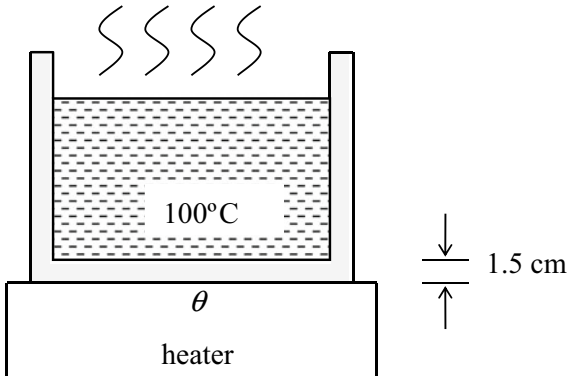


Fig. 18.2 A water boiler, Problem 18.2

$$kA\frac{(\theta-100)}{x}=\frac{mL}{t},$$

$$\left(50.2\,\mathrm{W\,m^{-1}\,K^{-1}}\right)\left(1500\times10^{-4}\,\mathrm{m^2}\right)\frac{(\theta-100)\,\mathrm{K}}{1.5\times10^{-2}\,\mathrm{m}}=\frac{\left(750\times10^{-3}\,\mathrm{kg}\right)\left(2.268\times10^{6}\,\mathrm{J/kg}\right)}{5\times60\,\mathrm{s}}.$$

By solving the last equation, the temperature of the outer surface of the boiler is,

$$\theta=111^{\circ}\mathrm{C}.$$

- wxMaxima codes:

```
(%i8)fpprintprec:5; ratprint:false; x:1.5e-2; A:1500e-4;
m:750e-3; t:5*60; k:50.2; L:2.268e6;
(fpprintprec)5
(ratprint)false
(x)0.015
(a) 0.15
(m)0.75
(t)300
(k)50.2
(L)2.268*10^6
(%i10) solve(k*A*(theta-100)/x = m*L/t, theta)$ float(%);
(%o10) [theta = 111.29]
```

Comments on the codes:

(%i8) Set floating point print precession to 5, internal rational number print to false, assign values of x, A, m, t, k, and L.
(%i10) Solve $kA(\theta-100)/x=mL/t$ for θ.

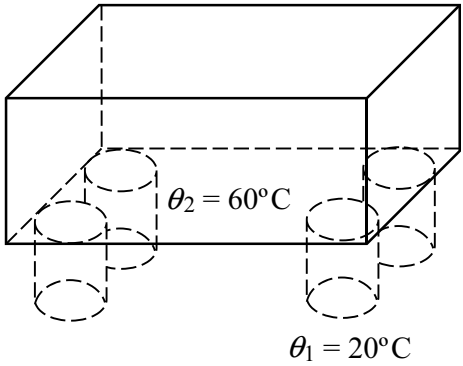


Fig. 18.3 A hot water tank with four legs, Problem 18.3

Problem 18.3 The four legs of a hot water tank are 15 cm in length and 1.25 cm in radius, as shown in Fig. 18.3. The legs are in contact with the floor whose temperature is 20°C. Assume heat can only travel from the legs to the floor.

(a) Calculate the thermal conductivity of the steel legs if 22 W are required to keep the water in the tank at 60°C.
(b) When 1.5 mm of asbestos is placed between the four legs and the floor, only 5.0 W is required to maintain the 60°C temperature of the water in the tank. Determine asbestos' thermal conductivity.

Solution

(a) The heat from the water is transferred by the four legs to the floor. Applying the conservation of energy, at steady state,

$$\begin{array}{c}\text{power supplied to keep the water} \\ \text{at a constant temperature}\end{array} = \text{rate of heat flow through the four legs}$$

This implies (Eq. 18.2), Fig. 18.3,

$$\begin{aligned} P &= H \\ &= k_{\text{steel}}\, A \frac{(\theta_2 - \theta_1)}{x_{\text{steel}}}, \\ 22\,\text{W} &= k_{\text{steel}}\,(4)\pi\left(1.25 \times 10^{-2}\,\text{m}\right)^2 \frac{(60-20)\,\text{K}}{15 \times 10^{-2}\,\text{m}}. \end{aligned}$$

Solving the equation gives the coefficient of thermal conductivity of steel as,

$$k_{\text{steel}} = 42\,\text{W}\,\text{m}^{-1}\,\text{K}^{-1}.$$

(b) Figure 18.4 shows the tank with the asbestos between the leg and the floor. When the asbestos is in place, we have,

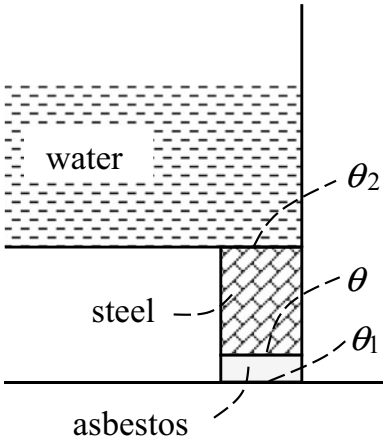


Fig. 18.4 Asbestos between the floor and leg of hot water tank, Problem 18.3

$$\begin{aligned}\text{power supplied to keep the water at a constant temperature} &= \text{rate of heat flow through the four legs} \\ &= \text{rate of heat flow through the asbestos}\end{aligned}$$

This implies (Eq. 18.2),

$$P' = k_{\text{steel}}\, A \frac{(\theta_2 - \theta)}{x_{\text{steel}}} = k_{\text{asbestos}}\, A \frac{(\theta - \theta_1)}{x_{\text{asbestos}}}.$$

From these equations, θ can be eliminated and the coefficient of thermal conductivity of asbestos is,

$$\begin{aligned} k_{\text{asbestos}} &= \frac{P' x_{\text{asbestos}}}{A\left(\theta_2 - \theta_1 - \frac{P' x_{\text{steel}}}{A k_{\text{steel}}}\right)} \\ &= \frac{5.0\,\text{W}\left(1.5 \times 10^{-3}\,\text{m}\right)}{4\pi\left(1.25 \times 10^{-2}\,\text{m}\right)^2 \left(60^\circ\text{C} - 20^\circ\text{C} - \frac{5.0\,\text{W}\left(15 \times 10^{-2}\,\text{m}\right)}{4\pi\left(1.25 \times 10^{-2}\,\text{m}\right)^2 \left(42.02\,\text{W m}^{-1}\,\text{K}^{-1}\right)}\right)} \\ &= 0.12\,\text{W m}^{-1}\,\text{K}^{-1}. \end{aligned}$$

- wxMaxima codes:

```
(%i5) fpprintprec:5;          ratprint:false;          P:22;
A:float(4*%pi*(1.25e-2)^2); xsteel:15e-2;
(fpprintprec) 5
(ratprint) false
(P) 22
```

```
(a) 0.0019635
(xsteel) 0.15
(%i7)solve(P=ksteel*A*(60-20)/xsteel,ksteel)$float(%);
(%o7)[ksteel=42.017]
(%i10) ksteel: 42.017; Pprime: 5; xasbestos: 1.5e-3;
(ksteel) 42.017
(Pprime) 5
(xasbestos)0.0015
(%i12) solve([Pprime=ksteel*A*(60-theta)/xsteel,
Pprime       =       kasbestos*A*(theta-20)/xasbestos],
[kasbestos,theta])$ float(%);
(%o12) [[kasbestos=0.12358,theta=50.909]]
```

Comments on the codes:

(%i5) Set floating point print precession to 5, internal rational number print to false, assign values of P, A, and x_{steel}.
(%i7) Solve $P = k_{steel} A \frac{(\theta_2 - \theta_1)}{x_{steel}}$ for k_{steel}.
(%i10) Assign values of k_{steel}, P', $x_{asbestos}$.
(%i12) Solve $P' = k_{steel} A \frac{(\theta_2 - \theta)}{x_{steel}}$ and $P' = k_{asbestos} A \frac{(\theta - \theta_1)}{x_{asbestos}}$ for $k_{asbestos}$ and θ.

Problem 18.4 The inner surface of a window of a house is at 20°C while the outer surface is at 5.0°C. Compare the rates of heat loss through,

(a) a single pane window of 5.0 mm thick glass, and
(b) a double pane window. The double pane window has two glasses each of 2.5 mm thick separated by 5.0 mm thick air.

Thermal conductivities of glass and air are 1.0 W m^{-1} K^{-1} and 0.025 W m^{-1} K^{-1}, respectively.

Solution

Figure 18.5 shows the windows considered in this problem, (a) is the window with one sheet of glass, and (b) is the window with two sheets of glass.

The rate of heat loss through window (a) is (Eq. 18.2),

$$P_1 = \frac{A(\theta_2 - \theta_1)}{\frac{x_{glass1}}{k_{glass}}}.$$

The rate of heat loss through window (b) is (Eq. 18.3),

$$P_2 = \frac{A(\theta_2 - \theta_1)}{\frac{x_{glass2}}{k_{glass}} + \frac{x_{air}}{k_{air}} + \frac{x_{glass2}}{k_{glass}}},$$

as the heat has to flow through a combination of glass-air-glass sheet. Comparing the two rates of heat loss,

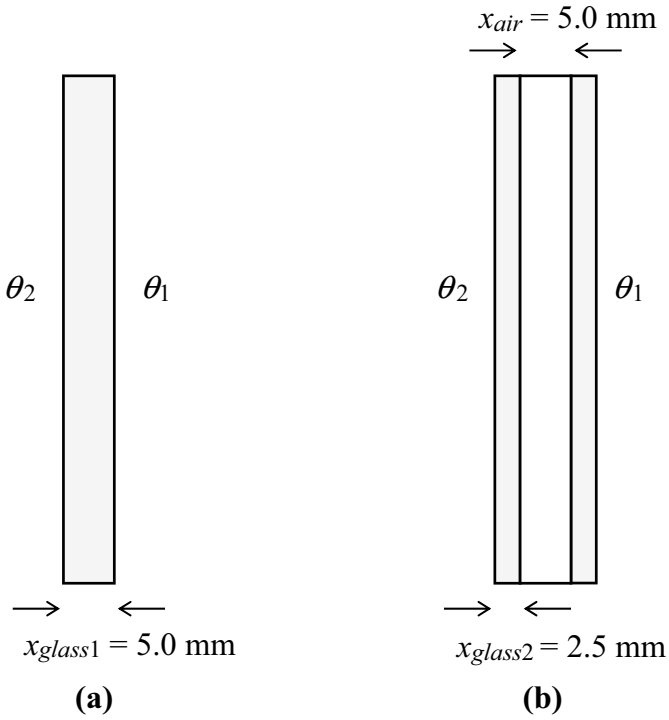


Fig. 18.5 A single (**a**) and double pane (**b**) windows, Problem 18.4

$$\begin{aligned}\frac{P_1}{P_2} &= \frac{\frac{x_{glass2}}{k_{glass}} + \frac{x_{air}}{k_{air}} + \frac{x_{glass2}}{k_{glass}}}{\frac{x_{glass1}}{k_{glass}}} \\ &= \frac{\frac{2.5\times10^{-3}\,\mathrm{m}}{1.0\,\mathrm{W\,m^{-1}\,K^{-1}}} + \frac{5.0\times10^{-3}\,\mathrm{m}}{0.025\,\mathrm{W\,m^{-1}\,K^{-1}}} + \frac{2.5\times10^{-3}\,\mathrm{m}}{1.0\,\mathrm{W\,m^{-1}\,K^{-1}}}}{\frac{5.0\times10^{-3}\,\mathrm{m}}{1.0\,\mathrm{W\,m^{-1}\,K^{-1}}}} \\ &= 41.\end{aligned}$$

The heat loss through window (a) is 41 times that of window (b). Window (b) is better in reducing heat loss.

- wxMaxima codes:

```
(%i1) (2.5e-3 + 5e-3/0.025 + 2.5e-3)/5e-3;
(%o1) 41.0
```

Comments on the codes:

(%i1) Calculate P_1/P_2.

Alternative calculation:

```
(%i8)fpprintprec:5;  theta2:20;  theta1:5;  xglass1:5e-3;
xglass2:2.5e-3; xair:5e-3; kglass:1; kair:0.025;
(fpprintprec)5
(theta2)20
(theta1)5
(xglass1)0.005
(xglass2)0.0025
(xair)0.005
(kglass)1
(kair)0.025
(%i9)P1: A*(theta2-theta1)/(xglass1/kglass);
(P1) 3000.0*a
(%i10) P2: A*(theta2-theta1)/(xglass2/kglass + xair/kair +
xglass2/kglass);
(P2) 73.171*a
(%i11) P1/P2;
(%o11) 41.0
```

Comments on the codes;

(%i8) Set floating point print precision to 5, assign values of θ_2, θ_1, x_{glass1}, x_{glass2}, x_{air}, k_{glass}, and k_{air}.
(%i9), (%i10), (%i11) Calculate P_1, P_2, and P_1/P_2.

Problem 18.5 The heat capacity of two identical containers is 12 J K^{-1}. One container contains 8.0×10^{-5} m^3 of water, while the other contains the same volume of a liquid. Each container is heated and then allowed to cool. The times 150 s and 50 s are required for a container filled with water and liquid to cool from 325 to 320 K, respectively. Calculate the specific heat capacity of the liquid. Specific heat capacity of water is 4190 J kg^{-1} K^{-1}, density of water is 1000 kg m^{-3}, and density of the liquid is 800 kg m^{-3}.

Solution
From $\Delta Q = mc\ \Delta\theta$ (Eq. 17.3), we write $\Delta Q/\Delta t = mc\ \Delta\theta/\Delta t$, where $\Delta Q/\Delta t$ is the rate of heat loss and $\Delta\theta/\Delta t$ is the rate of temperature fall. The physical conditions of both containers are the same and we assume the rates of the heat loss from both are the same. Thus,

$$\begin{gathered}\text{rate of heat loss from container with water} = \text{rate of heat loss from container with liquid,}\\ (C_{container} + m_{water}c_{water})\left(\frac{\Delta\theta}{\Delta t}\right)_{water} = (C_{container} + m_{liquid}c_{liquid})\left(\frac{\Delta\theta}{\Delta t}\right)_{liquid},\\ (C_{container} + \rho_{water}V_{water}c_{water})\left(\frac{\Delta\theta}{\Delta t}\right)_{water} = (C_{container} + \rho_{liquid}V_{liquid}c_{liquid})\left(\frac{\Delta\theta}{\Delta t}\right)_{liquid}.\end{gathered}$$

Substituting known values into the equation gives,

$$[12\,\mathrm{J\,K^{-1}} + (1000\,\mathrm{kg\,m^{-3}})(8 \times 10^{-5}\,\mathrm{m^3})(4190\,\mathrm{J\,kg^{-1}\,K^{-1}})]\frac{(325-320)\,\mathrm{K}}{150\,\mathrm{s}}$$
$$= [12\,\mathrm{J\,K^{-1}} + (800\,\mathrm{kg\,m^{-3}})(8 \times 10^{-5}\,\mathrm{m^3})c_{liquid}]\frac{(325-320)\,\mathrm{K}}{50\,\mathrm{s}}. \qquad (18.1)$$

Solving Eq. (18.1) gives the specific heat capacity of the liquid,

$$c_{\mathrm{liquid}} = 1620\,\mathrm{J\,kg^{-1}\,K^{-1}}.$$

- wxMaxima codes:

```
(%i2)fpprintprec:5; ratprint:false;
(fpprintprec)5
(ratprint)false
(%i4)solve((12 + 1000*8e-5*4190)*(325-320)/150 = (12 +
800*8e-5 *cliquid)*(325-320)/50, cliquid)$ float(%);
(%o4)[cliquid = 1620.8]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve Eq. (18.1) for c_{liquid}.

Problem 18.6

(a) As heat is taken out of the water in a pond at 0°C, a layer of ice is formed. If the ice layer is of thickness x at time t, show that the rate of increase in thickness is,

$$\frac{dx}{dt} = \frac{k\,\Delta\theta}{\rho L x},$$

where k is the thermal conductivity of ice, ρ is the density of ice, L is the specific latent heat of fusion of ice, and $\Delta\theta$ is the temperature difference of the air and the water in the pond.

(b) What is the increase in ice thickness over 12 h, if the initial thickness is 80 mm? Thermal conductivity of ice is 2.2 W m^{-1} K^{-1}, the temperature difference of the air and water in the pond is 10 K, the density of the ice is 9.2×10^2 kg m^{-3}, and the specific latent heat of fusion of ice is 3.3×10^5 J kg^{-1}.

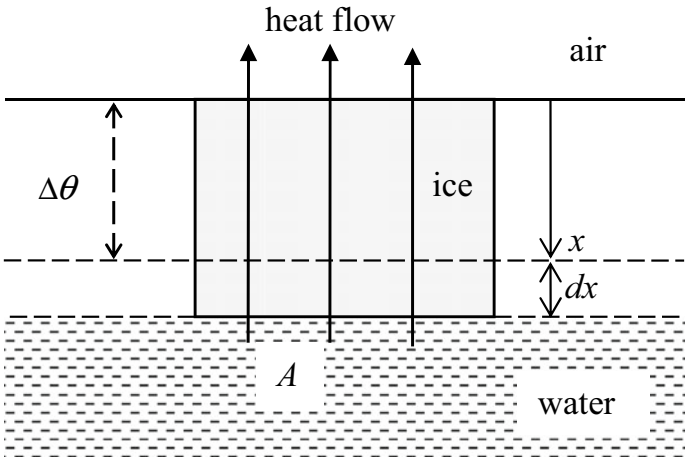


Fig. 18.6 Heat flow from the water, through the ice, to the air, Problem 18.6

Solution

(a) Figure 18.6 depicts the air, ice, and water layers. The heat travels from the water, through the ice, into the air layers. Ice layer thickness is x, while elemental thickness is dx.

The rate of heat loss from a layer of ice of thickness x is (Eq. 18.1),

$$H = kA\frac{\Delta\theta}{x},$$

where A and $\Delta\theta/x$ are the area and magnitude of the temperature gradient of the ice layer, respectively, and k is the thermal conductivity of ice.

The rate of ice formation is (Eq. 17.4),

$$L\frac{dm}{dt} = LA\rho\frac{dx}{dt},$$

where L is the specific latent heat of fusion of ice while m and ρ are the mass and density of ice, respectively. We assume that the rate of heat loss is equal to the rate of ice formation,

$$kA\frac{\Delta\theta}{x} = LA\rho\frac{dx}{dt}.$$

Thus, the thickness of the ice layer increases at the rate of,

$$\frac{dx}{dt} = \frac{k\ \Delta\theta}{\rho Lx}.$$

(b) From equation of part (a), we have,

$$\frac{dx}{dt} = \frac{k\Delta\theta}{\rho L x},$$

$$x\,dx = \frac{k\Delta\theta}{\rho L} dt,$$

$$\int_{h_0}^{h} x\,dx = \frac{k\Delta\theta}{\rho L} \int_{0}^{T} dt,$$

$$\left[\frac{x^2}{2}\right]_{h_0}^{h} = \left[\frac{k\ \Delta\theta}{\rho L} t\right]_{0}^{T},$$

$$\frac{h^2}{2} - \frac{h_0^2}{2} = \frac{k\ \Delta\theta}{\rho L} T,$$

$$\frac{h^2}{2} - \frac{\left(80 \times 10^{-3}\,\mathrm{m}\right)^2}{2} = \frac{\left(2.2\,\mathrm{W\,m^{-1}\,K^{-1}}\right)(10\,\mathrm{K})}{\left(9.2 \times 10^2\,\mathrm{kg\,m^{-3}}\right)\left(3.3 \times 10^5\,\mathrm{J\,kg^{-1}}\right)}(12 \times 60 \times 60\,\mathrm{s}).$$

Solving the last equation for h gives,

$$h = 0.113\,\mathrm{m}.$$

Therefore, the increase in the ice thickness is,

$$113\,\mathrm{mm} - 80\,\mathrm{mm} = 33\,\mathrm{mm}.$$

- wxMaxima codes:

```
(%i2)fpprintprec:5; ratprint:false;
(fpprintprec)5
(ratprint)false
(%i3)integrate(x, x, h0, h);
(%o3)h^2/2-h0^2/2
(%i4)integrate(k*deltatheta/(rho*L), t, 0, T);
(%o4)(T*deltatheta*k)/(L*rho)
(%i10)  h0:80e-3;  T:12*60*60;  k:2.2;  deltatheta:10;
rho:920; L:3.3e5;
(h0)0.08
(t) 43200
(k)2.2
(deltatheta) 10
(rho)920
```

```
(L)3.3*10^5
(%i12) solve(h^2/2 - h0^2/2 = (deltatheta*k*T)/(L*rho), h)$
float(%);
(%o12) [h = -0.11252,h = 0.11252]
(%i13) increase_in_ice_thickness: 113-80;
(increase_in_ice_thickness)33
```

Comments on the codes:

(%i2) Set floating point print precession to 5 and internal rational number print to false.
(%i3), (%i4) Definite integrations $\int_{h_0}^{h} x\ dx$ and $\int_{0}^{T} \frac{k\ \Delta\theta}{\rho L} dt$.
(%i10) Assign values of h_0, T, k, $\Delta\theta$, ρ, and L.
(%i12) Solve $\frac{h^2}{2} - \frac{h_0^2}{2} = \frac{k\ \Delta\theta}{\rho L} T$ for h.
(%i13) Calculate increase in ice thickness.

- Another calculation by wxMaxima:

```
(%i8)fpprintprec:5; ratprint:false; h0:80e-3; T:12*60*60;
k:2.2; deltatheta:10; rho:920; L:3.3e5;
(fpprintprec)5
(ratprint)false
(h0)0.08
(t) 43200
(k)2.2
(deltatheta) 10
(rho)920
(L) 3.3*10^5
(%i9) left: integrate(x, x, h0, h);
(left) h^2/2-2/625
(%i10) right: integrate(k*deltatheta/(rho*L), t, 0, T);
(right) 0.0031304
(%i12) solve(left = right, h)$ float(%);
(%o12) [h = -0.11252,h = 0.11252]
(%i13) h: rhs(%[2]);
(h) 0.11252
(%i14) h-h0;
(%o14) 0.032521
```

Comments on the codes:

(%i8) Set floating point print precision to 5, internal rational number print to false, assign values of h_0, T, k, $\Delta\theta$, ρ, and L.

(%i9), (%i10) Calculate $\int_{h_0}^{h} x\,dx$ and assign as *left*, calculate $\int_{0}^{T} \frac{k\,\Delta\theta}{\rho L} dt$ and assign as *right*.
(%i12) Solve *left* = *right* for *h*.
(%i13) Assign value of *h*.
(%i14) Calculate $h - h_0$.

Problem 18.7 A cup of tea cools down from 66 to 63°C in 1.0 min in a room of temperature 25°C. How long does it take to cool down from 43 to 40°C?

Solution

Applying the Newton's cooling law (Eq. 18.8), we have,

$$\frac{\text{change in temperature}}{\text{time}} \propto \text{(temperature of the body - temperature of surrounding)}$$

In the first case,

$$\frac{66°\text{C} - 63°\text{C}}{1.0\ \text{min}} = k(66°\text{C} - 25°\text{C}).$$

In the second case,

$$\frac{43°\text{C} - 40°\text{C}}{t} = k(43°\text{C} - 25°\text{C}).$$

By solving these two equations, we obtain the time taken to cool down as,

$$t = 2.3\,\text{min}.$$

- wxMaxima codes:

```
(%i2)fpprintprec:5; ratprint:false;
(fpprintprec)5
(ratprint)false
(%i4)solve([(66-63) = k*(66-25), (43-40)/t = k*(43-25)],
[t,k])$ float(%);
(%o4)[[t = 2.2778,k = 0.073171]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $(66 - 63) = k(66 - 25)$ and $(43 - 40)/t = k(43 - 25)$ for t and k.

Alternative solution: Using Eq. (18.8) and the calculus,

$$\begin{aligned}\frac{d\theta}{dt} &= -k(\theta-\theta_s),\\ \int \frac{d\theta}{\theta-\theta_s} &= \int -kdt,\\ \int \frac{d\theta}{\theta-25} &= \int -kdt.\end{aligned}$$

In the first case, the cooling is from 66 to 63°C and the time is 1.0 min,

$$\begin{aligned}\int_{66}^{63} \frac{d\theta}{\theta-25} &= \int_0^1 -kdt,\\ [\ln(\theta-25)]_{66}^{63} &= [-kt]_0^1,\\ \ln(63-25)-\ln(66-25) &= -k,\\ k &= 0.07599\,\text{min}^{-1}.\end{aligned}$$

In the second case, the cooling in from 43 to 40°C and the time is t,

$$\begin{aligned}\int_{43}^{40} \frac{d\theta}{\theta-25} &= \int_0^t -(0.07599\,\text{min}^{-1})dt,\\ [\ln(\theta-25)]_{43}^{40} &= \left[-(0.07599\,\text{min}^{-1})t\right]_0^t,\\ \ln(40-25)-\ln(43-25) &= -(0.07599\,\text{min}^{-1})t.\end{aligned}$$

The time is,

$$t = 2.4\,\text{min}.$$

- wxMaxima codes:

```
(%i2)fpprintprec:5; ratprint:false;
(fpprintprec)5
(ratprint)false
(%i3)log(63-25)-log(66-25) = -k;
(%o3)log(38)-log(41) = -k
(%i4)log(40-25)-log(43-25) = -k*t;
(%o4)log(15)-log(18) = -k*t
(%i6)solve([%o3, %o4], [k, t])$ float(%);
(%o6)[[k = 0.075986,t = 2.3994]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3) Set equation $\ln(63-25)-\ln(66-25)=-k$ as %o3.
(%i4) Set equation $\ln(40-25)-\ln(43-25)=-kt$ as %o4.
(%i6) Solve %o3 and %o4 for k and t.

- Another calculation by wxMaxima:

```
(%i2)fpprintprec:5; ratprint:false;
(fpprintprec)5
(ratprint)false
(%i4)left1:  integrate(1/(theta-25),  theta,  66,  63);
right1: integrate( -k, t, 0, 1);
(left1) log(38)-log(41)
(right1) -k
(%i6)left2:  integrate(1/(theta-25),  theta,  43,  40);
right2: integrate( -k, t, 0, t);
(left2) log(15)-log(18)
(right2) -k*t
(%i8)solve([left1 = right1, left2 = right2], [t,k])$
float(%);
(%o8)[[t = 2.3994,k = 0.075986]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Calculate $\int_{66}^{63}\frac{d\theta}{\theta-25}$ and assign as $left1$, calculate $\int_{0}^{1}-k\ dt$ and assign as $right1$.
(%i6) Calculate $\int_{43}^{40}\frac{d\theta}{\theta-25}$ and assign as $left2$, calculate $\int_{0}^{t}-k\ dt$ and assign as $right2$.
(%i8) Solve $left1=right1$ and $left2=right2$ for t and k.

Problem 18.8 A body is cooling down by forced convection in a room of temperature 20°C. The temperature of the body decreases from 70 to 50°C in 10 min. Calculate the time it cools down from 50 to 30°C.

Solution
Applying the Newton's cooling law, which is for forced convection, we have (Eq. 18.8),

$$\frac{d\theta}{dt}=-k(\theta-\theta_r),$$

where θ_r is the room temperature and k is a constant.

For cooling from 70 to 50°C, as an approximation, we write,

$$\frac{70°\text{C} - 50°\text{C}}{10\ \text{min}} = -k(70°\text{C} - 20°\text{C}). \tag{18.1}$$

For cooling from 50 to 30 °C, we write,

$$\frac{50°\text{C} - 30°\text{C}}{t} = -k(50°\text{C} - 20°\text{C}), \tag{18.2}$$

where t is the time to be calculated. Solving Eqs. (18.1) and (18.2), the cooling time is,

$$t = 17\,\text{min}.$$

- wxMaxima codes:

```
(%i2)fpprintprec:5; ratprint:false;
(fpprintprec)5
(ratprint)false
(%i4)solve([(70-50)/10 = -k*(70-20),(50-30)/t = -k*(50-
20)],[t,k])$ float(%);
(%o4)[[t = 16.667,k = -0.04]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $(70-50)/10 = -k(70-20)$ and $(50-30)/t = -k(50-20)$ for t and k.

Alternative solution: Using the calculus, the Newton's cooling law is (Eq. 18.8),

$$\frac{d\theta}{dt} = -k(\theta - \theta_r),$$
$$\int \frac{d\theta}{\theta - \theta_r} = \int -k\,dt.$$

For cooling from 70 to 50°C,

$$\int_{70}^{50} \frac{d\theta}{\theta - 20} = \int_{0}^{10} -k\,dt,$$
$$[\ln(\theta - 20)]_{70}^{50} = [-kt]_{0}^{10},$$

$$\ln(50-20)-\ln(70-20)=-10k,$$
$$k=\frac{1}{10}[\ln(70-20)-\ln(50-20)]=0.051082\,\text{min}^{-1}.$$

For cooling from 50 to 30°C,

$$\int_{50}^{30}\frac{d\theta}{\theta-20}=\int_{0}^{t}-(0.051082\,\text{min}^{-1})dt,$$
$$[\ln(\theta-20)]_{50}^{30}=\left[-(0.051082\,\text{min}^{-1})t\right]_{0}^{t},$$
$$\ln(30-20)-\ln(50-20)=-(0.051082\,\text{min}^{-1})t.$$

The time is,

$$t=22\ \text{min}\,.$$

- wxMaxima codes:

```
(%i2)fpprintprec:5; ratprint:false;
(fpprintprec)5
(ratprint)false
(%i3)log(50-20)-log(70-20) = -10*k;
(%o3)log(30)-log(50) = -10*k
(%i4)log(30-20)-log(50-20) = -t*k;
(%o4)log(10)-log(30) = -k*t
(%i6)solve([%o3,%o4], [t,k])$ float(%);
(%o6)[[t = 21.507,k = 0.051083]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3) Set equation $\ln(50-20)-\ln(70-20)=-10k$ as %o3.
(%i4) Set equation $\ln(30-20)-\ln(50-20)=-tk$ as %o4.
(%i6) Solve %o3 and %o4 for t and k.

- Another calculation by wxMaxima:

```
(%i2)fpprintprec:5; ratprint:false;
(fpprintprec)5
(ratprint)false
```

```
(%i4)left1:  integrate(1/(theta-20),  theta,  70,  50);
right1: integrate( -k, t, 0, 10);
(left1) log(30)-log(50)
(right1) -10*k
(%i6)left2:  integrate(1/(theta-20),  theta,  50,  30);
right2: integrate( -k, t, 0, t);
(left2) log(10)-log(30)
(right2) -k*t
(%i8)solve([left1 = right1, left2 = right2], [t,k])$
float(%);
(%o8)[[t = 21.507,k = 0.051083]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Calculate $\int_{70}^{50} \frac{d\theta}{\theta-20}$ and assign as *left*1, calculate $\int_{0}^{10} -k\ dt$ and assign as *right*1.
(%i6) Calculate $\int_{50}^{30} \frac{d\theta}{\theta-20}$ and assign as *left*2, calculate $\int_{0}^{t} -k\ dt$ and assign as *right*2.
(%i8) Solve *left*1 = *right*1 and *left*2 = *right*2 for t and k.

Note: The result of the first solution is different from the result of the exact alternative solution, that is, 17 min versus 22 min. This is due to large temperature difference (70°C − 50°C = 20°C) and long time interval (10 min), such that,

$$\frac{\Delta\theta}{\Delta t} \neq \frac{d\theta}{dt}.$$

Result of first solution can be made better by taking average temperature of 70 and 50°C (that is 60°C) and average temperature of 50 and 30°C (that is 40°C) as the body temperatures above the room temperature. The two equations become,

$$\frac{70°\text{C} - 50°\text{C}}{10\ \text{min}} = -k(60°\text{C} - 20°\text{C}),$$
$$\frac{50°\text{C} - 30°\text{C}}{t} = -k(40°\text{C} - 20°\text{C}).$$

Solving these two equations gives the time as,

$$t = 20\ \text{min},$$

which is closer to exact result.

- wxMaxima codes:

```
(%i2)fpprintprec:5; ratprint:false;
(fpprintprec)5
(ratprint)false
(%i4)solve([(70-50)/10 = -k*(60-20),(50-30)/t = -k*(40-
20)],[t,k])$ float(%);
(%o4)[[t = 20.0,k = -0.05]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $(70-50)/10 = -k(60-20)$ and $(50-30)/t = -k(40-20)$ for t and k.

Problem 18.9 The temperature of the Sun surface is T and that of planet Mars is t. Assume that the planet receives the radiation solely from the Sun and the planet is a black body, calculate the surface temperature of Mars. The Mars-Sun distance is S, the radius of Mars is r, and the radius of the Sun is R.

Solution
Figure 18.7 shows the Sun, planet Mars, and other quantities of the problem.

Applying the Stefan's law (Eq. 18.9), the power of the Sun radiation is,

$$P = e\sigma T^4 A = \sigma T^4 A = \sigma T^4 (4\pi R^2),$$

where the temperature of the sun is T, emissivity of the sun is taken as $e = 1$, and surface area of Sun is $4\pi R^2$.

The radiation power received by Mars from the Sun is only,

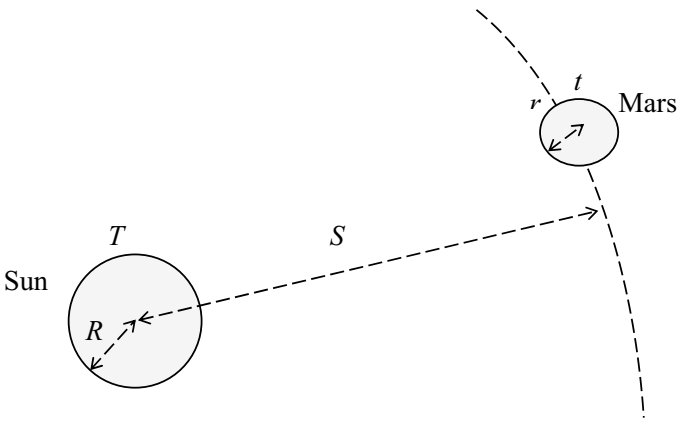


Fig. 18.7 Radiation received by Mars from the Sun, Problem 18.9

$$P_{received} = \frac{\text{area of Mars}}{\text{area of sphere of radius } S} \times P$$
$$= \frac{\pi r^2}{4\pi S^2} \times \sigma T^4 (4\pi R^2). \tag{18.1}$$

where r is the radius of Mars and S is Sun-Mars distance.

The power of Mars radiation is,

$$P_{radiation} = \sigma t^4 (4\pi r^2), \tag{18.2}$$

where r is the radius of Mars, t is its temperature, and emissivity of Mars is taken as $e = 1$.

In radiation equilibrium, the two quantities are the same,

$$P_{received} = P_{radiation},$$
$$\frac{\pi r^2}{4\pi S^2} \times \sigma T^4 (4\pi R^2) = \sigma t^4 (4\pi r^2).$$

Therefore, the temperature of Mars surface is,

$$t = \left(\frac{R}{2S}\right)^{1/2} T.$$

- wxMaxima codes:

```
(%i1) Preceived: %pi*r^2/(4*%pi*S^2)*sigma*T^4*4*%pi*R^2;
(Preceived) (%pi*R^2*T^4*r^2*sigma)/S^2
(%i2) Pradiation: sigma*t^4*4*%pi*r^2;
(Pradiation) 4*%pi*r^2*sigma*t^4
(%i3) solve(Preceived = Pradiation, t);
(%o3) [t = (%i*sqrt(R)*T)/(sqrt(2)*sqrt(S)),
t = -(sqrt(R)*T)/(sqrt(2)*sqrt(S)),
t = -(%i*sqrt(R)*T)/(sqrt(2)*sqrt(S)),
t = (sqrt(R)*T)/(sqrt(2)*sqrt(S))]
```

Comments on the codes:

(%i1) Assign $P_{received}$ as in Eq. (18.1).
(%i2) Assign $P_{radiation}$ as in Eq. (18.2).
(%i3) Solve $P_{received} = P_{radiation}$ for t.

Problem 18.10 A spherical black body of radius 5.0 cm is at temperature 327°C.

(a) What is the radiation power?

(b) Determine the wavelength of the radiation with the highest intensity. ($\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4})

Solution

(a) The power of a black body radiation is given by the Stefan's law (Eq. 18.9),

$$\begin{aligned} P &= e\sigma T^4 A = \sigma T^4 A = \sigma T^4 \times 4\pi r^2 \\ &= (5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}})[(327 + 273)\,\mathrm{K}]^4 \times 4\pi (5.0 \times 10^{-2}\,\mathrm{m})^2 \\ &= 231\,\mathrm{W}, \end{aligned}$$

where emissivity $e = 1$ for a black body.

(b) The wavelength of the maximum intensity radiation of a black body is given by the Wien's displacement law (Eq. 18.11),

$$\lambda_{max} T = 2898\,\mu\mathrm{m\,K}.$$

The wavelength is,

$$\begin{aligned} \lambda_{max} &= \frac{2898\,\mu\mathrm{m\,K}}{T} = \frac{2898\,\mu\mathrm{m\,K}}{(327 + 273)\,\mathrm{K}} \\ &= 4.83\,\mu\mathrm{m}. \end{aligned}$$

- wxMaxima codes:

```
(%i4) fpprintprec:5; sigma:5.67e-8; T:327 + 273; r:5e-2;
(fpprintprec) 5
(sigma) 5.67*10^-8
(t) 600
(r) 0.05
(%i6) P: sigma*T^4*4*%pi*r^2; float(%);
(P) 73.483*%pi
(%o6) 230.85
(%i8) lambdamax: 2898/T; float(%);
(lambdamax) 483/100
(%o8) 4.83
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of σ, T, and r.
(%i6) Calculate P, part (a).
(%i8) Calculate λ_{max}, part (b).

Problem 18.11 The Sun has a surface area of 6.1×10^{18} m^2 and it radiates the energy at 3.9×10^{26} W. Assuming the emissivity of the Sun surface is unity, what is the temperature of its surface?

Solution

The power of the Sun surface is (Eq. 18.9),

$$P = e\,\sigma T^4 A,$$

where A, T, and e are the area, temperature, and emissivity of the body, respectively; while σ is the Stefan-Boltzmann constant. Substituting known values into the equation, we have,

$$3.9 \times 10^{26}\,\mathrm{W} = (1)\left(5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}\right)T^4\left(6.1 \times 10^{-18}\,\mathrm{m^2}\right),$$

where emissivity of the Sun is taken as $e = 1$.

Solving the equation for T gives the temperature of the Sun surface as,

$$T = 5800\,\mathrm{K}.$$

- wxMaxima codes:

```
(%i6)fpprintprec:5; ratprint:false; e:1; sigma:5.67e-8;
A:6.1e18; P:3.9e26;
(fpprintprec)5
(ratprint)false
(e)1
(sigma) 5.67*10^-8
(A)6.1*10^18
(P)3.9*10^26
(%i8)solve(P = e*sigma*T^4*A, T)$ float(%);
(%o8)[T = 5794.8*%i,T = -5794.8,T = -5794.8*%i,T = 5794.8]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of e, σ, A, and P.
(%i8) Solve $P = e\,\sigma T^4 A$ for T.

18.3 Summary

- Heat conduction is the transfer of heat between two objects in direct contact with each other. The rate of conductive heat transfer is,

$$H = \frac{kA(\theta_2 - \theta_1)}{L}, \quad \theta_2 > \theta_1.$$

- Convection is heat transfer by the movement of hotter mass of the fluid to the less hot. Convection can be natural or forced.
- Radiation is heat transfer by the emission or absorption of electromagnetic waves. The rate of heat transfer from a body is

$$P = eA\sigma T^4,$$

where A, T, and e are the area, absolute temperature, and emissivity of the body, respectively, and σ is the Stefan-Boltzmann constant.

18.4 Exercises

Exercise 18.1 The inner and outer surface temperatures of a concrete wall 4.00 m by 2.50 m and 30 cm thick are 18.0°C and 6.0°C, respectively. How much heat flow through the wall in one hour? The thermal conductivity of concrete is 1.3 W m^{-1} K^{-1}.

(Answer: 1.9×10^6 J)

Exercise 18.2 A double paned window consists of two 60.00 cm by 100.00 cm glass panes of 3.0 mm thick, separated by a 1.50 cm layer of air. The rate of heat flow through the window is 25 W. Find the temperature difference between the indoors and the outdoors. Thermal conductivities of glass and air are 0.84 W m^{-1} K^{-1} and 0.023 W m^{-1} K^{-1}, respectively.

(Answer: 27 K)

Exercise 18.3 A body is cooling down by forced convection in a room of temperature 20°C. The temperature of the body decreases from 70 to 50°C in 10 min. Find an expression for temperature of the body against time.

(Answer: $\theta = 20 + 50\exp(-0.051083t)$)

Exercise 18.4 The Sun's radiation intensity is a maximum at wavelength of 0.50 μm. By assuming that the sun is a black body, calculate the surface temperature of the sun.

(Answer: 5800 K)

Exercise 18.5 A star radiates heat at the rate of 3.00×10^{30} W and has a surface temperature of 3000 K. Find the radius of the star by assuming that the star is a perfect emitter.

(Answer: 2.30×10^{11} m)

Chapter 19
Gas Laws and Kinetic Theory

19.1 Basic Concepts and Formulae

(1) At a constant temperature, Boyle's law states that the pressure p of a gas is inversely proportional to its volume V,

$$p \propto \frac{1}{V} \quad \text{or} \quad pV = \text{constant}. \tag{19.1}$$

(2) According to Charles and Gay-Lussac's law, the volume V of a gas at a constant pressure is directly proportional to its temperature T,

$$V \propto T \quad \text{or} \quad \frac{V}{T} = \text{constant}. \tag{19.2}$$

(3) The equation of state of an ideal gas is,

$$pV = \mu RT = Nk_B T, \tag{19.3}$$

where p, V, μ, T, and R are the pressure, volume, number of moles, temperature of the gas, and the universal gas constant, respectively, while N and k_B are the number of molecules and the Boltzmann constant, respectively. The universal gas constant and Boltzmann constant are,

$$R = 8.31\,\text{J}\,\text{mol}^{-1}\,\text{K}^{-1},$$

$$k_B = 1.38 \times 10^{-23}\,\text{J}\,\text{K}^{-1}.$$

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W. M. S. Wan Hassan, *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-42678-0_19

The mole (mol) is the SI base unit for the amount of substance. One mole is defined as containing exactly $6.02214076 \times 10^{23}$ elementary entities. The number is also called Avogadro number, N_A,

$$N_A = 6.02214076 \times 10^{23}\,\text{mol}^{-1}.$$

(4) The pressure of N molecules of an ideal gas in a container of volume V is,

$$p = \frac{2}{3}\frac{N}{V}\left(\frac{1}{2}m\overline{v^2}\right) = \frac{1}{3}\frac{N}{V}m\overline{v^2}, \tag{19.4}$$

where m is the mass, $\overline{v^2}$ is the average of square of speed, and $\frac{1}{2}m\overline{v^2}$ is the average kinetic energy of each molecule.

(5) The temperature of an ideal gas is directly proportional to the average kinetic energy of each molecule,

$$T = \frac{2}{3k_B}\left(\frac{1}{2}m\overline{v^2}\right), \tag{19.5}$$

where k_B is the Boltzmann constant.

(6) The average translational kinetic energy of each gas molecule is,

$$\frac{1}{2}m\overline{v^2} = \frac{3}{2}k_B T, \tag{19.6}$$

where every translational degree of freedom is associated with energy $\frac{1}{2}k_B T$.

(7) According to the equipartition of energy theorem, the energy of a system in thermal equilibrium is divided equally among all degrees of freedom.

(8) The total energy of N molecules of an ideal monatomic gas is,

$$U = \frac{3}{2}Nk_B T = \frac{3}{2}\mu RT = \frac{3}{2}pV. \tag{19.7}$$

(9) The change in internal energy of μ mol of an ideal gas as the temperature increases by ΔT is,

$$\Delta U = \mu c_v \Delta T, \tag{19.8}$$

where c_v is the molar heat capacity at constant volume.

(10) The molar heat capacity at constant volume of an ideal monatomic gas is,

$$c_v = \frac{3}{2}R, \tag{19.9}$$

and molar heat capacity at constant pressure is,

$$c_p = \frac{5}{2}R. \tag{19.10}$$

The ratio of both heat capacities is,

$$\gamma = c_p/c_v = \frac{5}{3}. \tag{19.11}$$

(11) In an adiabatic process, there is no heat transfer between the system and the environment. For an ideal gas undergoing adiabatic expansion or compression,

$$pV^\gamma = \text{ constant}. \tag{19.12}$$

The equation is obtained by applying the first law of thermodynamics and the state equation $pV = \mu RT$.

(12) The speed of sound, v, in a gas of density ρ and pressure p is,

$$v = \left(\frac{\gamma p}{\rho}\right)^{1/2}. \tag{19.13}$$

(13) The distribution of speed of N molecules of a gas at temperature T is given by the Maxwell's speed distribution function,

$$N_v = 4\pi N\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 e^{-mv^2/(2k_B T)}, \tag{19.14}$$

where m is the mass and v is the speed of the gas molecule. The root mean square speed v_{rms}, average speed $v_{average}$, and the most probable speed $v_{most\ probable}$ are,

$$v_{rms} = \left(\frac{3k_B T}{m}\right)^{1/2}, \tag{19.15}$$

$$v_{average} = \left(\frac{8k_B T}{\pi m}\right)^{1/2}, \tag{19.16}$$

$$v_{most\ probable} = \left(\frac{2k_B T}{m}\right)^{1/2}. \tag{19.17}$$

(14) For a gas with volume density n_v (number of molecules per unit volume) and molecule diameter d, the mean free path or the average distance between collisions is,

$$l = \frac{1}{\sqrt{2}\pi d^2 n_v}. \tag{19.18}$$

The frequency of collisions, f (number of collisions per second), is,

$$f = \sqrt{2}\pi d^2 \overline{v} n_v = \frac{\overline{v}}{l}. \tag{19.19}$$

19.2 Problems and Solutions

Problem 19.1 The pressure of a gas at 300 K is 0.15 MPa. The pressure is increased to 0.40 MPa and the density increases by three times by reducing the volume of the gas container. What is the final temperature?

Solution

From the equation of state of an ideal gas $pV = \mu RT$ (Eq. 19.3), for a gas in two different states we can write,

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}.$$

The density of a gas is inversely proportional to its volume. Thus, the final volume is 1/3 of the original volume as the density becomes 3 times the original density. Substitution into the equation gives,

$$\frac{(0.15 \times 10^6\,\text{Pa}) V_1}{300\,\text{K}} = \frac{(0.4 \times 10^6\,\text{Pa}) \frac{V_1}{3}}{T_2}.$$

The final temperature of the gas is,

$$T_2 = 267\,\text{K}.$$

- wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec) 5
(ratprint) false
(%i4)   solve(0.15e6*V1/300   =   0.4e6*V1/(T2*3),   T2)$
float(%);
(%o4) [T2 = 266.67]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i4) Solve $\frac{(0.15\times10^6)V_1}{300} = \frac{(0.4\times10^6)V_1/3}{T_2}$ for T_2.

Problem 19.2 The volume of an ideal gas at pressure 1.52 MPa and temperature 298.15 K is 10 L. What is,

(a) the number of mol of the gas?
(b) the density if it is a hydrogen gas, where the mass of a mol of hydrogen gas is 2.0 g?
(c) the density if it is an oxygen gas, where the mass of a mol of oxygen gas is 32 g?

Solution

(a) The ideal gas formula $pV = \mu RT$ (Eq. 19.3) gives the number of mol as,

$$\mu = \frac{pV}{RT} = \frac{(1.52 \times 10^6\,\text{Pa})(10 \times 10^{-3}\,\text{m}^3)}{(8.31\,\text{J mol}^{-1}\,\text{K}^{-1})(298.15\,\text{K})} = 6.13\,\text{mol}.$$

(b) The mass of a mol of hydrogen gas is 2.0 g $= 2.0 \times 10^{-3}$ kg, that is, $M = 2.0 \times 10^{-3}$ kg/mol. The density of the hydrogen gas is,

$$\rho_H = \frac{\text{mass}}{\text{volume}} = \frac{\mu M}{V} = \frac{(6.13\,\text{mol})(2.0 \times 10^{-3}\,\text{kg/mol})}{(10 \times 10^{-3}\,\text{m}^{-3})} = 1.23\,\text{kg m}^{-3}.$$

(c) The mass of a mol of oxygen gas is 32 g $= 32 \times 10^{-3}$ kg. The density of the oxygen gas is,

$$\rho_O = \frac{\mu M}{V} = \frac{(6.13\,\text{mol})(32 \times 10^{-3}\,\text{kg/mol})}{(10 \times 10^{-3}\,\text{m}^{-3})} = 19.6\,\text{kg m}^{-3}.$$

- wxMaxima codes:

```
(%i5) fpprintprec:5; p:1.52e6; V:10e-3; T:298.15; R:8.31;
(fpprintprec) 5
(p) 1.52*10^6
(V) 0.01
(t) 298.15
(R) 8.31
(%i6) mu: p*V/(R*T);
(mu) 6.1349
(%i7) rho_H: mu*2e-3/V;
(rho_H) 1.227
(%i8) rho_O: mu*32e-3/V;
(rho_O) 19.632
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of p, V, T, and R.
(%i6), (%i7), (%i8) Calculate μ, ρ_H, and ρ_O.

Problem 19.3 Helium gas is monatomic and its relative atomic mass is 4. What is,

(a) the molar mass of helium?
(b) the molar volume of helium at STP?
(c) the density of helium at STP?

Standard temperature and pressure (STP) means the standard temperature and pressure, namely the temperature of 273.15 K (0°C, 32°F) and the pressure of 1 atm (101 325 Pa).

Solution

(a) Relative atomic mass of helium is 4. This means the mass of one mol of helium is $4.0\,\text{g} = 4.0 \times 10^{-3}\,\text{kg}$. The molar mass of helium is $M = 4.0 \times 10^{-3}\,\text{kg mol}^{-1}$.
(b) The STP means the temperature of 273.15 K and pressure of 101 325 Pa. Applying the ideal gas equation $pV = \mu RT$ (Eq. 19.3), at STP the volume of one mol of helium is,

$$\begin{aligned} V &= \frac{\mu RT}{p} \\ &= \frac{(1.0\,\text{mol})(8.31\,\text{J mol}^{-1}\,\text{K}^{-1})(273.15\,\text{K})}{101,325\,\text{Pa}} \\ &= 0.0224\,\text{m}^3. \end{aligned}$$

(c) At STP the density of helium gas is,

$$\begin{aligned} \text{density} = \frac{\text{mass}}{\text{volume}} &= \frac{4 \times 10^{-3}\,\text{kg}}{0.0224\,\text{m}^3} \\ &= 0.179\,\text{kg m}^{-3}. \end{aligned}$$

- wxMaxima codes:

```
(%i5) fpprintprec:5; p:101325; T:273.15; R:8.31; mu:1;
(fpprintprec) 5
(p) 101325
(t) 273.15
(R) 8.31
(mu) 1
(%i6) V: mu*R*T/p;
(V) 0.022402
(%i7) density: 4e-3/V;
```

```
(density) 0.17856
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of p, T, R, and μ.
(%i6) Calculate V. Part (b).
(%i7) Calculate density. Part (c).

Problem 19.4 A sample of oxygen gas has 3.6×10^{24} molecules. Oxygen gas is diatomic and its relative atomic mass is 16. Calculate,

(a) the molar mass,
(b) the mass of the sample,
(c) the mass of an oxygen molecule,
(d) the volume of the sample at STP,
(e) the molecule number density at STP.

STP means the temperature of 273.15 K and the pressure of 101 325 Pa.

Solution

(a) Relative atomic mass of oxygen is 16, so relative atomic mass of oxygen molecule is 32 because an oxygen molecule has 2 atoms of oxygen. The mass of 1 mol of oxygen molecules is $32\ \text{g} = 32 \times 10^{-3}$ kg. The molar mass of oxygen molecule is $M = 32 \times 10^{-3}\ \text{kg mol}^{-1}$.
(b) The mass of the sample of oxygen gas is,

$$\begin{aligned}\text{mass of gas sample} &= \text{number of mol} \times \text{molar mass} \\ &= \frac{\text{number of molecules, } N}{N_A} \times \text{molar mass, } M \\ &= \frac{3.6 \times 10^{24}}{6.02 \times 10^{23}\ \text{mol}^{-1}} (32 \times 10^{-3}\ \text{kg/mol}) \\ &= 0.19\,\text{kg},\end{aligned}$$

where the Avogadro number is $N_A = 6.02 \times 10^{23}\ \text{mol}^{-1}$.
(c) Mass of one oxygen molecule is,

$$\begin{aligned}\text{mass of one oxygen molecule} &= \frac{\text{molar mass of oxygen molecules, } M}{N_A} \\ &= \frac{32 \times 10^{-3}\ \text{kg/mol}}{6.02 \times 10^{23}\ \text{mol}^{-1}} \\ &= 5.3 \times 10^{-26}\,\text{kg}.\end{aligned}$$

(d) At STP the pressure is $p = 1.01325 \times 10^5\ \mathrm{N\,m^{-2}}$ and the temperature is $T = 273.15\ \mathrm{K}$. Using the formula $pV = \mu RT$ (Eq. 19.3), the volume of the oxygen gas is,

$$\begin{aligned} V &= \frac{\mu RT}{p} = \left(\frac{N}{N_A}\right)\frac{RT}{p} \\ &= \left(\frac{3.6 \times 10^{24}}{6.02 \times 10^{23}\ \mathrm{mol^{-1}}}\right)\frac{(8.31\ \mathrm{J\,mol^{-1}\,K^{-1}})(273.15\ \mathrm{K})}{(101,325\ \mathrm{N/m^2})} \\ &= 0.13\ \mathrm{m^3}. \end{aligned}$$

(e) The molecule number density of the oxygen gas sample is,

$$\begin{aligned} \text{molecule number density} &= \frac{\text{number of molecules, } N}{\text{volume, } V} \\ &= \frac{3.6 \times 10^{24}}{0.13\ \mathrm{m^3}} \\ &= 2.7 \times 10^{25}\ \mathrm{m^{-3}}. \end{aligned}$$

- wxMaxima codes:

```
(%i7)  fpprintprec:5;  N:3.6e24;  NA:6.02e23;  M:32e-3;
p:1.01325e5; T:273.15; R:8.13;
(fpprintprec) 5
(N) 3.6*10^24
(NA) 6.02*10^23
(M) 0.032
(p) 1.0132*10^5
(t) 273.15
(R) 8.13
(%i8) mass_of_gas_sample: (N/NA)*M;
(mass_of_gas_sample) 0.19136
(%i9) mass_of_one_oxygen_molecule: M/NA;
(mass_of_one_oxygen_molecule) 5.3156*10^-26
(%i10) V: (N/NA)*R*T/p;
(V) 0.13106
(%i11) molecule_number_density: N/V;
(molecule_number_density) 2.7468*10^25
```

Comments on the codes:

(%i7) Set floating point print precision to 5, assign values of N, N_A, M, p, T, and R.
(%i8) Calculate mass of gas sample, part (b).
(%i9) Calculate mass of one oxygen molecule, part (c).

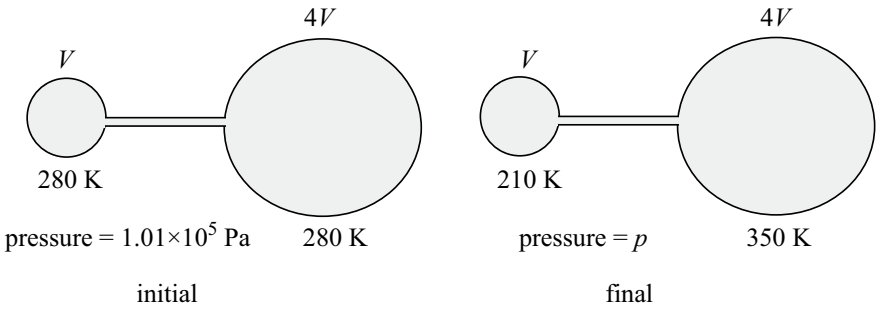


Fig. 19.1 Two flasks filled with air in initial and final states, Problem 19.5

(%i10) Calculate V, part (d).
(%i11) Calculate molecule number density, part (e).

Problem 19.5 Two flasks with volumes V and $4V$ are connected by a small tube to allow air to flow between them. Both flasks begin with a temperature of 280 K and an air pressure of 1.01×10^5 Pa. What is the pressure in the flasks if the large flask is heated and maintained at 350 K and the small flask is cooled and maintained at 210 K?

Solution
Figure 19.1 shows the two flasks in initial and final states. Applying the formula $\mu = pV/(RT)$, (Eq. 19.3), the total mol of air molecules in both flasks in the initial state is,

$$\frac{(1.01 \times 10^5\,\text{Pa})V}{R(280\,\text{K})} + \frac{(1.01 \times 10^5\,\text{Pa})4V}{R(280\,\text{K})}.$$

The total mol of air molecules in the flasks in the final state is,

$$\frac{pV}{R(210\,\text{K})} + \frac{p(4V)}{R(350\,\text{K})},$$

where p is the final pressure. The number of air molecules does not change, thus, the two expressions are equal, giving,

$$\frac{(1.01 \times 10^5\,\text{Pa})V}{R(280\,\text{K})} + \frac{(1.01 \times 10^5\,\text{Pa})4V}{R(280\,\text{K})} = \frac{pV}{R(210\,\text{K})} + \frac{p(4V)}{R(350\,\text{K})}.$$

Solving this equation gives the air pressure in the final state as,

$$p = 111,000\,\text{Pa}.$$

- wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec) 5
(ratprint) false
(%i4) solve(1.01e5*V/(R*280) + 1.01e5*4*V/(R*280) = p*V/
(R*210) + p*4*V/(R*350), p)$ float(%);
(%o4) [p = 1.114*10^5]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $\frac{(1.01\times10^5)V}{R(280)} + \frac{(1.01\times10^5)4V}{R(280)} = \frac{pV}{R(210)} + \frac{p(4V)}{R(350)}$ for p.

Problem 19.6 A tube connects two identical flasks at a temperature of 300 K so that the pressures of the trapped gas in both flasks are identical. Later, one flask is maintained at 350 K while the other is maintained at 250 K. What is the percentage change in pressure?

Solution
Figure 19.2 shows the flasks at the beginning and final states. Applying the formula $\mu = pV/(RT)$ (Eq. 19.3), the number of gas molecules in the flasks at the beginning is,

$$\frac{pV}{R(300\,\mathrm{K})} + \frac{pV}{R(300\,\mathrm{K})}. \tag{19.1}$$

Finally, the number of gas molecules in the flasks is,

$$\frac{p'V}{R(350\,\mathrm{K})} + \frac{p'V}{R(250\,\mathrm{K})}. \tag{19.2}$$

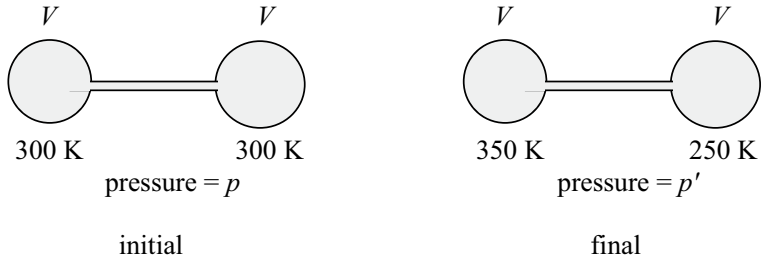


Fig. 19.2 Two flasks filled with a gas in initial and final states, Problem 19.6

Expressions (19.1) and (19.2) are equal, because the number of gas molecules in initial and final states is the same. From the two expressions we get,

$$\frac{p'}{p} = \frac{\left(\frac{1}{300\,\mathrm{K}} + \frac{1}{300\,\mathrm{K}}\right)}{\left(\frac{1}{350\,\mathrm{K}} + \frac{1}{250\,\mathrm{K}}\right)} = 0.9722.$$

The pressure has decreased. The percentage of pressure decrease is,

$$\begin{aligned}\frac{p' - p}{p} \times 100\% &= (0.9722 - 1) \times 100\% \\ &= -3\%.\end{aligned}$$

- wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec) 5
(ratprint) false
(%i4) solve(p*V/(R*300) + p*V/(R*300) = pprime*V/(R*350) +
pprime*V/(R*250), pprime)$ float(%);
(%o4) [pprime = 0.97222*p]
(%i5) pprime: 0.97222*p;
(pprime) 0.97222*p
(%i6) (pprime-p)/p*100;
(%o6) -2.778
```

Comments on the codes:

(%i4) Solve $\frac{pV}{R(300)} + \frac{pV}{R(300)} = \frac{p'V}{R(350)} + \frac{p'V}{R(250)}$ for p'.
(%i5) Assign p' in terms of p.
(%i6) Calculate percentage of change in pressure.

Problem 19.7 Calculate the volume of,

(a) 0.5 mol of oxygen gas at temperature of 300 K and pressure of 2.0 atm,
(b) 1.0 mol of nitrogen gas at temperature of 273 K and pressure of 1.0 atm.

Solution

(a) The equation of an ideal gas is $pV = \mu RT$ (Eq. 19.3). Applying the equation, the volume of the oxygen gas is,

$$V = \frac{\mu RT}{p} = \frac{(0.5\,\mathrm{mol})(8.31\,\mathrm{J\,mol^{-1}\,K^{-1}})(300\,\mathrm{K})}{2 \times 1.013 \times 10^5\,\mathrm{Pa}} = 6.16 \times 10^{-3}\,\mathrm{m^3}.$$

where the conversion 1 atm $= 1.013 \times 10^5$ Pa has been used.

- wxMaxima codes:

```
(%i5) fpprintprec:5; mu:0.5; R:8.31; T:300; p:2*1.013e5;
(fpprintprec) 5
(mu) 0.5
(R) 8.31
(t) 300
(p) 2.026*10^5
(%i6) V: mu*R*T/p;
(V) 0.0061525
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of μ, R, T, and p.
(%i6) Calculate V.

(b) Similarly applying the ideal gas equation, the volume of the nitrogen gas is,

$$V = \frac{\mu RT}{p} = \frac{(1\,\text{mol})(8.31\,\text{J mol}^{-1}\,\text{K}^{-1})(273\,\text{K})}{1.013 \times 10^5\,\text{Pa}} = 2.24 \times 10^{-2}\,\text{m}^3.$$

This result is the same as the fact that says the volume of one mol of any gas at STP is $22.4\,\text{L} = 2.24 \times 10^{-2}\,\text{m}^3$.

- wxMaxima codes:

```
(%i5) fpprintprec:5; mu:1; R:8.31; T:273; p:1.013e5;
(fpprintprec) 5
(mu) 1
(R) 8.31
(t) 273
(p) 1.013*10^5
(%i6) V: mu*R*T/p;
(V) 0.022395
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of μ, R, T, and p.
(%i6) Calculate V.

Problem 19.8 The speeds of four particles are 3.00, 4.00, 5.00, and 5.00 m s^{-1}. Calculate the average speed and the root mean square speed of the particles.

Solution

The average speed or the mean speed of the particles is,

$$\overline{v} = \frac{\text{sum of all speeds}}{\text{number of particles}} = \frac{\sum_i v_i}{n} = \frac{(3.00 + 4.00 + 5.00 + 5.00)\,\text{m s}^{-1}}{4}$$
$$= 4.25\,\text{m s}^{-1}.$$

The mean of squares of speeds is,

$$\overline{v^2} = \frac{\text{sum of all squares of speeds}}{\text{number of particles}}$$
$$= \frac{\sum_i v_i^2}{n} = \frac{(3.00^2 + 4.00^2 + 5.00^2 + 5.00^2)\,\text{m}^2\,\text{s}^{-2}}{4}$$
$$= 18.7\,\text{m}^2\,\text{s}^{-2}.$$

The root mean square speed of the particles is the square root of the above,

$$v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\frac{\sum_i v_i^2}{n}} = \sqrt{\frac{(3^2 + 4^2 + 5^2 + 5^2)\,\text{m}^2\,\text{s}^{-2}}{4}} = \sqrt{18.75\,\text{m}^2\,\text{s}^{-2}}$$
$$= 4.33\,\text{m s}^{-1}.$$

- wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec) 5
(%i2) v: [3,4,5,5];
(v) [3,4,5,5]
(%i4) mean(v); float(%);
(%o3) 17/4
(%o4) 4.25
(%i5) v_square: v^2;
(v_square) [9,16,25,25]
(%i7) mean(v_square); float(%);
(%o6) 75/4
(%o7) 18.75
(%i8) vrms: sqrt(%);
(vrms) 4.3301
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i2) Input speeds v as a list.
(%i4) Calculate mean speed $\overline{v}$.
(%i5) Calculate v^2.
(%i7) Calculate mean of v^2 i.e. $\overline{v^2}$.
(%i8) Calculate v_{rms}.

Problem 19.9 Compare the root mean square speeds of hydrogen and oxygen molecules at the same temperature. Relative molecular masses of hydrogen and oxygen are 2 and 32, respectively.

Solution

The root mean square speed of molecules at temperature T is (Eq. 19.15),

$$v_{rms} = \sqrt{\frac{3k_BT}{m}},$$

where m is the mass of the molecule and k_B is the Boltzmann constant. For hydrogen and oxygen we have,

$$\begin{aligned}\frac{v_{rms,H}}{v_{rms,O}} &= \sqrt{\frac{3k_BT}{m_H}} \Big/ \sqrt{\frac{3k_BT}{m_O}} \\ &= \sqrt{\frac{m_O}{m_H}} = \sqrt{\frac{32}{2}} \\ &= 4.\end{aligned}$$

The hydrogen molecules are four times as fast as the oxygen molecules. The mass of a hydrogen molecule is 1/16 that of oxygen.

Problem 19.10 A container contains oxygen gas at 300 K. Calculate the root mean square, the average, and the most probable speeds of the oxygen molecules. The mass of an oxygen molecule is 5.35×10^{-26} kg.

Solution

The root mean square speed of the oxygen molecules is (Eq. 19.15),

$$v_{rms} = \sqrt{\frac{3k_BT}{m}} = \sqrt{\frac{3(1.38 \times 10^{-23}\,\text{J/K})(300\,\text{K})}{5.35 \times 10^{-26}\,\text{kg}}} = 482\,\text{m s}^{-1}.$$

The average speed of the oxygen molecules is (Eq. 19.16),

$$v_{average} = \sqrt{\frac{8k_BT}{\pi m}} = \sqrt{\frac{8(1.38 \times 10^{-23}\,\text{J/K})(300\,\text{K})}{\pi \times 5.35 \times 10^{-26}\,\text{kg}}} = 444\,\text{m s}^{-1}.$$

The most probable speed of the oxygen molecules is (Eq. 19.17),

$$v_{most\ probable} = \sqrt{\frac{2k_BT}{m}} = \sqrt{\frac{2(1.38 \times 10^{-23}\,\text{J/K})(300\,\text{K})}{5.35 \times 10^{-26}\,\text{kg}}} = 393\,\text{m}\,\text{s}^{-1}.$$

- wxMaxima codes:

```
(%i4) fpprintprec:5; kB:1.38e-23; T:300; m:5.35e-26;
(fpprintprec) 5
(kB) 1.38*10^-23
(t) 300
(m) 5.35*10^-26
(%i5) v_rms: float( sqrt(3*kB*T/m));
(v_rms) 481.82
(%i6) v_average: float( sqrt(8*kB*T/(%pi*m)));
(v_average) 443.91
(%i7) v_most_probable: float( sqrt(2*kB*T/m));
(v_most_probable) 393.4
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of k_B, T, and m.
(%i5), (%i6), (%i7) Calculate v_{rms}, $v_{average}$, and $v_{most\ probable}$.

Further question: Plot speed distributions of 1 mol of oxygen gas molecules at 300 K and 500 K.

Solution: The Maxwell's speed distribution of oxygen molecules is (Eq. 19.14),

$$N_v = 4\pi N\left(\frac{m}{2\pi k_BT}\right)^{3/2} v^2 e^{-mv^2/(2k_BT)},$$

where m is mass of oxygen molecule (5.35×10^{-26} kg), N is number of molecules ($6.02 \times 10^{23} = 1$ mol), and T is the temperature of the gas.

At 300 K,

$$N_{v300} = 4\pi(6.02 \times 10^{23})\left(\frac{5.35 \times 10^{-26}\,\text{kg}}{2\pi(1.38 \times 10^{-23}\,\text{J/K})(300\,\text{K})}\right)^{3/2} \times v^2 \exp\{-(5.35 \times 10^{-26}\,\text{kg})v^2/[2(1.38 \times 10^{-23}\,\text{J/K})(300\,\text{K})]\}.$$

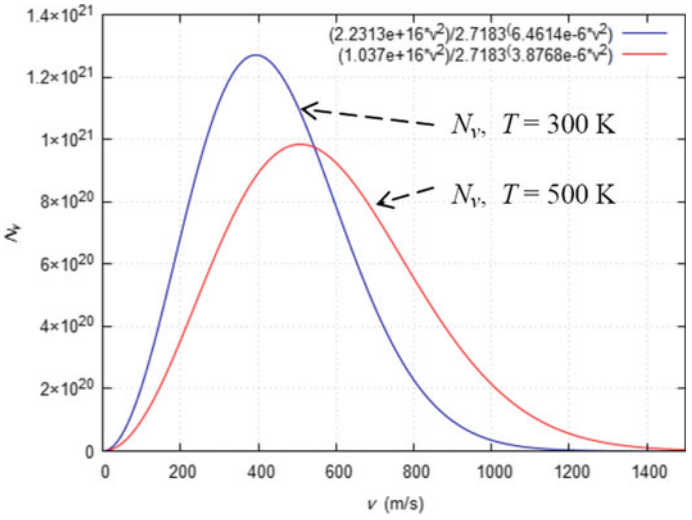
At 500 K,

$$N_{v500} = 4\pi(6.02 \times 10^{23})\left(\frac{5.35 \times 10^{-26}\,\text{kg}}{2\pi(1.38 \times 10^{-23}\,\text{J/K})(500\,\text{K})}\right)^{3/2} \times v^2 \exp\{-(5.35 \times 10^{-26}\,\text{kg})v^2/[2(1.38 \times 10^{-23}\,\text{J/K})(500\,\text{K})]\}.$$

We plot N_{v300} and N_{v500} against v for $0 \leq v \leq 1500$ m s^{-1} by wxMaxima.

- wxMaxima codes:

```
(%i5)  fpprintprec:5;  kB:1.38e-23;  T:300;  m:5.35e-26;
N:6.02e23;
(fpprintprec) 5
(kB) 1.38*10^-23
(t) 300
(m) 5.35*10^-26
(N) 6.02*10^23
(%i6) Nv300: float(4*%pi*N*(m/(2*%pi*kB*T))^(3/2)
*v^2*exp(-m*v^2/(2*kB*T)));
(Nv300) (2.2313*10^16*v^2)/2.7183^(6.4614*10^-6*v^2)
(%i7) T:500;
(t) 500
(%i8) Nv500: float(4*%pi*N*(m/(2*%pi*kB*T))^(3/2)
*v^2*exp(-m*v^2/(2*kB*T)));
(Nv500) (1.037*10^16*v^2)/2.7183^(3.8768*10^-6*v^2)
(%i9)   wxplot2d([Nv300,Nv500],   [v,0,1500],   grid2d,
[xlabel,"{/Helvetica-Italic   v}   (m/s)"],   [ylabel,"{/
Helvetica-Italic N_v}"]);
```



Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of k_B, T, m, and N.

(%i6) Define N_{v300}.
(%i7) Reassign $T = 500$ K
(%i8) Define N_{v500}.
(%i9) Plot N_{v300} and N_{v500} against v for $0 \leq v \leq 1500$ m s^{-1}.

Problem 19.11 A 2.00 m^3 tank has 5.00 mol of helium gas at a temperature of 273 K in it. Helium gas is monatomic. Mass of a helium atom is 6.69×10^{-27} kg. Calculate,

(a) the internal energy of the gas,
(b) the average kinetic energy per atom,
(c) the root mean square speed of helium atoms,
(d) the pressure of the helium gas.

Solution

(a) The internal energy of helium gas is (Eq. 19.7),

$$U = \frac{3}{2}\mu RT = \frac{3}{2}(5.00\,\text{mol})(8.31\,\text{J K}^{-1}\,\text{mol}^{-1})(273\,\text{K}) = 1.70 \times 10^4\,\text{J}.$$

(b) The average kinetic energy per atom is (Eqs. 19.6 and 19.7),

$$\begin{aligned}\frac{1}{2}m\overline{v^2} &= \frac{\text{internal energy}}{\text{number of atoms}} \\ &= \frac{U}{N} = \frac{U}{\mu N_A} = \frac{1.70 \times 10^4\,\text{J}}{(5.00\,\text{mol})(6.02 \times 10^{23}\,\text{mol}^{-1})} = 5.65 \times 10^{-21}\,\text{J}.\end{aligned}$$

Here, number of atoms is $N = \mu N_A$, where μ is number of mol and N_A is Avogadro number.

The average kinetic energy per atom can also be obtained by Eq. (19.6),

$$\frac{1}{2}m\overline{v^2} = \frac{3}{2}k_B T = \frac{3}{2}(1.38 \times 20^{-23}\,\text{J/K})(273\,\text{K}) = 5.65 \times 10^{-21}\,\text{J}.$$

(c) The root mean square speed of the helium atoms is (Eq. 19.15),

$$v_{rms} = \sqrt{\frac{3k_B T}{m}} = \sqrt{\frac{3(1.38 \times 10^{-23}\,\text{J/K})(273\,\text{K})}{6.69 \times 10^{-27}\,\text{kg}}} = 1.30 \times 10^3\,\text{m s}^{-1}.$$

(d) The pressure of the helium gas is (Eq. 19.7),

$$p = \frac{2U}{3V} = \frac{2(1.7 \times 10^4\,\text{J})}{3(2.0\,\text{m}^3)} = 5.67 \times 10^3\,\text{N m}^{-2}.$$

- wxMaxima codes:

```
(%i8) fpprintprec:5; kB:1.38e-23; R:8.31; T:273; m:6.69e-
27; mu:5; NA:6.02e23; V:2;
(fpprintprec) 5
(kB) 1.38*10^-23
(R) 8.31
(t) 273
(m) 6.69*10^-27
(mu) 5
(NA) 6.02*10^23
(V) 2
(%i9) U: 3/2*mu*R*T;
(U) 1.7015*10^4
(%i10) average_kinetic_energy_per_atom: U/(mu*NA);
(average_kinetic_energy_per_atom) 5.6527*10^-21
(%i11) average_kinetic_energy_per_atom2: 3/2*kB*T;
(average_kinetic_energy_per_atom2) 5.6511*10^-21
(%i12) v_rms: sqrt(3*kB*T/m);
(v_rms) 1299.8
(%i13) p: 2*U/(3*V);
(p) 5671.6
```

Comments on the codes:

(%i8) Set floating point print precision to 5, assign values of k_B, R, T, m, μ, and N_A.
(%i9) Calculate U, part (a).
(%i10) Calculate average kinetic energy per atom, part (b).
(%i11) Another calculation of average kinetic energy per atom, part (b).
(%i12) Calculate v_{rms}, part (c).
(%i13) Calculate p, part (d).

Problem 19.12 Plot p–V diagram for 1 mol of nitrogen gas at 300 K and 500 K.

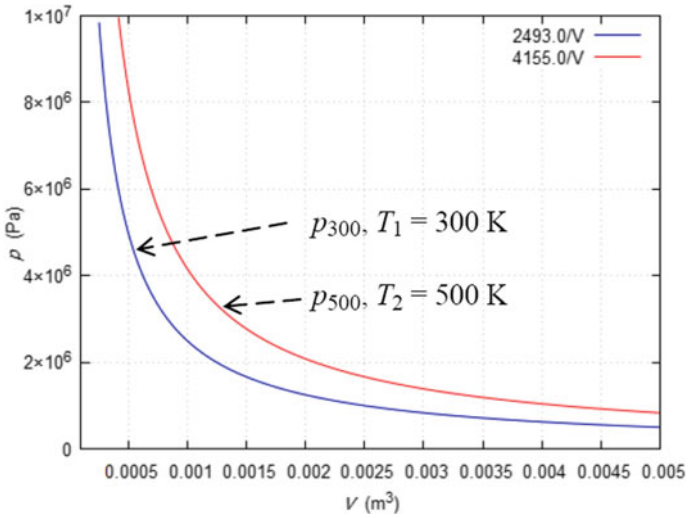
Solution
The ideal gas equation of state is $pV = \mu RT$ (Eq. 19.3). The p–V relations of 1 mol of nitrogen gas at $T_1 = 300$ K and $T_2 = 500$ K are written as,

$$p_{300} = \frac{\mu RT_1}{V} = \frac{(1\,\text{mol})(8.31\,\text{J K}^{-1}\,\text{mol}^{-1})(300\,\text{K})}{V},$$
$$p_{500} = \frac{\mu RT_2}{V} = \frac{(1\,\text{mol})(8.31\,\text{J K}^{-1}\,\text{mol}^{-1})(500\,\text{K})}{V}.$$

The p–V diagrams of 1 mol of nitrogen gas at 300 K and 500 K are plotted by wxMaxima.

- wxMaxima codes:

```
(%i5) fpprintprec:5; R:8.31; mu:1; T1:300; T2:500;
(fpprintprec) 5
(R) 8.31
(mu) 1
(T1) 300
(T2) 500
(%i7) p300: mu*R*T1/V; p500: mu*R*T2/V;
(p300) 2493.0/V
(p500) 4155.0/V
(%i8) wxplot2d([p300,p500], [V, 1e-4,0.005], [y,0,1e7],
grid2d,    [xlabel,"{/Helvetica-Italic    V}    (m^3)"],
[ylabel,"{/Helvetica-Italic p} (Pa)"]);
```



Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of R, μ, T_1, and T_2.
(%i7) Define $p_{300} = \frac{\mu R T_1}{V}$ and $p_{500} = \frac{\mu R T_2}{V}$.
(%i8) Plot p_{300} and p_{500} against V for $1 \times 10^{-4} \leq V \leq 0.005$ m^3 and $0 \leq y \leq 1 \times 10^7$ Pa.

Problem 19.13 At 400 K, a container contains 1.00 mol of oxygen gas. Determine the root mean square, average, and most probable speed of the oxygen molecules. Plot the speed distribution of oxygen molecules. The mass of an oxygen molecule is 5.35×10^{-26} kg.

Solution

The root mean square speed of the oxygen molecules is (Eq. 19.15),

$$v_{rms} = \sqrt{\frac{3k_BT}{m}} = \sqrt{\frac{3(1.38 \times 10^{-23}\,\text{J/K})(400\,\text{K})}{5.35 \times 10^{-26}\,\text{kg}}} = 556\,\text{m s}^{-1}.$$

The average speed of the oxygen molecules is (Eq. 19.16),

$$v_{average} = \sqrt{\frac{8k_BT}{\pi m}} = \sqrt{\frac{8(1.38 \times 10^{-23}\,\text{J/K})(400\,\text{K})}{\pi \times 5.35 \times 10^{-26}\,\text{kg}}} = 513\,\text{m s}^{-1}.$$

The most probable speed of the oxygen molecules is (Eq. 19.17),

$$v_{most\ probable} = \sqrt{\frac{2k_BT}{m}} = \sqrt{\frac{2(1.38 \times 10^{-23}\,\text{J/K})(400\,\text{K})}{5.35 \times 10^{-26}\,\text{kg}}} = 454\,\text{m s}^{-1}.$$

The Maxwell's speed distribution of oxygen molecules is (Eq. 19.14),

$$N_v = 4\pi N\left(\frac{m}{2\pi k_BT}\right)^{3/2} v^2 e^{-mv^2/(2k_BT)},$$

where m is mass of oxygen molecule (5.35×10^{-26} kg), N is number of molecules [$N = \mu N_A = (1.00\,\text{mol})(6.02 \times 10^{23}\,\text{mol}^{-1}) = 6.02 \times 10^{23}$], and $T = 400$ K is the temperature of the gas.

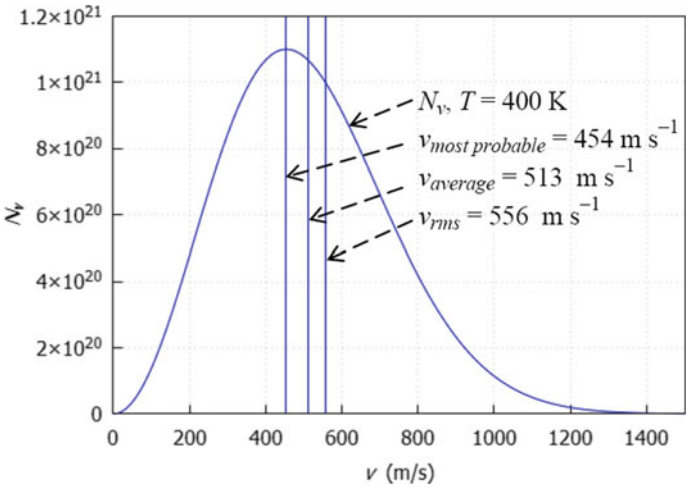
Substituting all the known numerical values, we have,

$$N_v = 4\pi(6.02 \times 10^{23})\left(\frac{5.35 \times 10^{-26}\,\text{kg}}{2\pi(1.38 \times 10^{-23}\,\text{J/K})(400\,\text{K})}\right)^{3/2} \times v^2 \exp\{-(5.35 \times 10^{-26}\,\text{kg})v^2/[2(1.38 \times 10^{-23}\,\text{J/K})(400\,\text{K})]\}.$$

- Plot of N_v against v for $0 \le v \le 1500$ m s^{-1} by wxMaxima:

```
(%i6) fpprintprec:5; ratprint:false; kB:1.38e-23; T:400;
m:5.35e-26; N:6.02e23;
(fpprintprec) 5
```

```
(ratprint) false
(kB) 1.38*10^-23
(t) 400
(m) 5.35*10^-26
(N) 6.02*10^23
(%i7) v_rms: float(sqrt(3*kB*T/m));
(v_rms) 556.36
(%i8) v_average: float(sqrt(8*kB*T/(%pi*m)));
(v_average) 512.58
(%i9) v_most_probable: float(sqrt(2*kB*T/m));
(v_most_probable) 454.26
(%i10) Nv: float(4*%pi*N*(m/(2*%pi*kB*T))^(3/2)*v^2*exp(-
m*v^2/(2*kB*T)));
(Nv) (1.4493*10^16*v^2)/2.7183^(4.846*10^-6*v^2)
(%i11) wxdraw2d( explicit(Nv, v, 0, 1500),
parametric(v_most_probable, t, t, 0, 1.2e21),
parametric(v_average, t, t, 0, 1.2e21),
parametric(v_rms, t, t, 0, 1.2e21),
grid = true,
xlabel = "{/Helvetica-Italic v} (m/s)",
ylabel = "{/Helvetica-Italic N_v}");
(%i12) N_test: float(integrate(Nv, v,0,1500));
(N_test) 6.0196*10^23
```



Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of k_B, T, m, and N.
(%i7), (%i8), (%i9) Calculate v_{rms}, $v_{average}$, and $v_{most\ probable}$.
(%i10) Define Nv as a function of v.
(%i11) Plot N_v against v for $0 \leq v \leq 1500$ m s^{-1} and draw vertical lines to mark v_{rms}, $v_{average}$, and $v_{most\ probable}$.
(%i12) The area under the curve should be $N = 6.02 \times 10^{23}$. We integrate N_v with respect to v from 0 to 1500 and obtained a close value of N_test = 6.0196 $\times 10^{23}$.

19.3 Summary

- The ideal gas law gives the relationship between pressure, volume, number of molecules, and temperature of a gas as,

$$pV = \mu RT = Nk_BT.$$

- Kinetic theory of a gas models the properties of a gas in terms of random motion of the molecules. The ideal gas law in terms of kinetic theory is,

$$pV = \frac{2}{3}N\left(\frac{1}{2}m\overline{v^2}\right) = \frac{1}{3}Nm\overline{v^2}.$$

- The temperature of a gas is proportional to average translational kinetic energy of the molecules and the mass of the molecule,

$$T = \frac{2}{3k_B}\left(\frac{1}{2}m\overline{v^2}\right).$$

19.4 Exercises

Exercise 19.1 A 0.025 m^3 tank contains 0.084 kg of nitrogen gas at a pressure of 4.00 atm. Find the temperature of the gas. Molar mass of nitrogen gas is 28.0 $\times$ 10^{-3} kg mol^{-1}. 1 atm = 1.013 $\times$ 10^5 Pa.
(Answer: 406 K)

Exercise 19.2 A balloon contains 500 m^3 of helium gas at 25°C and 1.0 atm pressure. What is the volume of the gas in the balloon at an altitude of 5500 m, where the temperature is −3.0°C and pressure is 0.50 atm? 1 atm = 1.013 $\times$ 10^5 Pa.
(Answer: 910 m^3)

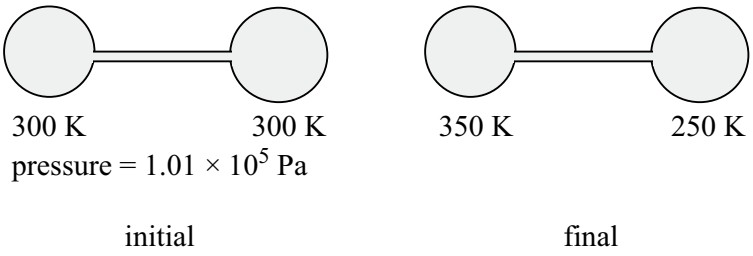


Fig. 19.3 Two flasks filled with a gas in initial and final states, Exercise 19.4

Exercise 19.3 A container contains 1.00 kmol of helium gas at 300 K. Calculate the number of helium atoms with speeds 400 to 403 m s^{-1}. The molar mass of helium is 4.00×10^{-3} kg mol^{-1}.

(Answer: 4.1×10^{23})

Exercise 19.4 Two identical flasks at temperature of 300 K are connected by a tube so that the pressures of the trapped gas in both flasks are the same at 1.01×10^5 Pa, as shown in Fig. 19.3. Later, one flask in heated and maintained at 350 K while the other is cooled and maintained at 250 K. What is the final pressure in the flasks?

(Answer: 9.82×10^4 Pa)

Exercise 19.5 A container contains hydrogen gas at a temperature of 400 K. What is the root mean square speed of the hydrogen molecules?

(Answer: 2230 m s^{-1})

Chapter 20
Thermodynamics

20.1 Basic Concepts and Formulae

(1) Heat, Q, is the transfer of energy from one body to another due to the temperature difference between the bodies. The SI unit of Q is joule (J).
(2) The internal energy, U, of a system represents its total energy content. It is the sum of all atomic and molecular energies in the system. The SI unit of U is J.
(3) The work ΔW performed by a system is positive if it loses energy to the surrounding environment. Work performed by a fluid under pressure p and expanding by a small volume ΔV is equal to:

$$\Delta W = p\ \Delta V. \tag{20.1}$$

The SI unit of ΔW is J.
(4) The first law of thermodynamics states that if heat ΔQ is given to a system, it will be transferred into the internal energy of the system ΔU and/or to do work ΔW by the system on the environment,

$$\Delta Q = \Delta U + \Delta W. \tag{20.2}$$

The law is a statement of conservation of energy.
(5)

(a) An isothermal process is a process occurring at a constant temperature. For an ideal gas undergoing an isothermal process, $\Delta U = 0$, and the first law becomes,

$$\Delta Q = \Delta W. \tag{20.3}$$

If the ideal gas changes from state (p_1, V_1, T) to state (p_2, V_2, T) in an isothermal process, then,

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W. M. S. Wan Hassan, *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-42678-0_20

$$\begin{aligned}\Delta Q = \Delta W &= \mu RT \ln\left(\frac{V_2}{V_1}\right) \\ &= p_1 V_1 \ln\left(\frac{V_2}{V_1}\right) \\ &= p_2 V_2 \ln\left(\frac{V_2}{V_1}\right),\end{aligned} \tag{20.4}$$

where μ, p, V, and T are the number of mol, pressure, volume, and temperature of the gas, respectively.

(b) An adiabatic process is a process occurring with no heat transfer to or from the system. For an adiabatic process, $\Delta Q = 0$, and the first law becomes,

$$0 = \Delta U + \Delta W. \tag{20.5}$$

For an ideal gas that changes adiabatically from state (p_1, V_1, T_1) to state (p_2, V_2, T_2),

$$\begin{aligned} p_1 V_1^{\gamma} &= p_2 V_2^{\gamma}, \\ p_1^{(1/\gamma)-1} T_1 &= p_2^{(1/\gamma)-1} T_2, \\ T_1 V_1^{\gamma-1} &= T_2 V_2^{\gamma-1}, \end{aligned} \tag{20.6}$$

where $\gamma = c_p/c_V$, c_p is the molar specific heat capacity at constant pressure and c_V is the molar specific heat capacity at constant volume, while p, V, and T are pressure, volume, and temperature, respectively.

(c) An isobaric process is a process occurring at a constant pressure. For an ideal gas undergoing an isobaric process,

$$\Delta Q = \mu c_p \Delta T, \tag{20.7}$$

$$\Delta W = p\ \Delta V. \tag{20.8}$$

where μ, c_p, and ΔT are the number of mol, molar specific heat capacity at constant pressure, and change in temperature, respectively, while p and ΔV are the pressure and change in volume, respectively.

(d) An isochoric process is a process occurring at a constant volume. For an ideal gas,

$$\Delta Q = \Delta U = \mu c_V \Delta T, \tag{20.9}$$

$$\Delta W = 0. \tag{20.10}$$

where μ, c_V, and ΔT are the number of mol, molar specific heat capacity at constant volume, and change in temperature, respectively,

(6) A heat engine is a device that changes heat to other forms of useful energy. Net work done by a heat engine in bringing a work material through a cyclic process ($\Delta U = 0$) is,

$$W = Q_h - Q_c, \tag{20.11}$$

where Q_h is the heat absorbed from a hot reservoir, and Q_c is the heat transferred to a cold reservoir. The thermal efficiency of a heat engine e, is defined as the ratio of net work done by the engine to the heat absorbed in each cycle,

$$e = \frac{W}{Q_h} = 1 - \frac{Q_c}{Q_h}. \tag{20.12}$$

(7) There are several ways to state the second law of thermodynamics

(a) No heat engine operating in a cycle can absorb heat from a heat reservoir and convert it all to work (Kelvin-Planck statement).
(b) It is impossible to build a cyclic heat engine that continuously transfers heat from a body to another at a higher temperature (Clausius statement).
(c) When a system undergoes a spontaneous change, its entropy either increases or remains constant.

(8) A process is reversible if the system changes from its initial to its final states through a series of equilibrium states.
(9) A process is irreversible if the system and its surrounding cannot be restored to their initial states.
(10) The efficiency of a heat engine operating in a Carnot cycle is,

$$e_C = 1 - \frac{T_c}{T_h}, \tag{20.13}$$

where T_c represents the absolute temperature of a cold reservoir and T_h is the absolute temperature of a hot reservoir.
(11) According to the second law of thermodynamics, when a real (irreversible) process occurs, the degree of disorder increases. The degree of disorder is called entropy, S.
(12) When a system changes between two equilibrium states, the infinitesimal change in entropy, dS, is,

$$dS = \frac{dQ}{T}, \tag{20.14}$$

and change in entropy, ΔS, is,

$$\Delta S = S_{final} - S_{init} = \int_{init}^{final} \frac{dQ}{T}. \tag{20.15}$$

The SI unit of entropy is J K^{-1}.

20.2 Problems and Solutions

Problem 20.1 Heat of 7000 calorie is given to a system and the system does 4000 J of work. What is the change in internal energy of the system?

Solution

Applying the first law of thermodynamics (Eq. 20.2), we have,

$$\begin{aligned} \Delta Q &= \Delta U + \Delta W, \\ (7000\,\text{cal})(4.18\,\text{J/cal}) &= \Delta U + 4000\,\text{J}, \\ \Delta U &= 25,260\,\text{J}, \end{aligned}$$

where for the unit conversion, 1 cal = 4.18 J, was used. The internal energy of the system increases by 25,260 J.

- wxMaxima codes:

```
(%i4)fpprintprec:5;     ratprint:false;     dQ:7000*4.18;
dW:4000;
(fpprintprec)5
(ratprint)false
(dQ)2.926*10^4
(dW)4000
(%i5)solve(dQ = dU + dW, dU);
(%o5)[dU = 25260]
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of ΔQ and ΔW.
(%i5) Solve $\Delta Q = \Delta U + \Delta W$ for ΔU.

Problem 20.2 What is the increase in the internal energy of 10 g of ice at 0°C as it melts into water at the same temperature? The change in volume of the phase change is negligible. The latent heat of fusion of ice is 3.336×10^5 J kg^{-1}.

Solution
Heat needed to melt the ice is (Eq. 5.4),

$$\begin{aligned}\Delta Q &= mL \\ &= (10 \times 10^{-3}\,\text{kg})(3.336 \times 10^5\,\text{J/kg}) \\ &= 3336\,\text{J}.\end{aligned}$$

where L is the latent heat of fusion of ice. The ice does not do any external work during the melting, that is, $\Delta W = 0$. Thus,

$$\begin{aligned}\Delta Q &= \Delta U + \Delta W, \\ 3336\,\text{J} &= \Delta U + 0, \\ \Delta U &= 3336\,\text{J}.\end{aligned}$$

The increase in internal energy is 3336 J.

- wxMaxima codes:

```
(%i4)fpprintprec:5; ratprint:false; m:10e-3; L:3.336e5;
(fpprintprec)5
(ratprint)false
(m)0.01
(L)3.336*10^5
(%i5)dQ: m*L;
(dQ)3336.0
(%i6)dW: 0;
(dW)0
(%i7)solve(dQ = dU + dW, dU);
(%o7)[dU = 3336]
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of m and L.
(%i5) Calculate ΔQ.
(%i6) Assign ΔW.
(%i7) Solve $\Delta Q = \Delta U + \Delta W$ for ΔU.

Problem 20.3 An iron cube with 6.0 cm sides and of mass 1700 g is heated from 20 to 320°C. What is its internal energy increase? Specific heat capacity of iron is 0.11 cal g^{-1} K^{-1}, thermal coefficient of volume expansion of iron is 3.6×10^{-5} K^{-1}, and the atmospheric pressure is 1.01×10^5 Pa.

Solution
Heat received by the iron cube is (Eq. 5.3),

$$\begin{aligned}\Delta Q &= mc\Delta\theta \\ &= (1700\,\mathrm{g})(0.11\,\mathrm{cal\,g^{-1}\,K^{-1}})(320-20)\,\mathrm{K}\times(4.18\,\mathrm{J/cal}) \\ &= 234,500\,\mathrm{J},\end{aligned}$$

where for the unit conversion, 1 cal = 4.18 J, was used.

The work done by the cube as it expands is,

$$\begin{aligned}\Delta W &= p\Delta V \\ &= p\gamma V\Delta\theta \\ &= (1.01\times10^5\,\mathrm{N/m^2})(3.6\times10^{-5}\,\mathrm{K^{-1}})(0.06\,\mathrm{m})^3(320-20)\,\mathrm{K} \\ &= 0.236\,\mathrm{J}.\end{aligned}$$

Here, the work done is the pressure times the change in volume. The change in volume is $\Delta V = \gamma V\Delta\theta$ (Eq. 4.6), that is, the change in volume is the product of thermal coefficient of volume expansion, original volume, and change in temperature.

Applying the first law of thermodynamics (Eq. 20.2), the increase in internal energy of the iron cube is,

$$\begin{aligned}\Delta U &= \Delta Q - \Delta W \\ &= 234,500\,\mathrm{J} - 0.236\,\mathrm{J} \\ &= 234,500\,\mathrm{J}.\end{aligned}$$

- wxMaxima codes:

```
(%i1)fpprintprec:5;
(fpprintprec)5
(%i2)dQ: 1700*0.11*(320-20)*4.18;
(dQ)2.345*10^5
(%i3)dW: 1.01e5*3.6e-5*0.06^3*(320-20);
(dW)0.23561
(%i4)dU: dQ-dW;
(dU)2.345*10^5
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2), (%i3), (%i4) Calculate ΔQ, ΔW, and ΔU.

Problem 20.4 One kilogram of nitrogen gas is heated and its temperature rises from 10 to 100°C under a constant gas pressure. What are the increase in internal energy of the gas and the work done by the gas? For nitrogen gas, specific heat capacity at constant volume $c_v = 741\,\mathrm{J\,kg^{-1}\,K^{-1}}$ and specific heat capacity at constant pressure $c_p = 1038\,\mathrm{J\,kg^{-1}\,K^{-1}}$.

Solution

Heat given to the nitrogen gas is (Eq. 5.3),

$$\begin{aligned}(\Delta Q)_p &= mc_p\Delta T\\ &= (1.0\,\mathrm{kg})(1038\,\mathrm{J\,kg^{-1}\,K^{-1}})(90\,\mathrm{K})\\ &= 93,420\,\mathrm{J}.\end{aligned}$$

According to the first law of thermodynamics, the heat is to increase the internal energy of the gas ΔU and to do work on the environment ΔW, Eq. (20.2),

$$(\Delta Q)_p = \Delta U + \Delta W.$$

The increase in internal energy of the gas is (Eq. 5.3),

$$\begin{aligned}\Delta U = (\Delta Q)_v &= mc_v\Delta T\\ &= (1.0\,\mathrm{kg})(741\,\mathrm{J\,kg^{-1}\,K^{-1}})(90\,\mathrm{K})\\ &= 66,690\,\mathrm{J}.\end{aligned}$$

The work done by the nitrogen gas is,

$$\begin{aligned}\Delta W &= (\Delta Q)_p - \Delta U\\ &= 93,420\,\mathrm{J} - 66,690\,\mathrm{J}\\ &= 26,730\,\mathrm{J}.\end{aligned}$$

- wxMaxima codes:

```
(%i5)fpprintprec:5; m:1; deltaT:100-10; cv:741; cp:1038;
(fpprintprec)5
(m)1
(deltaT)90
(cv)741
(cp)1038
(%i6)dQ: m*cp*deltaT;
(dQ)93420
(%i7)dU: m*cv*deltaT;
(dU)66690
(%i8)dW: dQ-dU;
```

```
(dW)26730
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of ΔT, c_v, and c_p.
(%i6), (%i7), (%i8) Calculate ΔQ, ΔU, and ΔW.

Problem 20.5 What is the work done by an ideal gas as it expands isothermally from volume of 5.0 m^3 and pressure of 10 atm to 7.0 m^3? (1 atm = 1.01×10^5 Pa).

Solution
For isothermal expansion of an ideal gas, the work done by the gas is (Eqs. 20.4 and 19.3),

$$\Delta W = \int p\, dV = \int_{V_1}^{V_2} \mu RT \frac{dV}{V} = \int_{V_1}^{V_2} p_1 V_1 \frac{dV}{V} = p_1 V_1 [\ln V]_{V_1}^{V_2} = p_1 V_1 \ln\left(\frac{V_2}{V_1}\right)$$
$$= (10 \times 1.01 \times 10^5\ \mathrm{N/m^2})(5.0\ \mathrm{m^3}) \ln\left(\frac{7.0\ \mathrm{m^3}}{5.0\ \mathrm{m^3}}\right)$$
$$= 1.7 \times 10^6\ \mathrm{J}.$$

- wxMaxima codes:

```
(%i1)fpprintprec:5;
(fpprintprec)5
(%i3)dW: 10*1.01e5*5*log(7/5); float(%);
(dW)5.05*10^6*log(7/5)
(%o3)1.6992*10^6
(%i6)V1: 5; p1:10*1.01e5; V2:7;
(V1) 5
(p1)1.01*10^6
(V2) 7
(%i8)dW: integrate( p1*V1/V, V, V1, V2); float(%);
(dW)5.05*10^6*(log(7)-log(5))
(%o8)1.6992*10^6
```

Comments on the codes:

(%i3) Calculate ΔW.
(%i6) Assign values of V_1, p_1, and V_2.

(%i8) Calculate $\Delta W = \int_{V_1}^{V_2} p_1 V_1 \frac{dV}{V}$.

Problem 20.6 One mol of an ideal gas is compressed isothermally at 300 K from volume V_0 to $V_0/3$. What is the work done by the gas?

Solution

Work done by an ideal gas is (Eq. 20.1),

$$W = \int p\, dV.$$

For an ideal gas undergoing an isothermal process, the work done by the gas is (Eq. 20.4),

$$\begin{aligned} W = \int p\, dV &= \int_{V_1}^{V_2} \frac{\mu RT}{V} dV = \mu RT [\ln V]_{V_1}^{V_2} = \mu RT \ln \frac{V_2}{V_1} \\ &= (1.0\,\text{mol})(8.31\,\text{J mol}^{-1}\,\text{K}^{-1})(300\,\text{K}) \ln\left(\frac{V_0/3}{V_0}\right) \\ &= -2740\,\text{J}. \end{aligned}$$

- wxMaxima codes:

```
(%i7)fpprintprec:5; ratprint:false; mu:1; R:8.31; T:300;
V1:V0; V2:V0/3;
(fpprintprec)5
(ratprint)false
(mu)1
(R) 8.31
(t) 300
(V1) V0
(V2) V0/3
(%i9)W: mu*R*T*log(1/3); float(%);
(W)-2493.0*log(3)
(%o9)-2738.8
(%i10) assume(V0 > 0);
(%o10) [V0 > 0]
(%i12) W: integrate( mu*R*T/V, V, V1, V2); float(%);
(W)-2493.0*(log(V0)-log(V0/3))
(%o12) -2493.0*(log(V0)-1.0*log(0.33333*V0))
(%i13) radcan(%);
(%o13) -2493*log(3)
(%i14) float(%);
(%o14) -2738.8
```

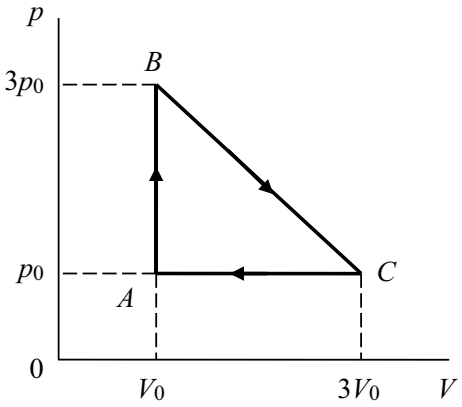

Fig. 20.1 p–V diagram of Problem 20.7

Comments on the codes:

(%i7) Set floating point print precision to 5, internal rational number print to false, assign values of μ, R, T, V_1, and V_2.
(%i9) Calculate W.
(%i10) Set $V_0 > 0$.
(%i12) Another calculation of work $W = \int_{V_1}^{V_2} \frac{\mu RT}{V} dV$.
(%i13) Simplify.
(%i14) Get the decimal value.

Problem 20.7 A gas undergoes a cyclic process $ABCA$ as in p–V diagram of Fig. 20.1. Determine the work done by the gas.

Solution

The work done by a gas is the area of triangle ABC (Eq. 20.1), that is,

$$\begin{aligned} W &= \frac{1}{2}(3V_0 - V_0)(3p_0 - p_0) \\ &= 2p_0V_0. \end{aligned}$$

Problem 20.8 Figure 20.2 shows two isothermal curves of helium gas for temperatures T_A and T_B. One mol of helium gas in state A, pressure of $p_A = 1.0$ atm, volume V_A and temperature $T_A = 273$ K, undergoes an isochoric process to state B with pressure reduced to $p_A/2$ and temperature to T_B.

(a) Calculate the work done by the gas in the AB process. What is the gas temperature at B?
(b) The helium gas then undergoes an isobaric expansion to volume $2V_A$. Calculate the work done by the gas in the BC process.
(c) Calculate the work done by the gas if it undergoes isothermal expansion from state A to C.

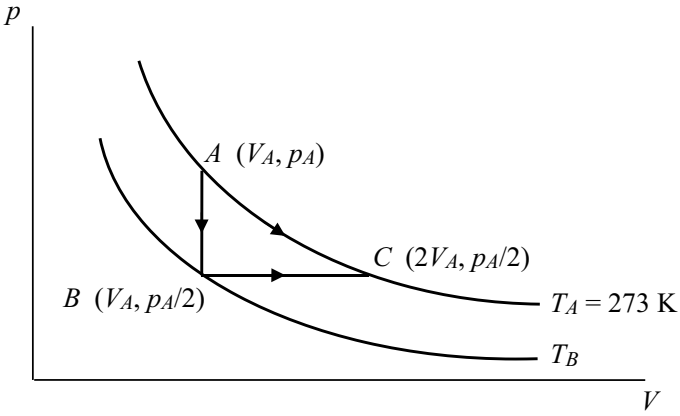

Fig. 20.2 Two isothermal curves of helium gas, Problem 20.8

Solution

(a) The work done by the gas in AB process is zero. The work $W_{AB} = 0$ because $W_{AB} = \int p\, dV$ and $dV = 0$. For a constant volume, the ideal gas equation is written as,

$$\frac{T}{p} = \text{constant} = \frac{T_A}{p_A} = \frac{T_B}{p_B}.$$

The temperature of the gas in state B is,

$$T_B = p_B \frac{T_A}{p_A} = \frac{p_A}{2}\frac{T_A}{p_A} = \frac{T_A}{2} = \frac{273\,\text{K}}{2} = 136\,\text{K}.$$

(b) The work done by the gas in BC process is,

$$W_{BC} = \int_{V_A}^{2V_A} p\, dV = \int_{V_A}^{2V_A} \frac{1}{2} p_A dV = \frac{1}{2} p_A V_A,$$

but,

$$p_A V_A = \mu R T_A.$$

The work done by the gas is,

$$W_{BC} = \frac{1}{2} p_A V_A = \frac{1}{2}\mu R T_A = \frac{1}{2}(1.0\,\text{mol})(8.31\,\text{J mol}^{-1}\,\text{K}^{-1})(273\,\text{K})$$
$$= 1134\,\text{J}.$$

(c) Along the isothermal curve, $T = T_A$, so $p = \mu RT_A/V$. The work done by the gas in AC process is,

$$\begin{aligned} W_{AC} &= \int_{V_A}^{2V_A} p\, dV = \mu RT_A \int_{V_A}^{2V_A} \frac{dV}{V} = \mu RT_A \ln(2) \\ &= (1.0\,\text{mol})(8.31\,\text{J K}^{-1}\,\text{mol}^{-1})(273\,\text{K})\ln(2) \\ &= 1572\,\text{J}. \end{aligned}$$

- wxMaxima codes:

```
(%i4)fpprintprec:5; mu:1; R:8.31; TA:273;
(fpprintprec)5
(mu)1
(R) 8.31
(Ta) 273
(%i6)TB: TA/2; float(%);
(Tb) 273/2
(%o6)136.5
(%i7)WBC: 0.5*mu*R*TA;
(Wbc) 1134.3
(%i9)WAC: mu*R*TA*log(2); float(%);
(WAC)2268.6*log(2)
(%o9)1572.5
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of μ, R, and T_A.
(%i6), (%i7), (%i9) Calculate T_B, W_BC, and W_AC.

Problem 20.9 A volume of 30.0 L of air at STP is isothermally compressed to 10.0 L. Calculate the heat flowing out of the air. STP means the standard temperature and pressure, namely the temperature of 273.15 K (0°C, 32°F) and the pressure of 1 atm (101 325 Pa).

Solution

The first law of thermodynamics (Eq. 20.2) becomes,

$$\Delta Q = \Delta U + \Delta W = \Delta W,$$

because in an isothermal process the temperature is constant and there is no change in the internal energy, $\Delta U = 0$. The work done by the gas is (Eq. 20.4),

$$\Delta W = \mu RT \ln(V_2/V_1) = p_1 V_1 \ln(V_2/V_1)$$

$$= (101,325\,\text{Pa})(30 \times 10^{-3}\,\text{m}^3)\ln(10/30)$$
$$= -3340\,\text{J}.$$

The heat that flows out of the gas is,

$$\Delta Q = \Delta W = -3340\,\text{J}.$$

- wxMaxima codes:

```
(%i4)fpprintprec:5; p1:101325; V1:30e-3; V2:10e-3;
(fpprintprec)5
(p1)101325
(V1) 0.03
(V2) 0.01
(%i6)dW: p1*V1*log(V2/V1); float(%);
(dW)-3339.5
(%o6)-3339.5
(%i7)dQ: dW;
(dQ)-3339.5
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of p_1, V_1, and V_2.
(%i6), (%i7) Calculate ΔW and ΔQ.

Problem 20.10 Helium gas of volume 1.0 m^3 at STP is cooled at constant pressure until its volume becomes 0.75 m^3. Specific heat capacity of helium at constant volume is $c_v = 3100\,\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}$. The mass of 1.0 mol of helium is 4.0 g. Helium is monatomic. Calculate,

(a) the final temperature,
(b) the heat that flows out of the gas. STP means the standard temperature and pressure, namely the temperature of 273.15 K (0°C, 32°F) and the pressure of 1 atm (101,325 Pa).

Solution

(a) Because the pressure is constant, the final temperature can be obtained as follows,

$$\frac{V_1}{T_1} = \frac{V_2}{T_2},$$

$$\frac{1.0\,\mathrm{m}^3}{273.15\,\mathrm{K}} = \frac{0.75\,\mathrm{m}^3}{T_2},$$
$$T_2 = 205\,\mathrm{K}.$$

- wxMaxima codes:

```
(%i2)fpprintprec:5; ratprint:false;
(fpprintprec)5
(ratprint)false
(%i4)solve(1/273.15=0.75/T2, T2)$ float(%);
(%o4)[T2=204.86]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $1/273.15 = 0.75/T_2$ for T_2.

(b) Applying the first law of thermodynamics (Eq. 20.2),

$$\begin{aligned}\Delta Q &= \Delta U + \Delta W = mc_v\Delta T + p\Delta V\\ &= \mu M c_v \Delta T + p\Delta V\\ &= \frac{1.0\,\mathrm{m}^3}{0.0224\,\mathrm{m}^3/\mathrm{mol}}(4.0\times10^{-3}\,\mathrm{kg/mol})(3100\,\mathrm{J\,kg^{-1}\,K^{-1}})(205\,\mathrm{K}-273.15\,\mathrm{K})\\ &\quad + (101,325\,\mathrm{Pa})(0.75\,\mathrm{m}^3 - 1.0\,\mathrm{m}^3)\\ &= -63,100\,\mathrm{J}.\end{aligned}$$

Here, the change in internal energy of the gas is $\Delta U = mc_v\Delta T$, where m is the mass of the gas, c_v is molar specific heat capacity at constant volume, and ΔT is change in temperature. The number of mol is $\mu = (1.0\ \mathrm{m}^3)/(0.0224\ \mathrm{m}^3/\mathrm{mol})$ because the volume of 1 mol of any gas at STP is $0.0224\ \mathrm{m}^3$. The molar mass of helium is $M = 4.0\times10^{-3}\ \mathrm{kg\ mol^{-1}}$. The work by the gas is $\Delta W = p\Delta V$, where p is the pressure and ΔV is change in volume. The heat that flows out of the helium gas is $-$ 63,100 J.

- wxMaxima codes:

```
(%i1)fpprintprec:5;
(fpprintprec)5
(%i2)dQ:   1/0.0224*4e-3*3100*(205-273.15)+101325*(0.75-
1);
```

```
(dQ)-6.3057*10^4
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2) Calculate ΔQ.

Problem 20.11 Obtain the relationship between the pressure p and the volume V of a monatomic ideal gas undergoing an adiabatic process.

Solution

The internal energy of μ mol of a monatomic ideal gas at temperature T is (Eq. 19.7),

$$U = \frac{3}{2}\mu RT = \frac{3}{2}Nk_BT,$$

where the number of atom is $N = \mu N_A$ and k_B is the Boltzmann constant. We write,

$$dU = \frac{3}{2}Nk_B dT.$$

The pressure of the gas is (Eq. 19.7),

$$p = \frac{2U}{3V} = \frac{Nk_BT}{V}.$$

The first law of thermodynamics for an adiabatic process becomes,

$$\begin{aligned} dQ &= dU + dW = dU + p\,dV, \\ 0 &= \frac{3}{2}Nk_B dT + \frac{Nk_BT}{V}dV, \\ \frac{dT}{T} &+ \frac{2}{3}\frac{dV}{V} = 0. \end{aligned}$$

Integrating the equation gives,

$$\begin{aligned} \ln T + \frac{2}{3}\ln V &= \text{constant}, \\ TV^{2/3} &= \text{constant}. \end{aligned}$$

But $T = pV/(\mu R)$. The equation becomes,

$$pV^{5/3} = \text{constant}.$$

20.3 Summary

- The first law of thermodynamics states that the increase in the internal energy of a thermodynamic system is the heat given to the system minus the work by the system,

$$\Delta U = \Delta Q - \Delta W.$$

- For isothermal process, $\Delta U = 0, \Delta Q - \Delta W$. For adiabatic process, $\Delta Q = 0, \Delta U = -\Delta W$. For isobaric process, $\Delta U = \Delta Q - \Delta W$. For isochoric process, $\Delta W = 0, \Delta U = \Delta Q$.
- The work done by a heat engine is the difference between the heat absorbed from the hot reservoir and the heat discharged to the cold reservoir. The ratio of the work done and the heat absorbed from the hot reservoir is the efficiency of the heat engine,

$$e = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h}.$$

- The Carnot cycle is the most efficient engine for a reversible cycle operating with reservoirs of temperature T_h and T_c,

$$e_C = 1 - \frac{T_c}{T_h}.$$

- The infinitesimal change in entropy for a reversible process at constant temperature is equal to the heat divided by the temperature,

$$dS = \frac{dQ}{T}.$$

20.4 Exercises

Exercise 20.1 A cylinder fitted with a piston contains 0.100 mol of air at temperature of 300 K, as shown in Fig. 20.3. The piston is pushed slowly that the temperature of the air remains the same and the volume of the air becomes one half of the initial volume. Find the work done by the air.

(Answer: -173 J)

Exercise 20.2 One hundred liter of nitrogen gas at a pressure of 1.0 atm expands adiabatically into a volume of 250 L and a new pressure. What is the new pressure? For nitrogen gas $\gamma = 1.4$.

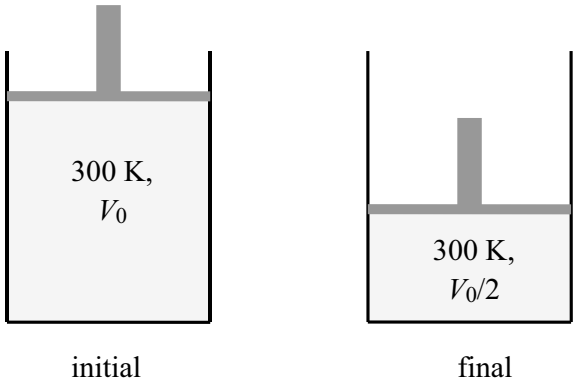

Fig. 20.3 Air trapped in cylinder and piston in initial and final states, Exercise 20.1

(Answer: 0.28 atm)

Exercise 20.3 One kilogram of water at a temperature of 100°C and pressure of 1.00 atm with a volume of 1.00×10^{-3} m^3 is converted into 1.700 m^3 of steam. Calculate the increase in the internal energy when the liquid changes into steam. The latent heat of vaporization of water is 2.26×10^6 J kg^{-1} and 1.00 atm $= 1.013 \times 10^5$ Pa.

(Answer: 2.09×10^6 J)

Exercise 20.4 A Carnot engine operates between 600 and 350 K. What is its efficiency?

(Answer: 42%)

Exercise 20.5 An iron bar of mass 5.00 kg at a temperature of 600 K is submerged in a pool of water at 300 K. Calculate the change in entropy of the iron, water, and the universe. Specific heat capacity of iron is 440 J kg^{-1} K^{-1}.

(Answer: $-$ 1525 J K^{-1}, 2200 J K^{-1}, 675 J K^{-1})

Appendix A
Introduction to wxMaxima

wxMaxima is an open computer algebra system software that can be installed on *Microsoft Windows* operating system, as well as on *Linux* and *OS X* operating systems. Other popular computer algebra systems are *Maple* and *Mathematica.* wxMaxima is a document based interface for the computer algebra system called *Maxima. Maxima* was developed from the *Macsyma* project since 1982 by the Department of Energy of the USA. This means wxMaxima gives menu and dialogue for various commands, plots, and animations of *Maxima.*

wxMaxima is distributed under *GNU General Public License.*

wxMaxima can be downloaded and installed on your pc from:
https://sourceforge.net/projects/maxima/files/Maxima-Windows/.

A manual of wxMaxima can be read from:
http://maxima.sourceforge.net/docs/manual/en/maxima.html.

A short and useful tutorial to start using wxMaxima can be obtained from:
http://math-blog.com/2007/06/04/a-10-minute-tutorial-for-solving-math-problems-with-maxima/.

In this book wxMaxima version 5.43.0 on *Microsoft Windows* was used. The installer file was maxima-clisp-sbcl-5.43.0-win64.exe. Older or newer versions wxMaxima would give minor changes in output display, but the calculation output should be almost the same.

Using wxMaxima

Figure A.1 shows wxMaxima window when it is started. On top of the window are *File*, *Edit*, *View*, *Cell*, *Maxima*, *Equations*, *Algebra*, *Calculus*, *Simplify*, *List*, *Plot*, *Numeric*, and *Help* menus. Under these menus, are other menus in icons. Discussion on using these menus is not done in this appendix.

To give a command, type the command, type ; to end it, and simultaneously press <shift> and <enter> keys to execute it. wxMaxima will display its response or output. To exit wxMaxima, press <ctrl-Q>, click *File* menu and choose *Exit*, or click the cross icon on the top right of the wxMaxima window.

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W. M. S. Wan Hassan, *Physics—Problems, Solutions, and Computer Calculations*,
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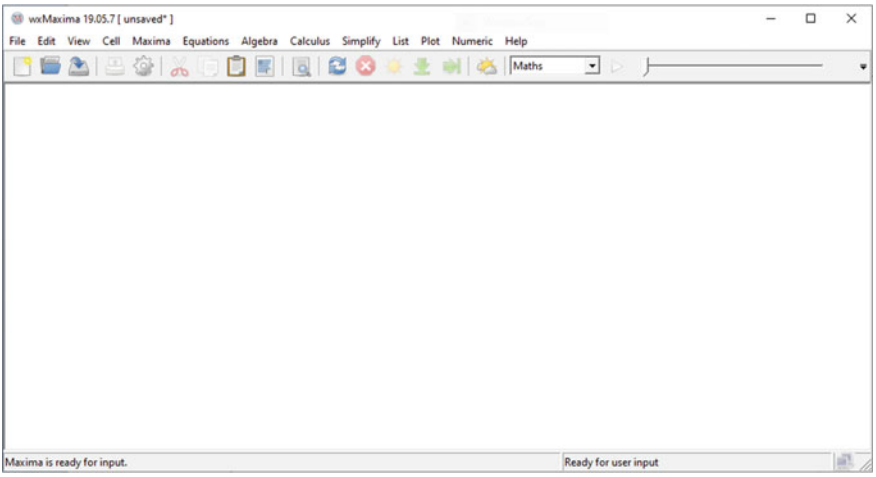

wxMaxima 19.05.7 [unsaved*]
File Edit View Cell Maxima Equations Algebra Calculus Simplify List Plot Numeric Help
Maths
Maxima is ready for input.
Ready for user input

Fig. A.1 wxMaxima window

For example, you are to calculate $(3 - 0.5) \times 6.54^3$ and plot the line $y = 2x + 3$ for $-5 \leq x \leq 5$. Type (3 – 0.5) * 6.54^3; and simultaneously press <shift> and <enter>. wxMaxima will display its result of calculation. Next, type y: 2 * x + 3; and simultaneously press <shift> and <enter> keys to define the line. Lastly, type wxplot2d(y, [x,−5,5]); and simultaneously press <shift> and <enter> again to instruct wxMaxima plot the line. Figure A.2 shows the wxMaxima window after the three commands were executed.

This appendix briefly discusses how to do the calculations as in this book. These are small set of calculations that wxMaxima can perform. Readers must study various sources about wxMaxima or Maxima from the internet for further applications.

Simple Calculations

To calculate 2×5, type 2*5; and simultaneously press <shift> and <enter> keys. For division, addition, and subtraction, use /, +, −. The result of calculating 2×5, $2 \div 5$, $2 + 5$, $2 - 5$ is as follows,

```
(%i1)  2*5;
(%o1)  10
(%i2)  2/5;
(%o2)  2/5
(%i3)  2+5;
(%o3)  7
(%i4)  2-5;
(%o4)  -3
```

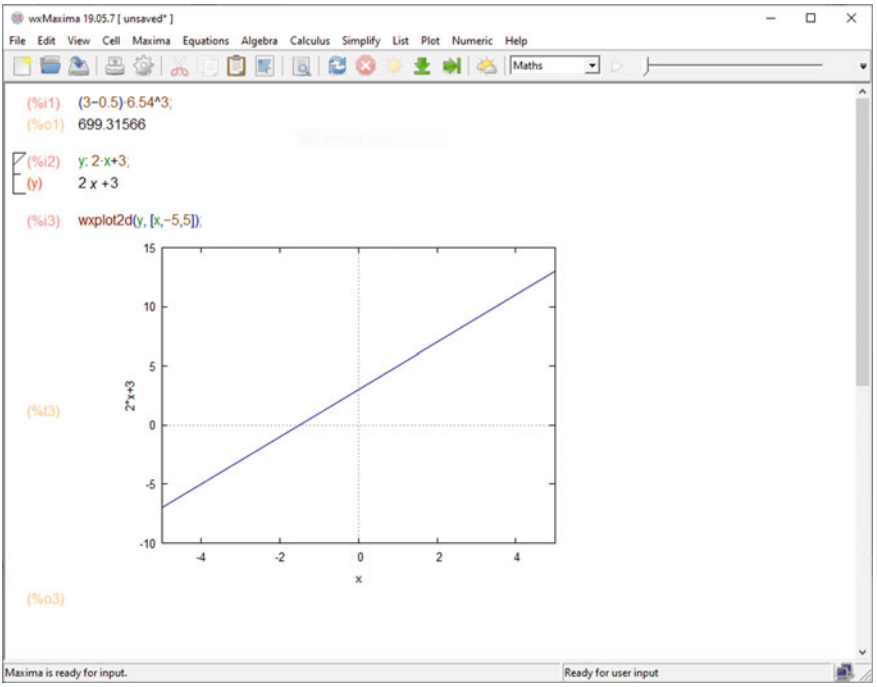


Fig. A.2 wxMaxima input and output in the window

wxMaxima tags the input typed by user as (%i---) and the output as (%o---). These tags could be used for further calculation. The last output is tagged as %. Typing %; will give -3, typing %o1; will give 10, and typing %o3; will give 7.

```
(%i5)  %;
(%o5)  -3
(%i6)  %o1;
(%o6)  10
(%i7)  %o3;
(%o7)  7
```

To hide the output type $ instead of ; at the end of a command, followed by simultaneously pressing <shift> and <enter> keys. wxMaxima executes the command and will not display the output. Other command to hide the output is *ratprint:false;*. This command will suppress display of output related to internal rational number calculation of wxMaxima. Command to limit number of digits of numerical value that is displayed is *fpprintprec,* floating point print precision. For example, the command *fpprintprec*:5; will display only 5 significant digits.

To calculate $\sqrt{3}$, type sqrt(3); and simultaneously press <shift> and <enter> keys. To get its decimal value type float(%); and simultaneously press <shift> and <enter> keys again, the result is,

```
(%i1)  sqrt(3);
(%o1)  sqrt(3)
(%i2)  float(%);
(%o2)  1.732050807568877
(%i3)  fpprintprec:5;
(fpprintprec)    5
(%i4)  float(%o2);
(%o4)  1.7321
```

To input 2^{10}, type 2^10; to input 3×10^8, type 3e8; and to input 1.6×10^{-19}, type 1.6e−19. To calculate $\frac{2^{10}}{3\times 10^8} \times 1.6 \times 10^{-19}$ for example, we only have to type 2^10/ 3e8 * 1.6e−19; and the result is,

```
(%i1)  2^10/3e8*1.6e-19;
(%o1)  5.461333333333334*10^-25
```

Let us say you want to change 2^{10} to 2^9 in the calculation. Use the computer mouse to go to 2^10, and do the editing to replace 10 with 9 using <del> or <backspace> and 9, followed by simultaneous press of <shift> and <enter> keys. The results is,

```
(%i2) 2^9/3e8*1.6e-19;
(%o2) 2.7306666666666667*10^-25
```

This way of editing and recalculation is very useful to correct typos and to recalculate. We do not have to retype the whole input, just correct the typos and simultaneously press <shift> and <enter> keys. This is an advantage of using a software or an apps as opposed to using a calculator to do calculations.

Restart

To start a new calculation, click *Maxima* menu at top of the window, and choose *Restart Maxima.* This will clear the wxMaxima memory and we can start a new session of the calculation. We will always *Restart Maxima* for a new calculation.

Assignment

To assign a value of 3 to m, i.e. $m = 3$, type m: 3; To assign $a = 11$, type a: 11; Do not forget to simultaneously press <shift> and <enter> for each command. There after m and a are always in the memory of the computer and can be used for further

calculation. To calculate $F = ma$ for example, type F: m * a; simultaneously press <shift> and <enter>.

```
(%i2)  m: 3; a: 11;
(m)    3
(a)    11
(%i3)  F: m*a;
(F)    33
```

We frequently use this method in solving, calculating, and checking physics problems in this book.

Substitution

The values of m and a in the previous example, can also be substituted into the formula $F = ma$ by function *subst* as follows:

```
(%i1)  F: m*a;
(F)    a*m
(%i2)  subst([m=3, a=11], F);
(%o2)  33
```

or more simply,

```
(%i1)  F: subst([m=3, a=11], m*a);
(F)    33
```

Function Definition

To define a function use :=. For example, define $g(x) = 2x^2 + x - 3$ and calculate $g(4)$. This is performed by wxMaxima as follows,

```
(%i1)  g(x) := 2*x^2 + x - 3;
(%o1)  g(x):=2*x^2+x-3
(%i2)  g(4);
(%o2)  33
```

Define

Another way to define a function is by predefined function *define*. Arguments of *define* are the function and its definition. The previous example can be realized as follows:

```
(%i1)  definition: 2*x^2 + x - 3;
(definition)    2*x^2+x-3
(%i2)  define(g(x), definition);
(%o2)  g(x):=2*x^2+x-3
(%i3)  g(4);
(%o3)  33
```

Solving an Equation

To solve an equation, use wxMaxima built in function called *solve*. For example, solve $12x + 3 = 5$, i.e. find x that satisfies the equation. Type solve (12 * x + 3 = 5,x); simultaneously press <shift> and <enter> keys, and wxMaxima gives $x = 1/6$ as a solution,

```
(%i1)  solve(12*x+3=5,x);
(%o1)  [x=1/6]
```

To use *solve* two arguments are needed, the first is the equation "12 * x + 3 = 5" and the second is the unknown variable to be found "x". The *solve* built in function is frequently used in this book.

The output of *solve* is a *list* as indicated by the square bracket […]. In this example, x is not yet assigned the value of 1/6. To pick the value 1/6 from a *list*, the right hand side *rhs*(…) built in function is useful. Thus, to solve the equation and assign the solution as x, the codes are,

```
(%i1)  solve(12*x + 3 = 5, x);
(%o1)  [x=1/6]
(%i2)  x: rhs(%o1[1]);
(x)   1/6
```

To solve a quadratic equation $5y^2 + y - 6 = 0$, type solve (5 * y^2 + y – 6 = 0, y); and simultaneously press <shift> and <enter> keys. The result is,

```
(%i1)  solve( 5*y^2 + y - 6 = 0, y);
(%o1)  [y=-6/5,y=1]
```

Therefore, the solutions of the quadratic equation are $y = -6/5$ and $y = 1$.

To solve the quadratic equation and assign the solutions as y_1 and y_2, the codes are,

```
(%i1)  solve( 5*y^2 + y - 6 = 0, y);
(%o1)  [y=-6/5,y=1]
(%i3)  y1: rhs(%o1[1]); y2: rhs(%o1[2]);
(y1)   -6/5
(y2)   1
```

Solving Simultaneous Equations

To solve simultaneous equations,

$$2x + 5y = 11,$$
$$x - 4y = 7,$$

type solve ([2 * x + 5 * y = 11, x − 4 * y = 7], [x, y]); and simultaneously press <shift> and <enter> keys. The result is,

```
(%i1)  fpprintprec:5;
(fpprintprec)      5
(%i3)  solve( [2*x+5*y=11, x-4*y=7], [x,y]); float(%);
(%o2)  [[x=79/13,y=-3/13]]
(%o3)  [[x=6.0769,y=-0.23077]]
```

Here, the first argument of *solve* is a *list* of two equations [2 * x + 5 * y = 11, x − 4 * y = 7], and the second argument is a *list* of two variables [x, y] that are to be calculated. The solutions of the system of equations are $x = 6.07\ldots$ and $y = -0.230\ldots$

To assign x and y the values of the solutions, the codes can be as follows,

```
(%i1)  fpprintprec:5;
(fpprintprec)     5
(%i3)  solve( [2*x+5*y=11, x-4*y=7], [x,y]); float(%);
(%o2)  [[x=79/13,y=-3/13]]
(%o3)  [[x=6.0769,y=-0.23077]]
(%i5)  x: rhs(%o3[1][1]); y: rhs(%o3[1][2]);
(x)    6.0769
(y)    -0.23077
```

As another example, solve the following system of equations,

$$3x + y - z = 0,$$
$$2x - 3y + z = 1,$$
$$2x + y + 2z = 7.$$

Type solve ([3 * x + y − z = 0, 2 * x − 3 * y + z = 1, 2 * x + y + 2 * z = 7], [x, y, z]); and simultaneously press <shift> and <enter> keys. The result is,

```
(%i1)  fpprintprec:5;
(fpprintprec)      5
(%i3)  solve([3*x+y-z=0,2*x-3*y+z=1,2*x+y+2*z=7], [x,y,z]);
float(%);
(%o2)  [[x=17/31,y=27/31,z=78/31]]
(%o3)  [[x=0.54839,y=0.87097,z=2.5161]]
```

Solutions of the system of equations are $x = 0.54\ldots$, $y = 0.87\ldots$, and $z = 2.51\ldots$.

To assign x, y, and z the values of the solutions, the codes can be as follows,

```
(%i1)  fpprintprec:5;
(fpprintprec)     5
(%i3)  solve([3*x+y-z=0,2*x-3*y+z=1,2*x+y+2*z=7],[x,y,z]);
float(%);
(%o2)  [[x=17/31,y=27/31,z=78/31]]
(%o3)  [[x=0.54839,y=0.87097,z=2.5161]]
(%i4)  x: rhs(%o3[1][1]);
(x)   0.54839
(%i5)  y: rhs(%o3[1][2]);
(y)   0.87097
(%i6)  z: rhs(%o3[1][3]);
(z)   2.5161
```

Angle

Angles are in radian. This means that cos(60) is cosine of 60 rad, and is not cosine of 60°. If cosine of 60° is needed, we type cos(60/180 * %pi). We convert angle in degree to angle in rad in the argument of the built in function *cos*. In wxMaxima π is typed as %pi.

```
(%i1)  cos(60);
(%o1)  cos(60)
(%i2)  float(%);
(%o2)  -0.9524129804151563
(%i3)  cos(60/180*%pi);
(%o3)  1/2
(%i4)  float(%);
(%o4)  0.5
```

The inverse cosine, inverse sine, and inverse tangent functions i.e. $\cos^{-1}$, $\sin^{-1}$, and $\tan^{-1}$ called *acos*, *asin*, and *atan* in wxMaxima will give angles in radians. If angles in degrees are needed, conversion must be made by multiplying the angle in radian by 180 and division by %pi.

```
(%i1)  acos(1/2);
(%o1)  %pi/3
(%i2)  float(%);
(%o2)  1.047197551196598
(%i3)  float(%*180/%pi);
(%o3)  60.0
```

The codes show that $\cos^{-1}(1/2) = 1.047$ rad $\equiv 60°$.

Logarithm

The built in function *log*(…) is the natural logarithm (logarithm of base *e*). Thus,

$$\log e = \ln e = 1,$$
$$\log(e \times e) = \ln e^2 = 2,$$
$$\log 10 = \ln 10 = 2.3026,$$

as in the following codes,

```
(%i1)  fpprintprec:5;
(fpprintprec)      5
(%i2)  log(%e);
(%o2)  1
(%i3)  log(%e*%e);
(%o3)  2
(%i5)  log(10); float(%);
(%o4)  log(10)
(%o5)  2.3026
```

Here, the Euler's number $e = 2.718\ldots$ is typed as %e in wxMaxima.
To get logarithm of base 10, you divide log(x) by log(10), because,

$$\log_{10} x = \frac{\log_e x}{\log_e 10} = \frac{\ln x}{\ln 10}.$$

The codes below show that $\log_{10}(0.2) = -0.69\ldots$, $\log_{10}(1) = 0$, $\log_{10}(2) = 0.30\ldots$, $\log_{10}(4) = 0.60\ldots$, $\log_{10}(10) = 1$, and $\log_{10}(151) = 2.1\ldots$.

```
(%i1)  fpprintprec:5;
(fpprintprec)     5
(%i2)  float(log(0.2)/log(10));
(%o2)  -0.69897
(%i3)  float(log(1)/log(10));
(%o3)  0.0
(%i4)  float(log(2)/log(10));
(%o4)  0.30103
(%i5)  float(log(4)/log(10));
(%o5)  0.60206
(%i6)  float(log(10)/log(10));
(%o6)  1.0
(%i7)  float(log(151)/log(10));
(%o7)  2.179
```

If one defines log10(x) as log(x)/log(10) at the beginning, then the defined function can be used repeatedly. The above codes become,

```
(%i1)  log10(x):= float( log(x)/log(10) );
(%o1)  log10(x):=float(log(x)/log(10))
(%i2)  fpprintprec:5;
(fpprintprec)      5
(%i3)  log10(0.2);
(%o3)  -0.69897
(%i4)  log10(1);
(%o4)  0.0
(%i5)  log10(2);
(%o5)  0.30103
(%i6)  log10(4);
(%o6)  0.60206
(%i7)  log10(10);
(%o7)  1.0
(%i8)  log10(151);
(%o8)  2.179
```

Differentiation

To differentiate a mathematical expression, use wxMaxima built in function *diff*. For example given $y = 2\sin(5x + \pi/4)$, differentiate y with respect to x, i.e. find dy/dx. We type diff(2 * sin(5 * x + %pi/4), x); and simultaneously press <shift> and <enter> keys. Alternatively, we can first define y followed by the differentiation with respect to x, that is, we input y:2*sin(5*x+%pi/4); followed by diff(y,x); The result is,

```
(%i1)  diff(2*sin(5*x + %pi/4), x);
(%o1)  10*cos(5*x+%pi/4)
(%i2)  y: 2*sin(5*x + %pi/4);
(y)   2*sin(5*x+%pi/4)
(%i3)  diff(y, x);
(%o3)  10*cos(5*x+%pi/4)
```

%pi is a predefined constant π. Other predefined constants are Euler's number %e, Euler–Mascheroni constant %gamma, and golden ratio %phi. Their values can be checked as follows,

```
(%i4)  float(%pi); float(%e); float(%gamma); float(%phi);
(%o1)  3.141592653589793
(%o2)  2.718281828459045
(%o3)  0.5772156649015329
(%o4)  1.618033988749895
```

Integration

To integrate, use built in function *integrate*. For example, calculate $\int_1^4 5x^2 dx$. Key in integrate (5 * x^2, x, 1, 4); and simultaneously press <shift> and <enter> keys. The result is,

```
(%i1)  integrate(5*x^2, x, 1, 4);
(%o1)  105
```

This means *integrate* needs four arguments: function, variable, lower limit, and upper limit.

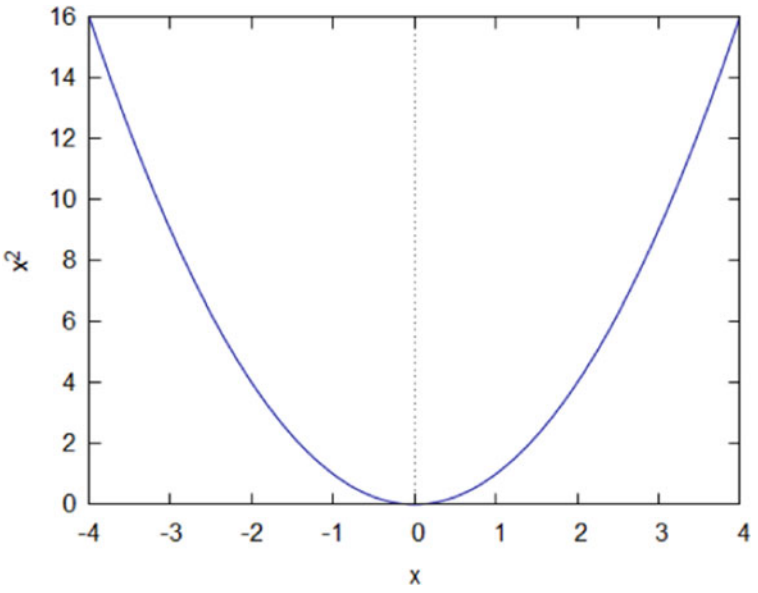
Another built in function to do definite integration is *romberg*. This is a numerical integration built in function. To calculate the same problem above, key in romberg(5*x^2, x, 1, 4); and simultaneously press <shift> and <enter> keys. The result is,

```
(%i1)  romberg(5*x^2, x, 1, 4);
(%o1)  105.0
```

Two Dimensional Plot

To plot in 2D, use the built in function *wxplot2d*. For example, plot the curve $y = x^2$ for $-4 \leq x \leq 4$. Key in wxplot2d(x^2, [x, − 4, 4]); and simultaneously press <shift> and <enter> keys. The result is,

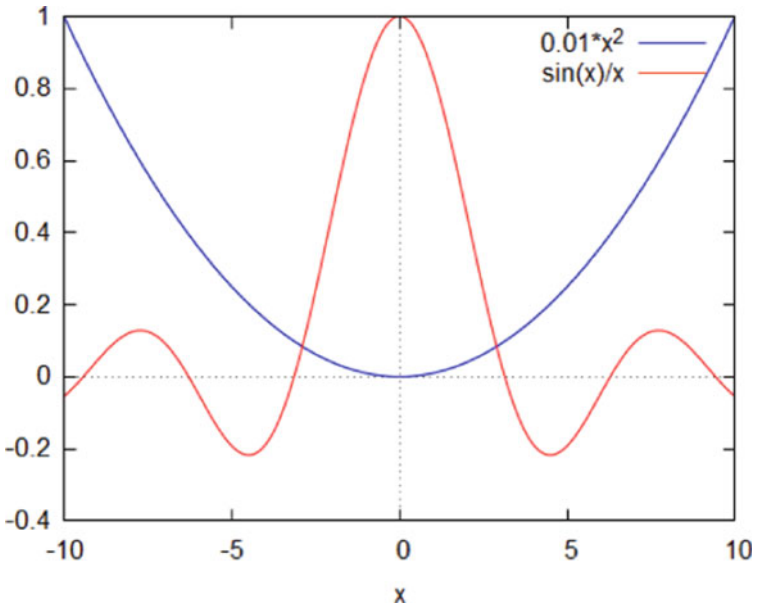
```
(%i1)  wxplot2d( x^2, [x,-4,4] );
```



The command *wxplot2d* needs two arguments. First, the expression of the curve. Second, a *list* consisting of the variable, the lower, and upper limits in square brackets.

Another example, plot two curves $y = 0.01x^2$ and $z = \sin(x)/x$, for $-10 \leq x \leq 10$. We define the two functions and use the definitions in the *wxplot2d* command. The result is,

```
(%i2)  y: 0.01*x^2;  z: sin(x)/x;
(y)    0.01*x^2
(z)    sin(x)/x
(%i3)  wxplot2d([y,z], [x,-10,10] );
```



Vector

For vector calculation, *vect* package or module has to be loaded by the command *load*("*vect*"); A vector is defined as a *list* in square bracket [...]. The operator for dot (scalar) product is the dot . and for vector (cross) product is ~ followed by *express*(%). For example, given vectors, $\boldsymbol{A} = 4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and $\boldsymbol{B} = 5\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}$, calculate $\boldsymbol{A} \cdot \boldsymbol{B}$, magnitude of A and B, the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{A} \times \boldsymbol{B}$, and magnitude of $\boldsymbol{A} \times \boldsymbol{B}$. The wxMaxima calculation is as follows,

```
(%i1)  fpprintprec:5;
(fpprintprec)      5
(%i2)  load("vect");
(%o2)   "C:/maxima-5.43.0/share/maxima/5.43.0/share/vector/
vect.mac"
(%i4)  A:[4,3,2]; B:[5,6,7];
(A)  [4,3,2]
(B)  [5,6,7]
(%i5) A.B;
(%o5) 52
(%i7) magnitudeA: sqrt(A.A); float(%);
(magnitudeA) sqrt(29)
(%o7) 5.3852
(%i9) magnitudeB: sqrt(B.B); float(%);
(magnitudeB) sqrt(110)
(%o9) 10.488
```

```
(%i11) acos(A.B/(magnitudeA*magnitudeB)); float(%);
(%o10) acos(52/(sqrt(29)*sqrt(110)))
(%o11) 0.40098
(%i12) float(%*180/%pi);
(%o12) 22.975
(%i14) A~B; express(%);
(%o13) [4,3,2]~[5,6,7]
(%o14) [9,-18,9]
(%i16) magnitudeAxB: sqrt(%.%); float(%);
(magnitudeAxB) 9*sqrt(6)
(%o16) 22.045
(%i18) asin(magnitudeAxB/(magnitudeA*magnitudeB)); float(%);
(%o17) asin((9*sqrt(6))/(sqrt(29)*sqrt(110)))
(%o18) 0.40098
(%i19) float(%*180/%pi);
(%o19) 22.975
```

The calculation gives, $\boldsymbol{A} \cdot \boldsymbol{B} = 52$, $A = 5.385$, $B = 10.49$, angle between $\boldsymbol{A}$ and $\boldsymbol{B}$ is 23°, $\boldsymbol{A} \times \boldsymbol{B} = 9\mathbf{i} - 18\mathbf{j} + 9\mathbf{k}$, $|\boldsymbol{A} \times \boldsymbol{B}| = 22.05$, angle between $\boldsymbol{A}$ and $\boldsymbol{B}$ is 23°.

Statistics

For statistics, the variable values are entered as a *list*. The built in functions *mean*, *var*, and *std* can be called to calculate mean, variance, and standard deviation of the variable. For example, calculate the mean, variance, and standard deviation of 20.4, 62.5, 61.3, 44.2, 11.1, and 23.7. The result is,

```
(%i1)  fpprintprec:5;
(fpprintprec)      5
(%i2)  x: [20.4, 62.5, 61.3, 44.2, 11.1, 23.7];
(x)   [20.4,62.5,61.3,44.2,11.1,23.7]
(%i3)  mean(x);
(%o3)  37.2
(%i4)  var(x);
(%o4)  402.6
(%i5)  std(x);
(%o5)  20.065
```

The wxMaxima calculation says that the mean of x is 37.2, the variance of x is 402.6, and the standard deviation of x is 20.06.

Linear Least Square Fitting

Determine the best line $y = mx + c$ by the linear least square method of (x, y) data in Table A.1.

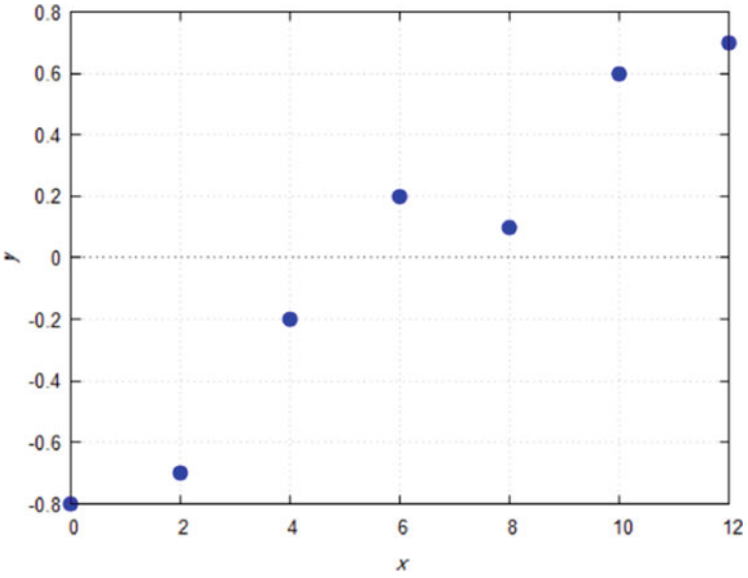
To do least square line fit by wxMaxima, the (x, y) data are entered as *matrix*, then load the *lsquares* routine by *load*("*lsquares*"); command, lastly the predefined command *lsquares_estimates* is called.

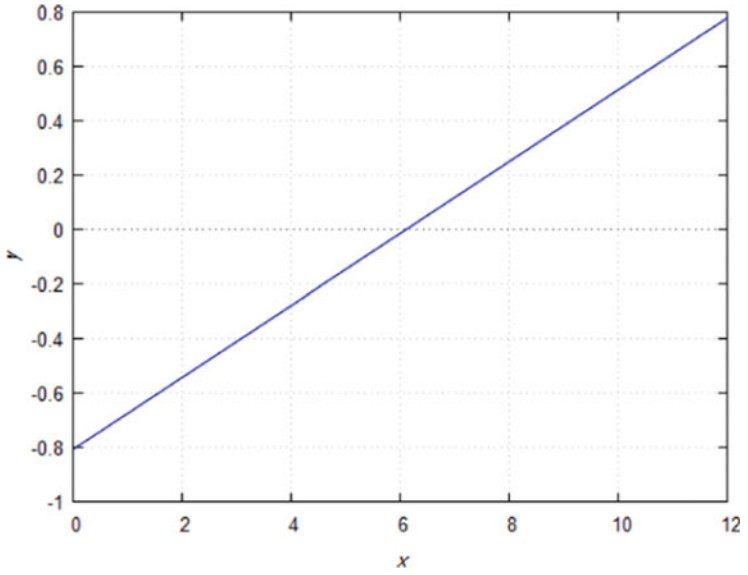
Table A.1 List of (x, y) values

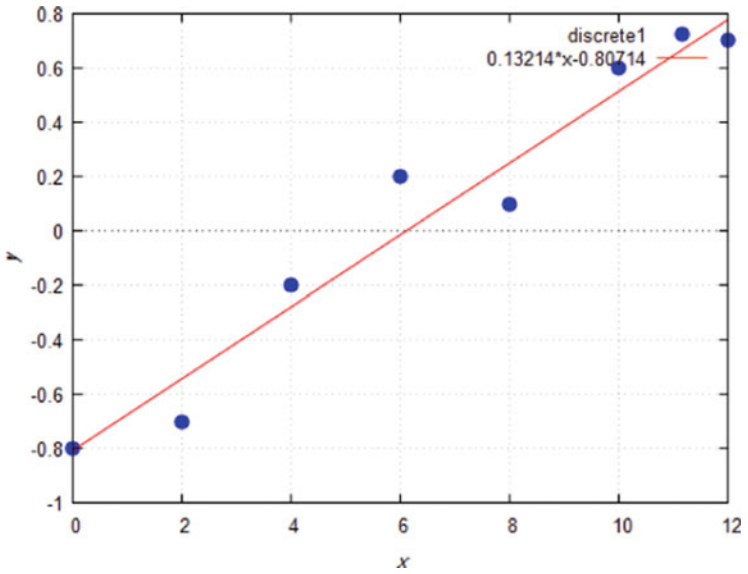
x	y
0	− 0.8
2	− 0.7
4	− 0.2
6	0.2
8	0.1
10	0.6
12	0.7

wxMaxima codes:

```
(%i1)  fpprintprec:5;
(fpprintprec)      5
(%i2) data: matrix([0,-0.8], [2,-0.7], [4,-0.2],
    [6,0.2], [8,0.1], [10,0.6], [12,0.7]);
(data)    matrix(
       [0,  -0.8],
       [2,  -0.7],
       [4,  -0.2],
       [6,   0.2],
       [8,   0.1],
       [10,  0.6],
       [12,  0.7])
(%i3)  load("lsquares");
(%o3)          "C:/maxima-5.43.0/share/maxima/5.43.0/share/
lsquares/lsquares.mac"
(%i5) lsquares_estimates(data,[x,y],y=m*x+c,[m,c])$float(%);
(%o5)  [[m=0.13214,c=-0.80714]]
(%i7)  m: rhs(%o5[1][1]); c: rhs(%o5[1][2]);
(m)   0.13214
(c)   -0.80714
(%i8)  xy: [[0,-0.8], [2,-0.7], [4,-0.2], [6,0.2], [8,0.1],
[10,0.6], [12,0.7]];
(xy)     [[0,-0.8],  [2,-0.7],  [4,-0.2],  [6,0.2],  [8,0.1],
[10,0.6], [12,0.7]]
(%i9)  wxplot2d([discrete, xy],[style, points], grid2d,
[xlabel,"{/Helvetica-Italic   x}"],   [ylabel,"{/Helvetica-
Italic y}"]);
(%i10) y: m*x + c;
(y)  0.13214*x-0.80714
(%i11)  wxplot2d(y,[x,0,12],  grid2d,  [xlabel,"{/Helvetica-
Italic x}"], [ylabel,"{/Helvetica-Italic y}"]);
(%i12) wxplot2d([[discrete,xy],y],[x,0,12], [style,
[points], [lines]], grid2d,[xlabel,"{/Helvetica-Italic x}"],
[ylabel,"{/Helvetica-Italic y}"]);
```


0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
y
0
2
4
6
8
10
12
x


0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
y
0
2
4
6
8
10
12
x



Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2) Assign "data" as matrix of (x, y) values.
(%i3) Load "lssquares" routine.
(%i5) Calculate m and c by the least square fit.
(%o4), (%o5) The results.
(%i7) Assign values of m and c.
(%i8), (%i9) Assign xy as data points and plot the points.
(%i10), (%i11) Assign line y and plot the line.
(%i12) Plot the data point and the line.

The calculation by wxMaxima says that the line fit has the slope $m = 0.13$ and the y axis intercept $c = -0.81$. The line fit is $y = 0.13x - 0.81$. We plot the data points, the fitted line, and the data points with the fitted line in three separate plots.

Simplify

To simplify an expression use *ratsimp*(*expression*); or *radcan*(*expression*); For example, the following codes show that,

$$bx + b\left(\frac{a}{b} - x\right) + a = 2a.$$

```
(%i1)  b*x + b*(a/b - x) + a;
(%o1)  b*x+b*(a/b-x)+a
(%i2)  ratsimp(%);
(%o2)  2*a
(%i3)  radcan(%o1);
(%o3)  2*a
```

Example 1, calculate $\int_0^L \frac{dx}{(L+a-x)^2}$.

```
(%i2)  assume(L>0); assume(a>0);
(%o1)  [L>0]
(%o2)  [a>0]
(%i3)  integrate(1/(L+a-x)^2,x,0,L);
(%o3)  1/a-1/(a+L)
(%i4)  ratsimp(%);
(%o4)  L/(a^2+L*a)
```

This means that,

$$\int_0^L \frac{dx}{(L+a-x)^2} = \frac{1}{a} - \frac{1}{a+L} = \frac{L}{a^2+aL}.$$

Example 2, calculate $\int_0^L \frac{dx}{(L+a-x)^{3/2}}$.

```
(%i2)  assume(L>0); assume(a>0);
(%o1)  [L>0]
(%o2)  [a>0]
(%i3)  integrate(1/(L+a-x)^(3/2), x,0,L);
(%o3)  2/sqrt(a)-2/sqrt(a+L)
(%i4)  radcan(%);
(%o4)  (2*sqrt(a+L)-2*sqrt(a))/(sqrt(a)*sqrt(a+L))
```

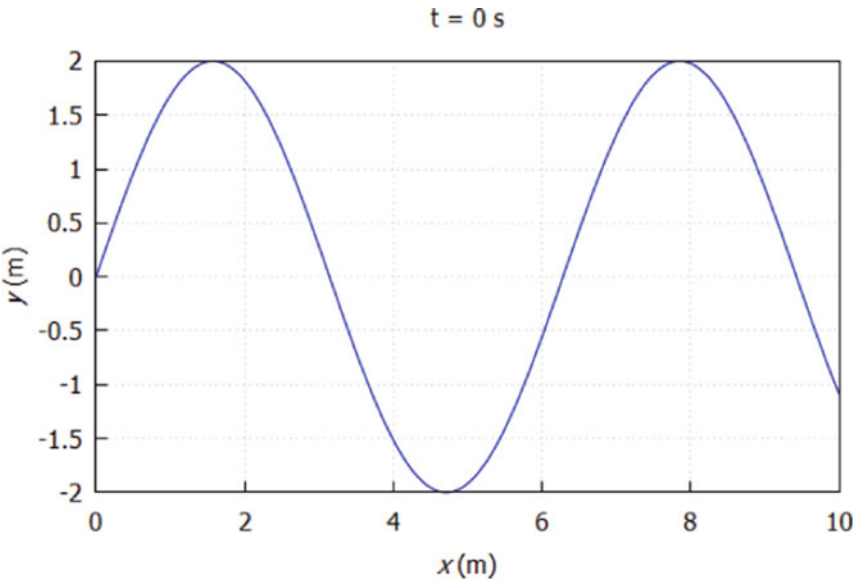
This means that,

$$\int_0^L \frac{dx}{(L+a-x)^{3/2}} = \frac{2}{\sqrt{a}} - \frac{2}{\sqrt{a+L}} = \frac{2\sqrt{a+L}-2\sqrt{a}}{\sqrt{a}\sqrt{a+L}}.$$

Animation

A simple animation of a harmonic wave $y(x, t) = 2 \sin(x - 10t)$ travelling to the right is as follows,

```
(%i1) with_slider_draw(
        t,[0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1],
        title=concat("t = ",t," s"),
        grid=true,
        xlabel="{/Helvetica-Italic x} (m)",
        ylabel="{/Helvetica-Italic y} (m)",
        explicit(2*sin(x - 10*t),x,0,10));
```



To run the animation, right click the graphic and choose *Start Animation.*

Solving Second Order Ordinary Differential Equation

To solve a second order ordinary differential equation (ODE), use predefined functions *ode*2 and *ic*2. Function *ode*2 solves the ODE and gives a general solution of the ODE. Its format is *ode*2(ODE, dependent variable, independent variable). Function *ic*2 sets the initial conditions and gives a particular solution of the ODE. Its format is *ic*2(output of *ode*2, independent variable value, dependent variable value, first derivative of dependent variable value).

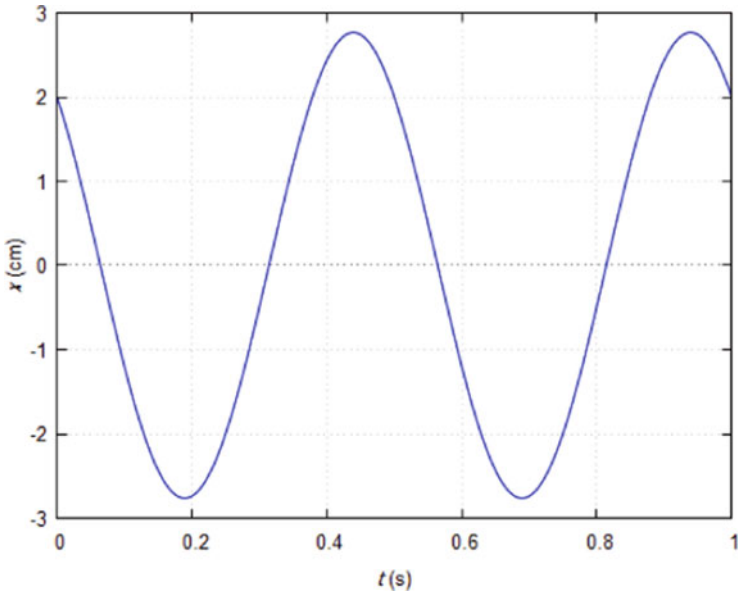
As an example, solve,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$

where $\omega = 4\pi$ rad/s, x is displacement (dependent variable) in cm, and t is time (independent variable) in second. The initial conditions at $t = 0$ s, is $x = 2$ cm, and $dx/dt = -24$ cm/s.

The codes are:

```
(%i2)  fpprintprec:5; omega:4*%pi;
(fpprintprec)      5
(omega) 4*%pi
(%i3)  soln: ode2('diff(x,t,2) + omega^2*x = 0, x, t);
(soln) x=%k1*sin(4*%pi*t)+%k2*cos(4*%pi*t)
(%i5)  ic2(soln, t=0, x=2, 'diff(x,t)=-24); float(%);
(%o4)  x=2*cos(4*%pi*t)-(6*sin(4*%pi*t))/%pi
(%o5)  x=2.0*cos(12.566*t)-1.9099*sin(12.566*t)
(%i6)  x: rhs(%);
(x)   2.0*cos(12.566*t)-1.9099*sin(12.566*t)
(%i7)  wxplot2d(x, [t,0,1], grid2d,
[xlabel,"{/Helvetica-Italic t} (s)"],
[ylabel,"{/Helvetica-Italic x} (cm)"]);
```



Comments on the codes:

(%i2) Set floating point print precision to 5 and assign value of ω.
(%i3) Get a general solution of $d^2x/dt^2 + \omega^2 x = 0$.
(soln) A general solution of the ODE is $x =$ constant $1 \times \sin(4\pi t) +$ constant $2 \times \cos(4\pi t)$.
(%i5) Set the initial conditions and get a particular solution.
(%o4), (%o5) The particular solution.
(%i6) Assign x.
(%i7) Plot x against t for $0 \leq t \leq 1$ s.

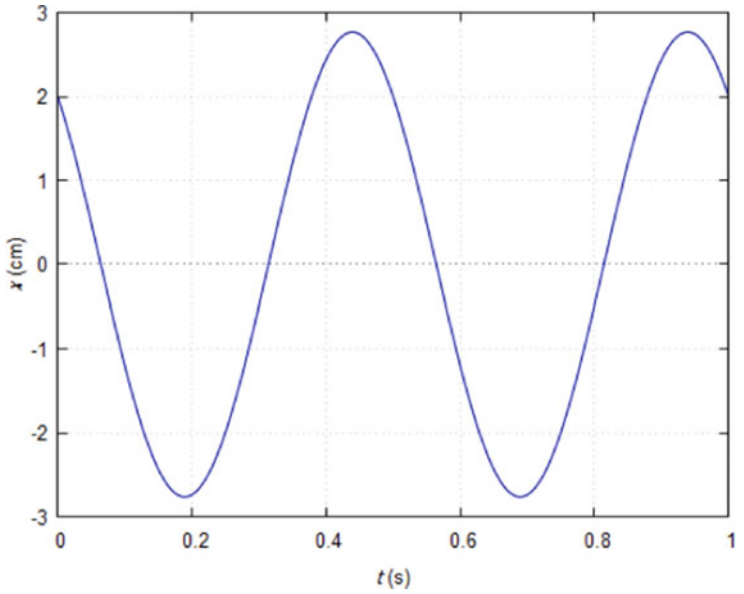
The codes say that the solution of the ODE is, (%o4) or (%o5),

$$\begin{aligned} x &= 2\cos(4\pi t) - \frac{6}{\pi}\sin(4\pi t) \\ &= 2\cos(12.566t) - 1.9099\sin(12.566t). \end{aligned}$$

Another way to solve second order ordinary differential equation by wxMaxima is to use predefined functions *atvalue* and *desolve*. The arguments of *desolve* are the ODE and the function we want a solution. The arguments of *atvalue* are the initial conditions: the dependent variable, the independent variable value, and the dependent variable value. We solve again the above problem.

- wsMaxima codes:

```
(%i2)  fpprintprec:5; omega:4*%pi;
(fpprintprec)      5
(omega) 4*%pi
(%i3)  equation: 'diff(x(t),t,2)+omega^2*x(t)=0;
(equation)  'diff(x(t),t,2)+16*%pi^2*x(t)=0
(%i5)  atvalue(x(t), t=0, 2); atvalue(diff(x(t),t), t=0, -
24);
(%o4)  2
(%o5)  -24
(%i7)  desolve(equation, x(t)); float(%);
(%o6)  x(t)=2*cos(4*%pi*t)-(6*sin(4*%pi*t))/%pi
(%o7)  x(t)=2.0*cos(12.566*t)-1.9099*sin(12.566*t)
(%i8)  define(x(t), rhs(%));
(%o8)  x(t):=2.0*cos(12.566*t)-1.9099*sin(12.566*t)
(%i9)  wxplot2d(x(t), [t,0,1], grid2d,
[xlabel,"{/Helvetica-Italic t} (s)"],
[ylabel,"{/Helvetica-Italic x} (cm)"]);
```



Comments on the codes:

(%i2) Set floating point print precision to 5 and assign value of ω.
(%i3) Define the differential equation.
(%i5) Set the initial conditions.
(%i7) Solve the differential equation and get a particular solution.
(%o6), (%o7) The solution.
(%i8) Define $x(t)$;
(%i9) Plot $x(t)$ for $0 \leq t \leq 1$ s.

Solving First Order Ordinary Differential Equation

To solve a first order ordinary differential equation (ODE), use predefined functions *ode*2 and *ic*1. Function *ode*2 solves the ODE and gives a general solution of the ODE. Its format is *ode*2(ODE, dependent variable, independent variable). Function *ic*1 sets the initial condition and gives a particular solution of the ODE. Its format is *ic*1 (output of *ode*2, independent variable value, dependent variable value).

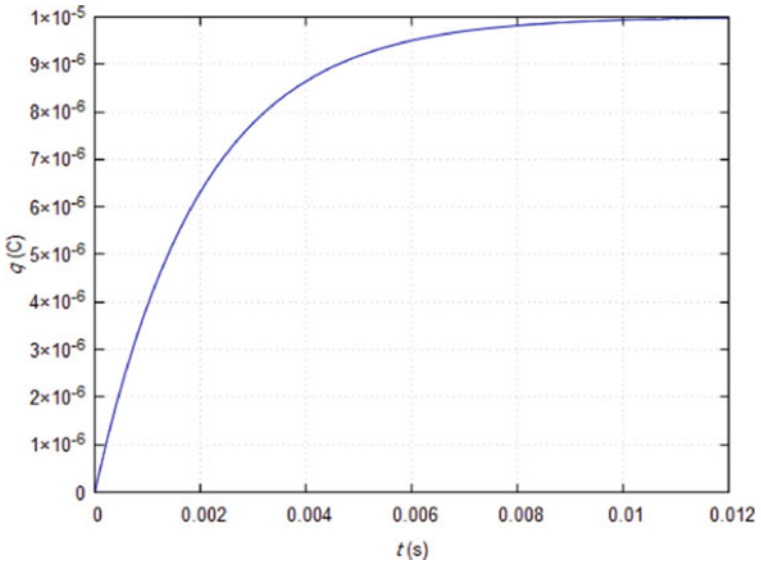
As an example, solve a direct current RC circuit equation,

$$R\frac{dq}{dt} + \frac{q}{C} = \mathcal{E},$$

where resistance $R = 2000\ \Omega$, capacitance $C = 1 \times 10^{-6}$ F, emf $= 10$ V, charge q (dependent variable) is in coulomb and time t (independent variable) is in second. The initial condition is, at $t = 0$ s, $q = 0$ C.

The codes are:

```
(%i2)  fpprintprec:5; ratprint:false;
(fpprintprec)      5
(ratprint)  false
(%i3)  soln: ode2(R*'diff(q,t) + q/C =emf, q, t);
(soln) q=%e^(-t/(C*R))*(C*emf*%e^(t/(C*R))+%c)
(%i4)  ic1(soln, t=0, q=0);
(%o4)  q=%e^(-t/(C*R))*(C*emf*%e^(t/(C*R))-C*emf)
(%i5)  q: rhs(%);
(q)  %e^(-t/(C*R))*(C*emf*%e^(t/(C*R))-C*emf)
(%i8)  R:2000; C:1e-6; emf:10;
(R)   2000
(C)   10.0*10^-7
(emf)  10
(%i9)  wxplot2d(q, [t,0,12e-3], grid2d,
[xlabel,"{/Helvetica-Italic t} (s)"], [ylabel,"{/Helvetica-
Italic q} (C)"]);
```



Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3) Solve ODE $R\frac{dq}{dt} + \frac{q}{C} = \mathcal{E}$, and get a general solution.
(soln) A general solution is $q = e^{-t/(RC)}(C\mathcal{E}\ e^{t/(RC)} + \text{constant})$.

(%i4) Set the initial condition and get a particular solution.
(%o4) The particular solution.
(%i5) Assign the solution to q.
(%i8) Assign values of R, C, and emf.
(%i9) Plot q against t for $0 \leq t \leq 12 \times 10^{-3}$ s.

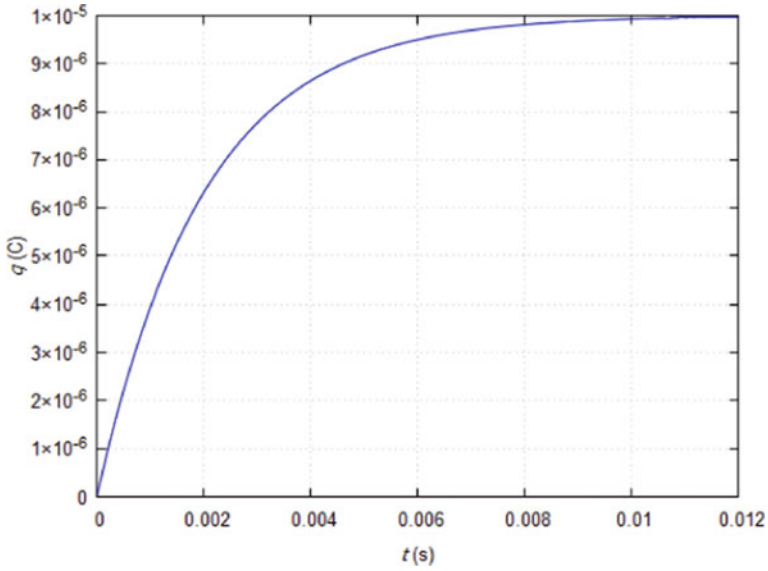
The codes say that the solution of the ODE is, (%o4),

$$\begin{aligned} q &= e^{-t/(RC)}(C\mathcal{E}\, e^{t/(RC)} - C\mathcal{E}) \\ &= C\mathcal{E}\,(1 - e^{-t/(RC)}). \end{aligned}$$

Another way to solve first order ordinary differential equation by wxMaxima is to use predefined functions *atvalue* and *desolve*. The arguments of *desolve* are the ODE and the function we want a solution. The arguments of *atvalue* are the initial conditions: the dependent variable, the independent variable value, and the dependent variable value. We solve again the above problem.

- wsMaxima codes:

```
(%i1)  fpprintprec:5;
(fpprintprec)      5
(%i2)  equation: R*'diff(q(t),t)+ q(t)/C = emf;
(equation)   R*('diff(q(t),t,1))+q(t)/C=emf
(%i3)  atvalue(q(t), t=0, 0);
(%o3)  0
(%i5)  desolve(equation, q(t)); float(%);
(%o4)  q(t)=C*emf-C*emf*%e^(-t/(C*R))
(%o5)  q(t)=C*emf-(1.0*C*emf)/2.7183^(t/(C*R))
(%i6)  define(q(t), rhs(%));
(%o6)  q(t):=C*emf-(1.0*C*emf)/2.7183^(t/(C*R))
(%i9)  R:2000; C:1e-6; emf:10;
(R)   2000
(C)   10.0*10^-7
(emf)  10
(%i10) wxplot2d(q(t), [t,0,12e-3], grid2d,
[xlabel,"{/Helvetica-Italic t} (s)"],
[ylabel,"{/Helvetica-Italic q} (C)"]);
```



Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2) Define the differential equation.
(%i3) Set the initial condition.
(%i5) Solve the differential equation and get a particular solution.
(%o4), (%o5) The solution.
(%i6) Define $q(t)$.
(%i9) Assign values of R, C, and emf.
(%i10) Plot $q(t)$ for $0 \leq t \leq 12 \times 10^{-3}$ s.

We end this *Introduction to wxMaxima* here. For further use and application of wxMaxima the readers are required to look for internet sources and the *Help* menu at the top right of the wxMaxima window.

Appendix B
Physical Constants

Symbol	Value	Physical quantity
c	3.0×10^{8} m s^{-1}	Speed of light
μ_0	$4\pi \times 10^{-7}$ H m^{-1}	Permeability of free space, Permeability constant
ε_0	8.85×10^{-12} F m^{-1}	Permittivity of free space, Permittivity constant
e	-1.6022×10^{-19} C	Electron charge
h	6.63×10^{-34} J s	Planck constant
m_e	9.11×10^{-31} kg 5.48×10^{-4} u	Mass of electron
m_p	1.673×10^{-27} kg 1.007825 u	Mass of proton
m_d	3.34×10^{-27} kg 2.014102 u	Mass of deuteron
R	8.31 J K^{-1} mol^{-1}	Molar gas constant, Universal gas constant
R_H	1.097×10^{7} m^{-1}	Rydberg constant
N_A	6.02×10^{23} mol^{-1}	Avogadro number
$k_B = \frac{R}{N_A}$	1.38×10^{-23} J K^{-1}	Boltzmann constant
G	6.67×10^{-11} N m^2 kg^{-2}	Universal gravitational constant
g	9.8 m s^{-2}	Acceleration of gravity, Gravitational acceleration
$k = \frac{1}{4\pi\varepsilon_0}$	9×10^{9} N m^2 C^{-2}	Electrostatic constant
$k_m = \frac{\mu_0}{4\pi}$	10^{-7} Wb A^{-1} m^{-1}	Magnetic constant

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Appendix C
Conversion Factors

Quantity	Conversion
Length	1 m = 39.37 in. = 3.28 ft = 100 cm
	1 ft = 30.48 cm
	1 in. = 2.54 cm
	1 mile = 5280 ft = 1.609 km
	1 Å = 10^{-8} cm = 10^{-10} m
	1 μm = 10^{-4} cm = 10^{-6} m 1 nm = 10^{-7} cm = 10^{-9} m
Mass	1 slug = 14.59 kg = 32.2 lb
	1 g = 10^{-3} kg = 6.85 × 10^{-5} slug
	1 u = 1.66 × 10^{-27} kg = 931.5 MeV/c^2
Time	1 year = 365 day = 3.16 × 10^7 s
	1 day = 24 h = 1.44 × 10^3 min = 8.64 × 10^4 s 1 h = 60 min = 3600 s
Area	1 cm^2 = 0.155 $in.^2$
	1 $in.^2$ = 6.452 cm^2
	1 m^2 = 10.76 ft^2
	1 ft^2 = 144 $in.^2$ = 0.0929 m^2 1 ha = 10^4 m^2 = 2.471 acre 1 acre = 4047 m^2 = 0.4047 ha = 4840 $yard^2$ = 43,560 ft^2
Volume	1 m^3 = 10^6 cm^3 =10^3 dm^3 = 10^3 L 1 dm^3 = 1 L 1 cm^3 = 1 mL
	1 m^3 = 35.3 ft^3 = 6.1 × 10^4 $in.^3$
	1 ft^3 = 2.83 × 10^{-2} m^3 = 28.32 L
Speed	1 mile h^{-1} = 1.47 ft s^{-1} = 0.447 m s^{-1} = 1.609 km h^{-1}
	1 m s^{-1} = 100 cm s^{-1}= 3.281 ft s^{-1} = 3.6 km h^{-1} = 2.237 mile h^{-1}

(continued)

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(continued)

Quantity	Conversion
	1 km h^{-1} = 0.278 m s^{-1} = 0.621 mile h^{-1} = 0.911 ft s^{-1} 1 mile min^{-1} = 60 mile h^{-1} = 88 ft s^{-1}
Acceleration	1 m s^{-2} = 3.28 ft s^{-2} = 100 cm s^{-2}
	1 ft s^{-2} = 0.3048 m s^{-2} = 30.48 cm s^{-2}
Force	1 N = 10^5 dyne = 0.2247 lb
Pressure	1 N m^{-2} = 1 Pa = 10 dyne cm^{-2} = 1.45 × 10^{-4} lb $in.^{-2}$
	1 atm = 1.013 × 10^5 Pa
Energy and power	1 J = 10^7 erg = 0.239 cal = 0.738 ft lb
	1 eV = 1.6 × 10^{-19} J = 1.6 × 10^{-12} erg
	1 cal = 4.18 J
	1 horse power = 745 W = 550 ft lb s^{-1}
Magnetic field	1 T = 10^4 gauss 1 T = 1 Wb m^{-2}

Appendix D
Mathematical Formulae

Roots of a quadratic equation

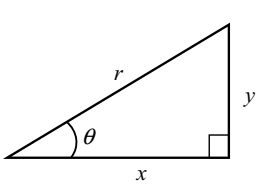
$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

Trigonometric identity

$\sin\theta = \frac{y}{r}$ $\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$

$\cos\theta = \frac{x}{r}$ $\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$

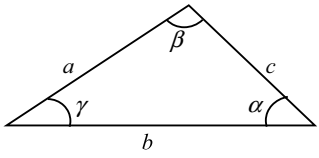
$\tan\theta = \frac{y}{x}$ $\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$



$\sin^2\theta + \cos^2\theta = 1$	$x^2 + y^2 = r^2$	$1 + \tan^2\theta = \sec^2\theta$
$\sin 2\theta = 2\sin\theta\cos\theta$	$\sin(-\theta) = -\sin\theta$	$\sin^2\theta = \frac{1-\cos 2\theta}{2}$
$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$	$\cos(-\theta) = \cos\theta$	$\cos^2\theta = \frac{1+\cos 2\theta}{2}$
$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$	$\tan(-\theta) = -\tan\theta$	$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$
$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$	$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$	
$\sin\theta\sin\phi = \frac{1}{2}[\cos(\theta-\phi) - \cos(\theta+\phi)]$	$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$	
$\cos\theta\cos\phi = \frac{1}{2}[\cos(\theta-\phi) + \cos(\theta+\phi)]$	$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$	
$\sin\theta\cos\phi = \frac{1}{2}[\sin(\theta+\phi) + \sin(\theta-\phi)]$	$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$	
$\cos\theta\sin\phi = \frac{1}{2}[\sin(\theta+\phi) - \sin(\theta-\phi)]$	$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$	

Cosine rule
$c^2 = a^2 + b^2 - 2ab\cos\gamma$

Sine rule
$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$



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Series expansion	
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$(1-x)^n = 1 - nx + \frac{n(n-1)x^2}{2!} - \frac{n(n-1)(n-2)x^3}{3!} + \cdots$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$
$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$	$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots$
$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \cdots$	$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + \cdots$

Differentiation	Integration
$\frac{d}{dx}\sin x = \cos x$	$\int \sin x\, dx = -\cos x$
$\frac{d}{dx}\cos x = -\sin x$	$\int \cos x\, dx = \sin x$
$\frac{d}{dx}e^x = e^x$	$\int e^x\, dx = e^x$
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n\, dx = \frac{x^{n+1}}{n+1}$
$\frac{d}{dx}\ln(x + \sqrt{x^2+a^2}) = \frac{1}{\sqrt{x^2+a^2}}$	$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2})$
$\frac{d}{dx}\ln(x + \sqrt{x^2-a^2}) = \frac{1}{\sqrt{x^2-a^2}}$	$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln(x + \sqrt{x^2-a^2})$
$\frac{d}{dx}\frac{1}{(x^2+a^2)^{1/2}} = -\frac{x}{(x^2+a^2)^{3/2}}$	$\int \frac{x\,dx}{(x^2+a^2)^{3/2}} = -\frac{1}{(x^2+a^2)^{1/2}}$
$\frac{d}{dx}\frac{x}{a^2(x^2+a^2)^{1/2}} = \frac{1}{(x^2+a^2)^{3/2}}$	$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{1/2}}$
$\frac{d}{dx}(a^2-x^2)^{3/2} = -3x(a^2-x^2)^{1/2}$	$\int x(a^2-x^2)^{1/2}dx = -\frac{1}{3}(a^2-x^2)^{3/2}$

Dot (scalar) product of two vectors
Given two vectors, $A = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$, $\boldsymbol{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$, then, $\boldsymbol{A} \cdot \boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z$, also the magnitudes of A and B are, $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$, $B = \sqrt{B_x^2 + B_y^2 + B_z^2}$ Also, $\boldsymbol{A} \cdot \boldsymbol{B} = AB\cos\theta$, where θ is the small angle between $\boldsymbol{A}$ and $\boldsymbol{B}$

Cross (vector) product of two vectors

Given two vectors,
$\boldsymbol{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$,
$\boldsymbol{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$,
then,

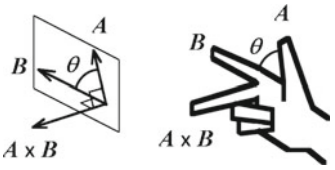
$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$
$$= (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

Also,
$|\boldsymbol{A} \times \boldsymbol{B}| = AB\sin\theta$,
where θ is the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$, and,

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2},$$

$$B = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

The right hand rule



Cramer's rule

The solutions for
$a_1x + b_1y = c_1$,
$a_2x + b_2y = c_2$,
are

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1},$$
$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

Logarithm

$\log_b x = \frac{\log_{10} x}{\log_{10} b} = \frac{\log_e x}{\log_e b}$	$x = b^a \Rightarrow \log_b x = a$
$\log_x x = \log_{10} 10 = \log_e e = \log_2 2 = 1$	$1000 = 10^3 \Rightarrow \log_{10} 1000 = 3$
$2 = e^{0.693} \Rightarrow \log_e 2 = 0.693$	$512 = 2^8 \Rightarrow \log_2 512 = 8$

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